Journal of Algebra and Related Topics
Vol. 14, No. 2, 2026, pp. 21-32. Research Article

Identity meet graph of elements of lattices

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Abstract. Let $\mathcal{L}$ be a complete lattice. The identity meet graph of elements of $\mathcal{L}$, denoted by $\mathbb{IMG}(\mathcal{L})$, is an undirected simple graph whose vertices are all nontrivial elements of $\mathcal{L}$ and two distinct elements x and y are adjacent if and only if $x \vee y = 1$ and $x \wedge y \neq 0$. The basic properties and possible structures of the graph $\mathbb{IMG}(\mathcal{L})$ are investigated. The connectedness, clique number, domination number, independence number, chromatic number of $\mathbb{IMG}(\mathcal{L})$ and their relations to algebraic properties of $\mathcal{L}$ are explored.

Keywords: Lattice, Atom, Coatom, Identity meet graph.
AMS Subject Classification 2010: 97H50, 06A11, 16G30, 06D25.

1 Introduction

In abstract algebra, structures composed of certain subobjects tend to have their distinct qualities by way of lattice theoretic properties. This is the case (to mention only the presumably most prominent example) for the ideals of a commutative ring. The present paper can be placed in this context. The study of algebraic structures, using the properties of graph theory, tends to an exciting research topic in last decade. Associating a graph with an algebraic structure allows us to obtain characterizations and representations of special classes of algebraic structures in terms of graphs and vice versa (see for example [1–7], [10], [12–19], [21]).

Beck [4], Anderson and Naseer [2], and Anderson and Livingston [1] et. al. have studied graphs on commutative rings. One of the most important graphs which have been studied is the intersection graph. Bosak [5] defined the intersection graph of semigroups. *Csàkàny* and *Pollàk* [7] studied the graph of subgroups of a finite group. The intersection graph of ideals of a ring was considered by Chakrabarty, Ghosh, Mukherjee and Sen [6]. Comaximal intersection graph of ideals of rings, i.e., a simple graph whose vertices are nontrivial ideals of a ring R and two vertices I, J are adjacent if and only if $I + J = R$ and $I \cap J \neq 0$, was investigated by Roy,

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Received: 30 January 2025/ Revised: 16 November 2025/ Accepted: 14 February 2026
DOI: 10.22124/JART.2026.29711.1766

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Budhraja and Rajkhowa in [17]. The intersection graph of ideals of rings, submodules of modules and lattices has been investigated by several authors (see for example [3], [15–17], [19], [21]).

Let $\mathcal{L}$ be a complete distributive lattice. The purpose of this paper is to investigate a graph associated to a lattice $\mathcal{L}$ called the identity meet graph of elements of $\mathcal{L}$. The identity meet graph of elements of $\mathcal{L}$ is a simple graph $\mathbb{IMG}(\mathcal{L})$ whose vertices are all nontrivial elements and two distinct elements x and y are adjacent if and only if $x \vee y = 1$ and $x \wedge y \neq 0$. Among many results in this paper, Section 2 contains elementary observations needed later on. In Section 3, We characterize the lattices for which the identity meet graph of elements of lattices are empty, disconnected, connected, complete graph and complete bipartite. The diameter and girth of $\mathbb{IMG}(\mathcal{L})$ is also described. In Section 4, the clique number, domination number, global domination number, independence number, chromatic number of $\mathbb{JN}(\mathcal{L})$ and their relations to algebraic properties of $\mathcal{L}$ are explored.

2 Preliminaries

Let G be a simple graph with vertex set $\mathcal{V}(G)$ and edge set $\mathcal{E}(G)$. The degree of a vertex v of the graph G, denoted by $\deg_G(v)$, is the number of edges incident to v. A vertex of degree zero (no edges going in or out of it) is said to be an isolated vertex. The (open) neighborhood $N(v)$ of a vertex v of $\mathcal{V}(G)$ is the set of vertices that are adjacent to v. A graph G is said to be connected if there exists a path between any two distinct vertices, G is a complete graph if every pair of distinct vertices of G is adjacent, and K_n will stand for a complete graph with n vertices. A graph which is not a connected graph is called a disconnected graph. A graph that does not contain any edges is called a totally disconnected graph. If the vertices of G can be partitioned into two disjoint sets V_1 and V_2 with every vertex of V_1 is adjacent to any vertex of V_2 and no two vertices belonging to same set are adjacent, then G is called a complete bipartite graph. If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$. If one of the partite sets contains exactly one element, then the graph becomes a star graph. If G graph does not have K_5 or $K_{3,3}$ as its subgraph, then G is planar. Let u and v be elements of $\mathcal{V}(G)$. We say that u is a universal vertex of G if u is adjacent to all other vertices of G and write $u \backsim v$ if u and v are adjacent. In a connected graph G, the distance $d(u, v)$ is the length of the shortest path from u to v if such path exists, otherwise, $d(a, b) = \infty$. The diameter of G is $\operatorname{diam}(G) = \sup\{d(a, b) : a, b \in \mathcal{V}(G)\}$. If G is not connected, then $\operatorname{diam}(G) = \infty$. The complement graph of G denoted by $\widehat{G}$ is the graph with vertex set $\mathcal{V}(G)$ such that two vertices are adjacent in $\widehat{G}$ if and only if they are not adjacent in G. The girth of a graph G, denoted by $\operatorname{gr}(G)$, is the length of a shortest cycle in G. If G has no cycles, then $\operatorname{gr}(G) = \infty$. A clique of a graph is its maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the clique number of G. A subset S of $\mathcal{V}(G)$ is said to be an independent set if no two vertices of S are adjacent. The independence number $\alpha(G)$ is the maximum size of an independent set in G. If $\mathcal{V}(G)$ can be partitioned in an independent set and a clique then G is said to be split. A set $D \subseteq \mathcal{V}(G)$ is said to be a dominating set if every vertex not in D is adjacent to at least one of the members of D. The cardinality of smallest dominating set is the domination number of the graph G and is denoted by $\gamma(G)$. A set $D \subseteq \mathcal{V}(G)$ is called a global dominating set of G if it is a dominating set for both the graphs G

and its complement G. The minimum cardinality of a global dominating set is called the global domination number of G and is denoted by $\gamma_g(G)$. The chromatic number of G, denoted by $\chi(G)$ is the minimum number of colors assigning to the vertices of G so that no two adjacent vertices have the same color. Note that a graph whose vertices set is empty is a null graph and a graph whose edge set is empty is an empty graph. For a connected graph G, x is a cut vertex of G if $G \setminus \{x\}$ is not connected [20].

A poset $(\mathcal{L}, \leq)$ is a lattice if $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$ exist for all $a, b \in \mathcal{L}$ (and call $\wedge$ the meet and $\vee$ the join). A lattice $\mathcal{L}$ is complete when each of its subsets X has a least upper bound and a greatest lower bound in $\mathcal{L}$. Setting $X = \mathcal{L}$, we see that any nonvoid complete lattice contains a least element 0 and a greatest element 1 (in this case, we say that $\mathcal{L}$ is a lattice with 0 and 1). A lattice $\mathcal{L}$ is called a distributive lattice if $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ for all $x, y, z \in \mathcal{L}$ (equivalently, $\mathcal{L}$ is distributive if $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ for all $x, y, z \in \mathcal{L}$). We say that an element x in a lattice $\mathcal{L}$ is an atom (resp. coatom) if there is no $y \in \mathcal{L}$ such that $0 < y < x$ (resp. $x < y < 1$). The set of all coatom (resp. atom) elements of $\mathcal{L}$ is denoted by $\mathcal{CA}(\mathcal{L})$ (resp. $\mathcal{A}(\mathcal{L})$). If $\mathcal{L}$ is a complete lattice, then the meet of all the coatoms of $\mathcal{L}$, denoted $\mathrm{J}(\mathcal{L})$, is called the Jacobson radical of the lattice $\mathcal{L}$ (i.e. $\mathrm{J}(\mathcal{L}) = \bigwedge_{c \in \mathcal{CA}(\mathcal{L})} c$). An element x of a lattice $\mathcal{L}$ is nontrivial (resp. proper) if $x \neq 0, 1$ (resp. $x \neq 1$). An element x of a lattice $\mathcal{L}$ is called essential (written $x \trianglelefteq \mathcal{L}$), if there is no nonzero $y \in \mathcal{L}$ such that $x \wedge y = 0$. A lattice $\mathcal{L}$ is called uniform if every non-zero element in $\mathcal{L}$ is essential. A lattice $\mathcal{L}$ is called $\mathcal{L}$-domain if $a \vee b = 1$ $(a, b \in \mathcal{L})$, then $a = 1$ or $b = 1$ [9]. A lattice $\mathcal{L}$ is said to be an Artinian lattice if every descending chain of elements in $\mathcal{L}$ is stationary.

Assume that $(\mathcal{L}_1, \leq_1), (\mathcal{L}_2, \leq_2), \ldots, (\mathcal{L}_n, \leq_n)$ are lattices $(n \geq 2)$ and let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$. We set up a partial order $\leq_c$ on $\mathcal{L}$ as follows: for each $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathcal{L}$, we write $x \leq_c y$ if and only if $x_i \leq_i y_i$ for each $i \in \{1, 2, \ldots, n\}$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}, \leq_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n)$ and $x \wedge_c y = (x_1 \wedge y_1, \ldots, x_n \wedge y_n)$. In this case, we say that $\mathcal{L}$ is a decomposable lattice. We need the following lemma proved Lemma 2.1 in [14] and Proposition 2.3 in [11], respectively.

Lemma 1. *(1) Let $\mathcal{L}$ be a complete lattice. If a is a nontrivial element of $\mathcal{L}$, then there exists a coatom element c of $\mathcal{L}$ such that $a \leq c$.*

(2) Suppose that $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice and let $a = (a_1, a_2)$ be a proper element of $\mathcal{L}$. Then a is an atom element of $\mathcal{L}$ if and only if $a = (a_1, 0)$ for some atom element a_1 of $\mathcal{L}_1$ or $a = (0, a_2)$ for some atom element a_2 of $\mathcal{L}_2$.

The undefined terms related to lattice theory are taken from [8] and terms related to graph theory are taken from [20].

3 Basic properties of $\mathbb{IMG}(\mathcal{L})$

Throughout this paper we shall assume, unless otherwise stated, that $\mathcal{L}$ is a *complete distributive lattice*. In this section, we collect some basic properties concerning the identity meet graph of elements of $\mathcal{L}$. We remind the reader with the following definition.

Definition 1. *The identity meet graph of elements $\mathbb{IMG}(\mathcal{L})$ of $\mathcal{L}$ is simple undirected graph whose vertices are all nontrivial elements of $\mathcal{L}$ (i.e. different from 0 and 1)) and any two distinct vertices a and b are adjacent if and only if $a \vee b = 1$ and $a \wedge b \neq 0$.*

Proposition 1. *Let $\mathcal{L}$ be a lattice. Then $\mathbb{IMG}(\mathcal{L})$ is an empty graph if and only if $\mathcal{L}$ is a $\mathcal{L}$-domain.*

Proof. This follows directly from the definitions. □

Henceforth we will assume that all considered lattices $\mathcal{L}$ are not $\mathcal{L}$-domain and have at least two nontrivial elements, i.e. $|\mathcal{V}\ (\mathbb{IMG}(\mathcal{L}))| \geq 2$. The next theorem gives a more explicit description of the isolated vertices of $\mathbb{IMG}(\mathcal{L})$.

Compare the next theorem with Theorem 2.2 in [17].

Theorem 1. *For the lattice $\mathcal{L}$, the following statements are equivalent:*

(1) There is an isolated vertex a in $\mathbb{IMG}(\mathcal{L})$;

(2) Either $a \leq c$ for all $c \in \mathcal{CA}(\mathcal{L})$ or $a \wedge c = 0$ for some $c \in \mathcal{CA}(\mathcal{L})$.

Proof. (1) ⇒ (2) Let a be an isolated vertex in $\mathbb{IMG}(\mathcal{L})$. If $a \leq c$ for all $c \in \mathcal{CA}(\mathcal{L})$, then we are done. So we may assume that there is a coatom element c such that $a \nleq c$ which implies that $a \vee c = 1$, as c is a coatom. Therefore, a is an isolated vertex gives $a \wedge c = 0$.

(2) ⇒ (1) If there is an element a of $\mathcal{L}$ such that $a \leq c$ for all $c \in \mathcal{CA}(\mathcal{L})$ and $x \in \mathcal{V}(\mathbb{IMG}(\mathcal{L}))$, then there exists $c' \in \mathcal{CA}(\mathcal{L})$ such that $x \vee a \leq c' \neq 1$ by Lemma 1; hence x and a are not adjacent in $\mathbb{IMG}(\mathcal{L})$ and so a is an isolated vertex in $\mathbb{IMG}(\mathcal{L})$. Suppose that there is an element a of $\mathcal{L}$ such that $a \nleq c$ for some coatom element c with $a \wedge c = 0$ and let $y \in \mathcal{V}(\mathbb{IMG}(\mathcal{L}))$. If $y \leq c$, then $y \wedge a \leq a \wedge c = 0$ gives y and a are not adjacent in $\mathbb{IMG}(\mathcal{L})$. If $y \nleq c$, then $y \vee c = 1$ which implies that $y \vee a = (y \vee a) \wedge (y \vee c) = y \vee (a \wedge c) = y \neq 1$; so y and a are not adjacent in $\mathbb{IMG}(\mathcal{L})$, i.e. a is an isolated vertex in $\mathbb{IMG}(\mathcal{L})$. □

Corollary 1. *For the lattice $\mathcal{L}$, if $x \leq \mathrm{J}(\mathcal{L})$, then x is an isolated vertex in $\mathbb{IMG}(\mathcal{L})$.*

Proof. If $a \leq \mathrm{J}(\mathcal{L}) = \bigwedge_{c \in \mathcal{CA}(\mathcal{L})} c$, then $a \leq c$ for all $c \in \mathcal{CA}(\mathcal{L})$. Now the assertion follows from Theorem 1. □

Theorem 2. *If the lattice $\mathcal{L}$ has an atom element a, then a is an isolated vertex in $\mathbb{IMG}(\mathcal{L})$.*

Proof. If $b \in \mathcal{V}(\mathbb{IMG}(\mathcal{L}))$ with $a \neq b$, then either $a \wedge b = 0$ or $a \wedge b \neq 0$. If $a \wedge b = 0$, then a and b are not adjacent in $\mathbb{IMG}(\mathcal{L})$. If $a \wedge b \neq 0$, then $a \wedge b \leq a$ gives $a \leq b$ and so $a \vee b = b \neq 1$. This shows that a is an isolated vertex in $\mathbb{IMG}(\mathcal{L})$. □

Lemma 2. *For an Artinian lattice $\mathcal{L}$, the following hold:*

(1) If e is a nontrivial element of $\mathcal{L}$, then there is an atom element a of $\mathcal{L}$ such that $a \leq e$. In particular, $\mathcal{A}(\mathcal{L}) \neq \emptyset$;

(2) If $\mathcal{L}$ has a unique atom element a, then $\mathrm{J}(\mathcal{L}) \neq 0$.

Proof. (1) If e is atom, we are done. So, assume that e is not atom. Suppose the result is false. Then there exists $e_1 \in \mathcal{L}$ such that $e_1 < e$. By repeated use of this observation we get a descending chain $1 > e > e_1 > e_2 \cdots$ which is a contradiction, i.e. (1) holds. The "in particular" statement is clear.

(2) Since for each coatom element c of $\mathcal{L}$ we have $a \leq c$ by (1), we conclude that $0 < a \leq \mathrm{J}(\mathcal{L})$ and so $\mathrm{J}(\mathcal{L}) \neq 0$. □

Corollary 2. *If $\mathcal{L}$ is an Artinian lattice, then $\mathbb{IMG}(\mathcal{L})$ has at least one isolated vertex.*

Proof. Since $\mathcal{L}$ is Artinian, we conclude that $\mathcal{A}(\mathcal{L}) \neq \emptyset$ by Lemma 2. Now the assertion follows from Theorem 2. □

Definition 2. *A lattice $\mathcal{L}$ is called colocal if it has exactly one coatom element c such that $x \leq c$ for all $1 \neq x \in \mathcal{L}$.*

In the following theorem, we give a condition under which the graph $\mathbb{IMG}(\mathcal{L})$ is totally disconnected. Compare the next theorem with Theorem 2.1 in [17].

Theorem 3. *For a lattice $\mathcal{L}$, the following statements are equivalent:*

(1) $\mathbb{IMG}(\mathcal{L})$ is totally disconnected;

(2) $\mathcal{L}$ is colocal or every vertex in $\mathbb{IMG}(\mathcal{L})$ is coatom as well as atom.

Proof. (1) $\Rightarrow$ (2) Let $\mathbb{IMG}(\mathcal{L})$ be totally disconnected. Consider two distinct nontrivial elements a and b of $\mathcal{L}$. By the hypothesis, either $a \vee b \neq 1$ or $a \wedge b = 0$. If $a \vee b \neq 1$, then Lemma 1 shows that $a \vee b \lneqq c$, where c is a coatom element of $\mathcal{L}$. In this case, $a \leq c$ and $b \leq c$ and so $\mathcal{L}$ is a colocal. So, we may assume that $a \vee b = 1$, $a \wedge b = 0$, and $\mathcal{L}$ is not colocal. Suppose that both a and b are not coatom. If a is not a coatom, then $a \lneqq c'$ for some coatom element c' of $\mathcal{L}$. Then $\mathbb{IMG}(\mathcal{L})$ is totally disconnected and $1 = a \vee b \leq b \vee c'$ gives $b \vee c' = 1$ and $b \wedge c' = 0$ which implies that $c' = c' \wedge 1 = c' \wedge (a \vee b) = (c' \wedge a) \vee (c' \wedge b) = a$ which is impossible. Therefore, every nontrivial element is a coatom (as well as atom).

(2) $\Rightarrow$ (1) Take two distinct nontrivial elements a and b of $\mathcal{L}$. If $\mathcal{L}$ is a colocal lattice with unique coatom c, then $a \vee b \leq c \neq 1$ gives there is no edge between any pair of distinct nontrivial elements. Let a and b be coatoms as well as atoms. Then $a \vee b = 1$. If $a \wedge b \neq 0$, then $a \wedge b \leq a, b$ gives $a = b$, which is a contradiction. Thus, $a \wedge b = 0$; so there is no edge between any pair of distinct nontrivial elements. Hence, the theorem holds. □

Theorem 4. *For the lattice $\mathcal{L}$, $\mathbb{IMG}(\mathcal{L})$ is not a complete graph.*

Proof. If $\mathcal{L}$ is a colocal lattice, then $\mathbb{IMG}(\mathcal{L})$ is totally disconnected by Theorem 3. Suppose that $\mathcal{L}$ is not a colocal lattice. If $\mathrm{J}(\mathcal{L}) = 0$ and $a \in \mathcal{V}(\mathbb{IMG}(\mathcal{L}))$, then $a \nleq \mathrm{J}(\mathcal{L})$ gives $a \nleq c$ for some $c \in \mathcal{CA}(\mathcal{L})$ (so $a \vee c = 1$). If $a \wedge c = 0$, then a and c are not adjacent. If $a \wedge c \neq 0$, then $a \wedge c$ is a nontrivial element of $\mathcal{L}$ and $c \vee (a \wedge c) = c \neq 1$, which implies that a and $a \wedge c$ are not adjacent. Suppose that $\mathrm{J}(\mathcal{L}) \neq 0$ and $c \in \mathcal{CA}(\mathcal{L})$ such that $\mathrm{J}(\mathcal{L}) < c$. Since $\mathrm{J}(\mathcal{L})$ is a nontrivial element of $\mathcal{L}$, we infer that $\mathrm{J}(\mathcal{L}) \vee c = c \neq 1$, and so $\mathrm{J}(\mathcal{L})$ and c are not adjacent. In either case, $\mathbb{IMG}(\mathcal{L})$ is not a complete graph. □

Theorem 5. *For an uniform lattice $\mathcal{L}$, the following statements are equivalent:*

(1) $\mathbb{IMG}(\mathcal{L})$ *is connected;*

(2) $\mathcal{L}$ *is not a colocal lattice.*

Proof. (1) $\Rightarrow$ (2) If $\mathbb{IMG}(\mathcal{L})$ is a connected graph, then $\mathcal{L}$ is not colocal by Theorem 3.

(2) $\Rightarrow$ (1) Suppose that $\mathcal{L}$ is not colocal and let $a, b \in \mathcal{V}(\mathbb{IMG}(\mathcal{L}))$. If $a \backsim b$, we are done. So we may assume that a and b are not adjacent (so $a \vee b \neq 1$ which gives a and b are not coatom elements of $\mathcal{L}$). By Lemma 1, $a \leq c_1$ and $b \leq c_2$ for some coatom elements c_1 and c_2. If $c_1 = c_2$, then there exists a coatom element c of $\mathcal{L}$ with $c \neq c_1$ such that $a \backsim c \backsim b$ is a path between a and b since $a \wedge c \neq 0$ and $b \wedge c \neq 0$. If $c_1 \neq c_2$, then $a \backsim c_2 \backsim c_1 \backsim b$ is a path between a and b, as $\mathcal{L}$ is uniform. Therefore, $\mathbb{IMG}(\mathcal{L})$ is a connected graph. □

Example 1. Let $\mathcal{L} = \{0, a, b, c, d, e, 1\}$ be a lattice with the relations $0 < e < a < b < c < 1$, $0 < e < a < d < c < 1$, $b \wedge d = a$ and $b \vee d = c$. An inspection will show that $\mathcal{L}$ is an uniform lattice with $\mathcal{CA}(\mathcal{L}) = \{a, b\}$; so $\mathbb{IMG}(\mathcal{L})$ is a connected graph by Theorem 5.

Theorem 6. *If $\mathcal{L}$ is an uniform lattice, then* $\operatorname{diam}(\mathbb{IMG}(\mathcal{L})) \leq 3$ *or* ∞.

Proof. If $\mathbb{IMG}(\mathcal{L})$ is not connected, then $\operatorname{diam}(\mathbb{IMG}(\mathcal{L})) = \infty$. So suppose that $\mathbb{IMG}(\mathcal{L})$ is connected and let $a, b \in \mathcal{V}(\mathbb{IMG}(\mathcal{L}))$. If a and b are adjacent, then $\mathrm{d}(a, b) = 1$. So we may assume that a and b are not adjacent (so $a \vee b \neq 1$ which gives a and b are not coatom elements of $\mathcal{L}$). By Lemma 1, $a \lneqq c_1$ and $b \lneqq c_2$ for some coatom elements c_1 and c_2. If $c_1 = c_2$, then there exists a coatom element c of $\mathcal{L}$ with $c \neq c_1$ such that $a \backsim c \backsim b$ is a path between a and b since $a \wedge c \neq 0$ and $b \wedge c \neq 0$; so $\mathrm{d}(a, b) = 2$. If $c_1 \neq c_2$, then $a \backsim c_2 \backsim c_1 \backsim b$ is a path between a and b, as $\mathcal{L}$ is uniform; hence $\mathrm{d}(a, b) = 3$. Therefore, $\operatorname{diam}(\mathbb{IMG}(\mathcal{L})) \leq 3$. □

Theorem 7. *For the lattice $\mathcal{L}$, the following hold:*

(1) If $\mathrm{J}(\mathcal{L}) \neq 0$, *then* $\operatorname{diam}(\mathbb{IMG}(\mathcal{L})) = \infty$.

(2) If $\mathcal{L}$ is an Artinian lattice, then $\operatorname{diam}(\mathbb{IMG}(\mathcal{L})) = \infty$.

Proof. (1) It is a direct consequence of Corollary 1.

(2) It is a direct consequence of Corollary 2.

□

Theorem 8. *For the lattice $\mathcal{L}$, the following hold:*

(1) If $\mathrm{J}(\mathcal{L}) \neq 0$, *then* $\operatorname{diam}(\widehat{\mathbb{IMG}}(\mathcal{L})) \leq 2$.

(2) If $\mathcal{L}$ is an Artinian lattice, then $\operatorname{diam}(\widehat{\mathbb{IMG}}(\mathcal{L})) \leq 2$.

Proof. (1) Let $a, b \in \mathcal{V}(\mathbb{IG}(\mathcal{L}))$. If $a \backsim b$ in $\mathbb{IMG}(\mathcal{L})$, then $\mathrm{d}(a, b) = 1$. If a and b are not adjacent in $\mathbb{IMG}(\mathcal{L})$, then there exists an isolated vertex x such that $a \backsim x \backsim b$ is a path in $\widehat{\mathbb{IMG}}(\mathcal{L})$ by Corollary 1; hence $\operatorname{diam}(\widehat{\mathbb{IMG}}(\mathcal{L})) \leq 2$.

(2) By an argument like that (1), (2) holds by Lemma 2. □

The following theorem is a lattice counterpart of Theorem 2.13 in [17] describing the structure of cycles of $\mathbb{IMG}(\mathcal{L})$.

Theorem 9. *If* $\mathrm{J}(\mathcal{L})$ *is an atom element of* $\mathcal{L}$*, then* $\mathbb{IMG}(\mathcal{L})$ *contains no cycle if and only if* $|\mathcal{CA}(\mathcal{L})| \leq 2$.

Proof. Suppose that $\mathbb{IMG}(\mathcal{L})$ contains no cycle; we show that $|\mathcal{CA}(\mathcal{L})| \leq 2$. On the contrary, assume that $|\mathcal{CA}(\mathcal{L})| \geq 3$. By the hypothesis, since $\mathrm{J}(\mathcal{L}) \neq 0$, we conclude that for all distinct coatom elements c and c' of $\mathcal{L}$ we have $c \wedge c' \neq 0$. Now, let c_1, c_2, c_3 be distinct coatom element of $\mathcal{L}$. Then $c_1 \vee c_2 = c_1 \vee c_3 = c_2 \vee c_3 = 1$ gives $c_1 \backsim c_2 \backsim c_3 \backsim c_1$ is a cycle of length 3 in $\mathbb{IMG}(\mathcal{L})$, which is a contradiction.

Conversely, suppose that $|\mathcal{CA}(\mathcal{L})| \leq 2$. Our objective is to show that $\mathbb{IMG}(\mathcal{L})$ contains no cycle. Assume to the contrary that $a_1 \backsim a_2 \backsim \cdots \backsim a_n \backsim a_1$ is a cycle in $\mathbb{IMG}(\mathcal{L})$. If $\mathcal{L}$ is a colocal lattice with unique coatom element c, then $a_1, a_2 \leq c$ by Lemma 1, and so $1 = a_1 \vee a_2 \leq c$, a contradiction. Suppose that $\mathcal{CA}(\mathcal{L}) = \{c_1, c_2\}$. Then for each $1 \leq i \leq n$, either $a_i \leq c_1$ or $a_i \leq c_2$. If $a_i \leq c_j$ and $a_{i+1} \leq c_j$ for some $j \in \{1, 2\}$, then $1 = a_i \vee a_{i+1} \leq c_j$, which is impossible since $a_i \backsim a_{i+1}$. It is possible $a_{i-1} \leq c_j$ and $a_{i+1} \leq c_j$ for some $j \in \{1, 2\}$. We may assume that $a_{i-1}, a_{i+1} \leq c_1$ and $a_i \leq c_2$. If $a_{i+1} \leq \mathrm{J}(\mathcal{L}) \leq c_2$, then $a_i, a_{i+1} \leq c_2$, which is a contradiction. Therefore, $a_{i+1} \nleq \mathrm{J}(\mathcal{L})$ (so $a_{i+1} \nleq c_2$). If $\mathrm{J}(\mathcal{L}) \wedge a_{i+1} = c_1 \wedge c_2 \wedge a_{i+1} = c_2 \wedge a_{i+1} = 0$, then $a_i \wedge a_{i+1} \leq a_{i+1} \wedge c_2 = 0$, a contradiction, as $a_i \backsim a_{i+1}$. Hence, $\mathrm{J}(\mathcal{L}) \wedge a_{i+1} \neq 0$, which implies that $\mathrm{J}(\mathcal{L}) \leq a_{i+1}$ since $\mathrm{J}(\mathcal{L})$ is an atom. Since $a_{i+1} \nleq c_2$, we infer that $a_{i+1} \vee c_2 = 1$. Then $c_1 = c_1 \wedge (a_{i+1} \vee c_2) = (c_1 \wedge a_{i+1}) \vee (c_1 \wedge c_2) = a_{i+1} \vee \mathrm{J}(\mathcal{L}) = a_{i+1}$. Similarly, $a_{i-1} = c_1$. This shows that $n = 2$ and so $\mathbb{IMG}(\mathcal{L})$ contains no cycle. □

Theorem 10. *For the lattice* $\mathcal{L}$ *with* $\mathrm{J}(\mathcal{L}) \neq 0$*,* $\mathbb{IMG}(\mathcal{L})$ *is a bipartite graph if and only if* $|\mathcal{CA}(\mathcal{L})| \leq 2$.

Proof. (1) Let $\mathbb{IMG}(\mathcal{L})$ be a bipartite graph; we show that $|\mathcal{CA}(\mathcal{L})| \leq 2$. Since $\mathrm{J}(\mathcal{L}) \neq 0$, we infer that $c \wedge c' \neq 0$ for all $c, c' \in \mathcal{CA}(\mathcal{L})$. On the contrary, assume that $|\mathcal{CA}(\mathcal{L})| \geq 3$ and let c_1, c_2 and c_3 be distinct coatom elements of $\mathcal{L}$ (so $c_1 \vee c_2 = c_1 \vee c_3 = c_2 \vee c_3 = 1$). Then $c_1 \backsim c_2 \backsim c_3 \backsim c_1$ is a cycle in $\mathbb{IMG}(\mathcal{L})$, which is impossible, as $\mathbb{IMG}(\mathcal{L})$ is a bipartite graph. For the opposite direction, if $|\mathcal{CA}(\mathcal{L})| = 1$, then we are done. So we may assume that $\mathcal{CA}(\mathcal{L}) = \{c_1, c_2\}$. It is easy to show that the vertex set $\mathcal{V}(\mathbb{IMG}(\mathcal{L}))$ can be partitioned into the two disjoint subsets $V_1 = \{a \in \mathcal{V}(\mathbb{IMG}(\mathcal{L})) : a \leq c_1\}$ and $V_2 = \{b \in \mathcal{V}(\mathbb{IMG}(\mathcal{L})) : b \leq c_2\}$. This completes the proof. □

Theorem 11. *For the uniform lattice* $\mathcal{L}$*,* $\mathbb{IMG}(\mathcal{L})$ *is a complete bipartite graph if and only if* $|\mathcal{CA}(\mathcal{L})| = 2$.

Proof. Let $\mathbb{IMG}(\mathcal{L})$ be a complete bipartite graph; we show that $|\mathcal{CA}(\mathcal{L})| = 2$. Since $\mathcal{L}$ is uniform, we conclude that $a \wedge b \neq 0$ for all $a, b \in \mathcal{V}(\mathbb{IMG}(\mathcal{L}))$. If $|\mathcal{CA}(\mathcal{L})| \geq 3$, then $c_1 \backsim c_2 \backsim c_3 \backsim c_1$ is a cycle in $\mathbb{IMG}(\mathcal{L})$, where $c_i \in \mathcal{CA}(\mathcal{L})$, which is a contradiction. For the other direction, assume that $V_1 = \{a \in \mathcal{V}(\mathbb{IMG}(\mathcal{L})) : a \leq c_1\}$ and $V_2 = \{b \in \mathcal{V}(\mathbb{IMG}(\mathcal{L})) : b \leq c_2\}$. If $a \in V_1$ and $b \in V_2$, then we claim that $a \vee b = 1$. On the contrary, suppose that $a \vee b \neq 1$. By Lemma 1, $a \vee b \leq c_i$ for some $i \in \{1, 2\}$, say $i = 1$. Since $b \leq a \vee b \leq c_1$, we infer that $b \in V_1$, a contradiction. Therefore each vertex of V_1 is joined to each vertex of V_2 by exactly one edge, i.e. $\mathbb{IMG}(\mathcal{L})$ is a complete bipartite graph. □

Theorem 12. *For a lattice $\mathcal{L}$, if $\mathrm{J}(\mathcal{L}) \neq 0$ and $\mathbb{IMG}(\mathcal{L})$ contains a cycle, then $\mathrm{gr}(\mathbb{IMG}(\mathcal{L})) \leq 4$.*

Proof. If $\mathcal{CA}(\mathcal{L}) = \{c_1, c_2\}$, then from proof of Theorem 9; if $\mathbb{IMG}(\mathcal{L})$ contains a cycle $a_1 \backsim a_2 \backsim \cdots \backsim a_n \backsim a_1$, the length of this cycle should be even as $a_i, a_{i+1} \not\leq c_i$ for some $i \in \{1, 2\}$. So $\mathrm{gr}(\mathbb{IMG}(\mathcal{L})) = 4$ by Theorem 10. If $|\mathcal{CA}(\mathcal{L})| \geq 3$, then $c_1 \backsim c_2 \backsim c_3 \backsim c_1$ is a cycle of length 3 in $\mathbb{IMG}(\mathcal{L})$, where $c_i \in \mathcal{CA}(\mathcal{L})$, as needed. □

4 Independence number, domination number, clique number and chromatic number of $\mathbb{IG}(\mathcal{L})$

We continue this section with the investigation of the stability of the identity meet elements of graph of lattices in various lattice-theoretic constructions.

If $a \in \mathcal{L}$, then we set $\mathcal{I}(a) = \{x \in \mathcal{V}\ (\mathbb{IMG}(\mathcal{L})) : x \leq a\}$. Let us begin this section with the following theorem:

Theorem 13. *If $\mathcal{L}$ is an Artinian lattice with a unique atom element a, then $\alpha(\mathbb{IMG}(\mathcal{L})) = \mathrm{Max}\{|\mathcal{I}(c)| : c \in \mathcal{CA}(\mathcal{L})\}$.*

Proof. Suppose that $c \in \mathcal{CA}(\mathcal{L})$ and let x and y be distinct nontrivial elements of $\mathcal{L}$ such that $x \leq c$ and $y \leq c$. Since $x \vee y \leq c \neq 1$, we conclude that x and y are not adjacent in $\mathbb{IMG}(\mathcal{L})$. This shows that $\mathcal{I}(c)$ is an independent set. Let z be an element of $\mathcal{L}$ such that $z \notin \mathcal{I}(c)$. Then $z \not\leq c$ gives $z \vee c = 1$ and $0 \neq a \leq z \wedge c$, which implies that z and c are adjacent in $\mathbb{IMG}(\mathcal{L})$. Therefore, $\{z\} \cup \mathcal{I}(c)$ is not an independent set. Hence, $\mathcal{I}(c)$ is a maximal independent set in $\mathbb{IMG}(\mathcal{L})$, i.e. $\alpha(\mathbb{IMG}(\mathcal{L})) = \mathrm{Max}\{|\mathcal{I}(c)| : c \in \mathcal{CA}(\mathcal{L})\}$. □

We need the following proposition proved Proposition 1 in [18].

Proposition 2. *A dominating set S of a graph G is a global dominating set if and only if for each $v \in \mathcal{V}\ (G) \setminus S$, there exists a $u \in S$ such that u is not adjacent to v.*

Compare the next theorem with Theorem 3.3 in [17].

Theorem 14. *For an Artinian lattice $\mathcal{L}$ with a unique atom element a, $|\mathcal{I}(J(\mathcal{L}))| \leq \gamma(\mathbb{IMG}(\mathcal{L})) \leq |\mathcal{I}(J(\mathcal{L})) \cup \mathcal{CA}(\mathcal{L})|$.*

Proof. If $\mathcal{L}$ is a colocal lattice with unique coatom element c, then $\mathcal{I}(J(\mathcal{L})) = \mathcal{I}(c) = \mathcal{L} \setminus \{0, 1\}$, which implies that $|\mathcal{I}(J(\mathcal{L}))| = \gamma(\mathbb{IMG}(\mathcal{L}))$ (by definition), i.e., (1) holds. Suppose that $\mathcal{L}$ is not a colocal lattice. By the hypothesis, $a \leq J(\mathcal{L})$. Since a dominating set must contain all the isolated vertices, we infer that a dominating set of $\mathbb{IMG}(\mathcal{L}))$ contains at least $|\mathcal{I}(J(\mathcal{L}))|$ vertices by Corollary 1. Therefore, $|\mathcal{I}(J(\mathcal{L}))| \leq \gamma(\mathbb{IMG}(\mathcal{L}))$. Let x be a nontrivial element of $\mathcal{L}$ such that $x \not\leq J(\mathcal{L})$. Then $x \not\leq c$ for some coatom element c of $\mathcal{L}$. This shows that $x \vee c = 1$ and $0 \neq a \leq x \wedge c$, i.e. $x \backsim c$. Hence, the set $\mathcal{I}(J(\mathcal{L})) \cup \mathrm{Max}(\mathcal{L})$ of elements forms a dominating set for $\mathbb{IMG}(\mathcal{L})$, which implies that $\gamma(\mathbb{IMG}(\mathcal{L})) \leq |\mathcal{I}(J(\mathcal{L})) \cup \mathcal{CA}(\mathcal{L})|$. □

Theorem 15. *For an Artinian lattice $\mathcal{L}$ with a unique atom element a, $|\mathcal{I}(J(\mathcal{L}))| \leq \gamma_g(\mathbb{IMG}(\mathcal{L})) \leq |\mathcal{I}(J(\mathcal{L})) \cup \mathcal{CA}(\mathcal{L})|$.*

Proof. Let D be the smallest dominating set of $\mathbb{IMG}(\mathcal{L})$. Then D contains vertices $x \leq J(\mathcal{L})$ by Corollary 1, which implies that D is a global dominating set of $\mathbb{IMG}(\mathcal{L})$ by Proposition 2, i.e., the result holds by using Theorem 14. □

Proposition 3. *Suppose that $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice and let c be a nontrivial element of $\mathcal{L}$. Then c is a coatom element of $\mathcal{L}$ if and only if $c = (c_1, 1)$ for some coatom element c_1 of $\mathcal{L}_1$ or $c = (1, c_2)$ for some coatom element c_2 of $\mathcal{L}_2$.*

Proof. Clearly, $(0,1),(1,0) \notin \mathcal{CA}(\mathcal{L})$. Let $c = (c_1, c_2)$ be a coatom element of $\mathcal{L}$. Since either $c_1 \neq 1$ or $c_2 \neq 1$, and $c = (c_1, 1) \wedge_c (1, c_2) \leq_c (c_1, 1), (1, c_2)$, we conclude that either $c = (c_1, 1)$ or $c = (1, c_2)$. Without loss of generality, we can assume that $c = (c_1, 1)$. Let $e \leq c_1 < 1$. Then c is a coatom element of $\mathcal{L}$ and $(e, 1) \leq_c (c_1, 1) <_c (1, 1)$ gives $c_1 = e$. Thus c_1 is a coatom element of $\mathcal{L}_1$. Conversely, assume that $(a, b) \not\leq_c (c_1, 1)$ for some $(a, b) \in \mathcal{L}$ and coatom element c_1 of $\mathcal{L}_1$. Then $a \not\leq c_1$ gives $a \vee c_1 = 1$, as c_1 is coatom. Therefore we have $(a, b) \vee_c (c_1, 1) = (1, 1)$ and so, $(c_1, 1)$ is a coatom element of $\mathcal{L}$. Similarly, $(1, c_2)$ is a coatom element of $\mathcal{L}$ for some coatom element c_2 of $\mathcal{L}_2$. □

Compare the next theorem with Theorem 3.5 in [17].

Theorem 16. *If $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice with $\mathcal{L}_i \neq \{0, 1\}$ for $i \in \{1, 2\}$, then $\gamma(\mathbb{IMG}(\mathcal{L})) = 2 + |\mathcal{I}(\mathrm{J}(\mathcal{L}))|$.*

Proof. (1) By Proposition 3, the coatom elements of $\mathcal{L}$ are $(c_1, 1)$ and $(1, c_2)$, where c_i is a coatom element in $\mathcal{L}_i$ for $i \in \{1, 2\}$. Also, by Lemma 1, the atom elements of $\mathcal{L}$ are $(a_1, 0)$ and $(0, a_2)$, where a_i is an atom element in $\mathcal{L}_i$ for $i \in \{1, 2\}$. By Theorem 1 and Theorem 2, if $a \in \mathcal{A}(\mathcal{L})$, then $a \leq \mathrm{J}(\mathcal{L})$. Observe that any nontrivial element $x = (x_1, x_2) \not\leq \mathrm{J}(\mathcal{L})$ has the form $(x_1, 1)$ or $(1, x_2)$. If $(c_1, 1), (1, c_2) \in \mathcal{CA}(\mathcal{L})$, then $(x_1, 1) \vee_c (1, c_2) = (1, 1)$, $(x_1, 1) \wedge_c (1, c_2) \neq (0, 0)$, $(1, x_2) \vee_c (c_1, 1) = (1, 1)$ and $(1, x_2) \wedge_c (c_1, 1) \neq (0, 0)$ gives the elements $x = (x_1, x_2) \not\leq \mathrm{J}(\mathcal{L})$ are dominated by two elements and the induced subgraph

$$\{x \in \mathcal{V}(\mathbb{IMG}(\mathcal{L})) : x \not\leq \mathrm{J}(\mathcal{L})\}$$

is not a complete subgraph. Therefore, $\gamma(\mathbb{IMG}(\mathcal{L})) = 2 + |\mathcal{I}(\mathrm{J}(\mathcal{L}))|$. □

Corollary 3. *If $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice with $\mathcal{L}_i \neq \{0, 1\}$ for $i \in \{1, 2\}$, then $\gamma_g(\mathbb{IMG}(\mathcal{L})) = 2 + |\mathcal{I}(\mathrm{J}(\mathcal{L}))|$.*

Proof. By an argument like that Theorem 15 and Theorem 16 we have $\gamma_g(\mathbb{IMG}(\mathcal{L})) = 2 + |\mathcal{I}(\mathrm{J}(\mathcal{L}))|$. □

Theorem 17. *If $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice with $\mathcal{L}_1 \neq \{0, 1\}$ and $\mathcal{L}_2 = \{0, 1\}$, then $\gamma(\mathbb{IMG}(\mathcal{L})) = 1 + |\mathcal{I}(\mathrm{J}(\mathcal{L}))|$.*

Proof. By Proposition 3, the coatom elements of $\mathcal{L}$ are $(c_1, 1)$ and $(1, 0)$, where c_1 is a coatom element in $\mathcal{L}_1$. Also, by Lemma 1, the atom elements of $\mathcal{L}$ take the form $(a_1, 0)$, where a_1 is an atom element in $\mathcal{L}_1$. Observe that any nontrivial element $x = (x_1, x_2) \not\leq \mathrm{J}(\mathcal{L})$ is adjacent to $(1, 0)$. This shows that the coatom element $(1, 0)$ dominates all the nontrivial elements $x \not\leq \mathrm{J}(\mathcal{L})$. Thus, $\gamma(\mathbb{IMG}(\mathcal{L})) = 1 + |\mathcal{I}(\mathrm{J}(\mathcal{L}))|$. □

Corollary 4. *If $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice with $\mathcal{L}_1 \neq \{0,1\}$ and $\mathcal{L}_2 = \{0,1\}$, then $\gamma_g(\mathbb{IMG}(\mathcal{L})) = 1 + |\mathcal{I}(\mathrm{J}(\mathcal{L}))|$.*

Proof. By an argument like that Theorem 15 and Theorem 17 we have $\gamma_g(\mathbb{IMG}(\mathcal{L})) = 1 + |\mathcal{I}(\mathrm{J}(\mathcal{L}))|$. □

Theorem 18. *If $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice with $\mathcal{L}_1 = \mathcal{L}_2 = \{0,1\}$, then $\gamma(\mathbb{IMG}(\mathcal{L})) = 2$.*

Proof. Since $\mathcal{L}$ has only two nontrivial elements $(1,0)$ and $(0,1)$, we infer that they are coatom as well as atom. Therefore, $\gamma(\mathbb{IMG}(\mathcal{L})) = 2$ by Theorem 3 and Theorem 14. □

Corollary 5. *If $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice with $\mathcal{L}_1 = \mathcal{L}_2 = \{0,1\}$, then $\gamma_g(\mathbb{IMG}(\mathcal{L})) = 2$.*

Proof. By an argument like that Theorem 15 and Theorem 18 we have $\gamma_g(\mathbb{IMG}(\mathcal{L})) = 2$. □

Compare the next theorem with Theorem 3.14 in [17].

Theorem 19. *If $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ ($n \geq 3$) is a decomposable lattice with $\mathcal{L}_i = \{0,1\}$ for $i \in \{1,2,\ldots,n\}$, then $\gamma(\mathbb{IMG}(\mathcal{L})) = 2n-1$.*

Proof. By Proposition 3, any coatom element of $\mathcal{L}$ is of the form $c_i = (a_1, a_2, \ldots, a_n)$, where $a_j = 1$ for $i \neq j$ and $a_i = 0$. Set $d_i = (b_1, b_2, \ldots, b_n)$, where $b_j = 0$ for $i \neq j$ and $b_i = 1$. It suffices to show that

$$\{c_1, c_2, \ldots, c_{n-1}, d_1, d_2, \ldots, d_n\}$$

is a minimum dominating set for $\mathbb{IMG}(\mathcal{L})$. An inspection will show that $d_i \vee c_j \neq 1$ and $d_i \vee c_i = 1$ but $d_i \wedge c_i = 0$. This implies that d_i is an isolated vertex of $\mathbb{IMG}(\mathcal{L})$ for $i \in \{1,2,\ldots,n\}$. Consider the element $d_{i,j} = (e_1, e_2, \ldots, e_n)$ with $e_k = 1$ if $k = i, j$ and $e_k = 0$ if $k \neq i, j$. Then $d_{i,j}$ is dominated by c_i and c_j only. This completes the proof. □

Corollary 6. *If $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \times \mathcal{L}_n$ ($n \geq 3$) is a decomposable lattice with $\mathcal{L}_i = \{0,1\}$ for $i \in \{1,2,\ldots,n\}$, then $\gamma_g(\mathbb{IMG}(\mathcal{L})) = 2n-1$.*

Proof. By an argument like that Theorem 15 and and Theorem 19 we have $\gamma_g(\mathbb{IMG}(\mathcal{L})) = 2n-1$. □

Theorem 20. *If $\mathcal{L}$ is a lattice with $\mathrm{J}(\mathcal{L}) \neq 0$, $\omega(\mathbb{IMG}(\mathcal{L})) = \chi(\mathbb{IMG}(\mathcal{L})) = |\mathcal{CA}(\mathcal{L})|$.*

Proof. By assumption, since $\mathrm{J}(\mathcal{L}) \neq 0$, we conclude that for all distinct coatom elements c and c' of $\mathcal{L}$ we have $c \wedge c' \neq 0$ and $c \vee c' = 1$; hence the subgraph induced by $\mathcal{CA}(\mathcal{L})$ is a complete graph. This implies that $|\mathcal{CA}(\mathcal{L})| \leq \omega(\mathbb{IMG}(\mathcal{L}))$. Consider a nontrivial element e of $\mathcal{L}$. Then $e \leq c$ for some coatom element c of $\mathcal{L}$. If we take another element f such that $f \leq c$, then e and f are not adjacent, as $e \vee f \leq c \neq 1$. It follows that the vertex set of a complete subgraph of $\mathbb{IMG}(\mathcal{L})$ can contain atmost one vertex from each $\mathcal{I}(c)$ of $\mathcal{L}$ (i.e., a complete subgraph of $\mathbb{IMG}(\mathcal{L})$ can contain at most $|\mathcal{CA}(\mathcal{L})|$ vertices). This shows that $\omega(\mathbb{IMG}(\mathcal{L})) = |\mathcal{CA}(\mathcal{L})|$. Since $\omega(G) \leq \chi(G)$ for any graph G, we infer that $|\mathcal{CA}(\mathcal{L})| \leq \chi(\mathbb{IG}(\mathcal{L}))$. Moreover, for any two nontrivial elements $e, f \leq c \in \mathcal{CA}(\mathcal{L})$, we obtain e and f are not adjacent which gives $\chi(\mathbb{IMG}(\mathcal{L})) \leq |\mathcal{CA}(\mathcal{L})|$ and so we have equality. □

Corollary 7. *If $\mathcal{L}$ is an Artinian lattice with unique atom element, then $\omega(\mathbb{IMG}(\mathcal{L})) = \chi(\mathbb{IMG}(\mathcal{L})) = |\mathcal{CA}(\mathcal{L})|$.*

Proof. By Lemma 2 (2), $J(\mathcal{L}) \neq 0$. Now the assertion follows from Theorem 20. □

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