Journal of Algebra and Related Topics
Vol. , No. -, 2026, pp. -. Research Article

Planarity of the essential graph for modules

Samira Azlesh[†], Shiroyeh Payrovi[‡*], Fatemeh Soheilnia[§]

[†,‡,§] *Department of Mathematics, Imam Khomeini International University, Qazvin, Iran*
Emails: s.azlesh@edu.ikiu.ac.ir, shpayrovi@sci.ikiu.ac.ir, f.soheilnia@edu.ikiu.ac.ir

Abstract. Given a module $\mathcal{M}$ over a commutative ring $\mathcal{R}$, we can construct a simple graph $EG(\mathcal{M})$ with the vertex set $\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$. Two distinct vertices x, y are connected whenever $\mathrm{Ann}_{\mathcal{M}}(xy)$ is an essential submodule of $\mathcal{M}$. The present study provides a detailed analysis of planar zero-divisor and planar essential graphs, especially when they possess a universal vertex. It demonstrates that such graphs can be represented as join of some known graphs. Additionally, it examines that whether the zero-divisor and the essential graphs of $\mathbb{Z}_n$ are planar or not.

Keywords: Zero-divisor graph, Essential graph, Planar graph.
AMS Subject Classification 2010: 13Cxx, 05C25, 05C10.

1 Introduction and preliminaries

In the last few decades, algebraic graph theory in the context of ring theory has garnered significant interest due to its unique structural characteristics and the deep exploration opportunities it offers. The zero-divisor graph was formally introduced in [4] and later modified in [1]. Let us consider a commutative ring $\mathcal{R}$, and let $Z(\mathcal{R})$ be the set of zero divisors of $\mathcal{R}$. The zero-divisor graph of $\mathcal{R}$, denoted by $\Gamma(\mathcal{R})$, is a graph with the vertex set $Z(\mathcal{R})^* = Z(\mathcal{R}) \setminus \{0\}$ and two distinct vertices x, y are adjacent if $xy = 0$. The zero-divisor graph, as a topic in graph theory and commutative algebra, has been extensively studied by numerous investigators. For example, see [2, 3]. The essential graph of $\mathcal{R}$, a variant of the zero-divisor graph with modified adjacency criteria, has been defined and investigated in [7]. Given $Z^*(\mathcal{R})$ as the vertex set for the essential graph, which is denoted by $EG(\mathcal{R})$, and two different vertices x, y are connected whenever $\mathrm{Ann}_{\mathcal{R}}(xy)$ is an essential ideal of $\mathcal{R}$. It has been demonstrated, $\Gamma(\mathcal{R}) = EG(\mathcal{R})$ whenever $\mathcal{R}$ is a reduced ring, [7, Theorem 2.2]. Significant research has been dedicated to studying the zero-divisor graph of modules; for further details, see [5, 8, 9]. In this study, we adhere to the

*Corresponding author
Received: 07 May 2025/ Revised: 10 September 2025/ Accepted: 27 September 2025
DOI: 10.22124/JART.2025.30608.1799

© 2026 University of Guilan

definition used in [8]. Let us consider $\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$ as the vertex set for a graph over an $\mathcal{R}$-module $\mathcal{M}$. The zero-divisor graph denoted by $\Gamma(\mathcal{M})$, features of two vertices x, y from $\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$: they are adjacent if $xy\mathcal{M} = 0$.

The essential graph for $\mathcal{R}$-module $\mathcal{M}$ is another kind of simple graph which its structures and parameters have been defined and studied in [11, 12]. Let us consider $\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$ as the vertex set for a graph over $\mathcal{R}$-module $\mathcal{M}$. The essential graph of $\mathcal{M}$ denoted by $EG(\mathcal{M})$, features of two vertices x, y from $\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$: they are connected whenever $\mathrm{Ann}_{\mathcal{M}}(xy)$ is an essential submodule.

In section two, it provides a comprehensive analysis of the planarity of the zero-divisor and the essential graphs for an $\mathcal{R}$-module $\mathcal{M}$, specifically in cases where these graphs possess a universal vertex. It is shown that $\Gamma(\mathcal{M})$ is a planar graph if and only if it is a star graph. Furthermore, the planarity of the essential graph is studied when $\mathcal{M}$ is a Noetherian $\mathcal{R}$-module. More precisely, it is demonstrated that for a Noetherian $\mathcal{R}$-module $\mathcal{M}$ with $\mathrm{MinAss}_{\mathcal{R}}(\mathcal{M}) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$, the graph $EG(\mathcal{M})$ is planar if and only if $n \leq 3$ and one of the following statements holds:

(a) When $\mathrm{MinAss}_{\mathcal{R}}(\mathcal{M}) = \{\mathfrak{q}\}$.

  (i) $EG(\mathcal{M})$ is a null graph or $EG(\mathcal{M}) = K_4$.

  (ii) $EG(\mathcal{M}) = K_3 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M})\setminus\mathfrak{q}|}$, where $|\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathfrak{q}| \leq 2$.

  (iii) $EG(\mathcal{M}) = K_2 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M})\setminus\mathfrak{q}|}$ or $EG(\mathcal{M}) = K_1 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M})\setminus\mathfrak{q}|}$, where $|\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathfrak{q}| \leq 4$.

(b) When $\mathrm{MinAss}_{\mathcal{R}}(\mathcal{M}) = \{\mathfrak{q}_1, \mathfrak{q}_2\}$

  (i) $EG(\mathcal{M}) = K_4$ or $EG(\mathcal{M}) = K_2 \vee K_{1,2}$.

  (ii) $EG(\mathcal{M}) = P_1 \vee K_{2,2}$ or $EG(\mathcal{M}) = P_1 \vee K_{1,2}$ or $EG(\mathcal{M}) = P_2 \vee \overline{K}_{|\mathfrak{q}_t\setminus\mathfrak{q}_s|}$, where $1 \leq t, s \leq 2$.

  (iii) $EG(\mathcal{M}) = K_{1,|\mathfrak{q}_s\setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|}$ or $EG(\mathcal{M}) = K_{2,|\mathfrak{q}_s\setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|}$, where $1 \leq t, s \leq 2$.

(c) When $\mathrm{MinAss}_{\mathcal{R}}(\mathcal{M}) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3\}$

  (i) $EG(\mathcal{M}) = P_1 \vee (K_{1,|\mathfrak{q}_1\setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|} \cup K_{1,|\mathfrak{q}_2\setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|} \cup K_{1,|\mathfrak{q}_3\setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|})$.

  (ii) $EG(\mathcal{M}) = K_3 \circ \overline{K}_{|\mathfrak{q}_t\setminus\mathfrak{q}_s\cup\mathfrak{q}_r|}$ for each $1 \leq r, s, t \leq 3$.

  (iii) $EG(\mathcal{M}) = (K_3 \circ \overline{K}_{|\mathfrak{q}_t\setminus\mathfrak{q}_s\cup\mathfrak{q}_r|}) \cup (K_3 \circ \overline{K}_{|\mathfrak{q}_t\setminus\mathfrak{q}_s\cup\mathfrak{q}_r|})$, where $1 \leq r, s, t \leq 3$.

Consider a graph G with a vertex set $V(G)$ and an edge set $E(G)$. In the following section, we recall some important definitions related to G. A graph without any edges is called a null graph, and without any vertices is called an empty graph. A vertex of G is called universal if it is connected to all other vertices. If the elements of $V(G)$ can be listed in an ordered sequence $v_1, v_2, ..., v_n$ such that each v_i is adjacent to v_{i+1} for each $1 \leq i < n$, then it is called a path and denoted by P_n. A complete bipartite graph is a graph whose vertices can be partitioned into two subsets A and B such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph, this graph is denoted by $K_{|A|,|B|}$. Moreover, if we have $K_{1,n}$, it is called a star graph. A clique of G is a complete

subgraph of G, and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the clique number. G is characterized as planar if it can be drawn on the plane in such a way that no two edges intersect at a point other than a vertex. Consider two different graphs G and H. Then $G \vee H$ shows the join graph of G and H with the vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

In the whole of this article, $\mathcal{R}$ is a commutative ring with non-zero identity and $\mathcal{M}$ denotes a unitary $\mathcal{R}$-module. Recognize that $\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) = \{r \in \mathcal{R} : rm = 0 \text{ for some } 0 \neq m \in \mathcal{M}\}$, $\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) = \{r \in \mathcal{R} : r\mathcal{M} = 0\}$, $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) = \{r \in \mathcal{R} : r^t\mathcal{M} = 0 \text{ for some } t \in \mathbb{N}\}$ and the set of associated prime ideals of $\mathcal{M}$ is $\mathrm{Ass}_{\mathcal{R}}(\mathcal{M}) = \{\mathfrak{p} \in \mathrm{Spec}(\mathcal{R}) : \mathfrak{p} = \mathrm{Ann}_{\mathcal{R}}(m) \text{ for some } 0 \neq m \in \mathcal{M}\}$. Moreover, the set of minimal elements of $\mathrm{Ass}_{\mathcal{R}}(\mathcal{M})$ is denoted by $\mathrm{MinAss}_{\mathcal{R}}(\mathcal{M})$. For symbols and terms not provided in this article, refer to [6, 10].

2 Main results

This study explores the planarity of the zero-divisor and the essential graphs for an $\mathcal{R}$-module $\mathcal{M}$ whenever they have a universal vertex. To begin, we present the famous Kuratowski Theorem, which is central to this article.

Theorem 1. [13, Kuratowski Theorem] *G is a non-planar graph if and only if it contains a subdivision isomorphic to $K_{3,3}$ or K_5.*

In whole of the following section, let us assume that $\mathcal{M}$ is an $\mathcal{R}$-module over a commutative ring $\mathcal{R}$.

Definition 1. *Consider $\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$ as the vertex set of the zero-divisor graph over $\mathcal{M}$. The zero-divisor graph denoted by $\Gamma(\mathcal{M})$, features of two vertices x, y from $\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$: they are connected if $xy\mathcal{M} = 0$.*

Lemma 1. *Suppose that $x \in \mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))$ is a universal vertex of $\Gamma(\mathcal{M})$. Then $\Gamma(\mathcal{M})$ is a planar graph if and only if it is a star graph.*

Proof. Let $\Gamma(\mathcal{M})$ be planar and $x \in \mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))$ be a universal vertex of $\Gamma(\mathcal{M})$. Thus, from [8, Theorem 2.2], $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ and $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, whenever $\mathcal{R}_1$ and $\mathcal{R}_2$ are subrings of $\mathcal{R}$, and $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{R}$-submodules of $\mathcal{M}$ and $\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) = \{(1,0)\} \cup (\{0\} \oplus \mathcal{R}_2)$ moreover proof of the mentioned theorem shows that $\mathcal{Z}_{\mathcal{R}_2}(\mathcal{M}_2) = 0$. Assume that $(0,r), (0,s) \in (\{0\} \oplus \mathcal{R}_2) \setminus \{(0,0)\}$ and $(0,r)(0,s)M = 0 \oplus rs\mathcal{M}_2 = 0$. So either $0 \neq r \in \mathcal{Z}_{\mathcal{R}_2}(\mathcal{M}_2)$ or $0 \neq s \in \mathcal{Z}_{\mathcal{R}_2}(\mathcal{M}_2)$, which is a contradiction. Hence, $\Gamma(\mathcal{M}) = K_{1,|\mathcal{R}_2|}$ and $(1,0)$ is its only universal vertex. □

Consider an integral domain D. Then $\mathcal{Z}(\mathbb{Z}_2 \oplus D) = \{(1,0)\} \cup (\{0\} \oplus D)$, $\mathrm{Nil}(\mathbb{Z}_2 \oplus D) = \{(0,0)\}$ and $x = (1,0) \in \mathcal{Z}(\mathbb{Z}_2 \oplus D) \setminus \mathrm{Nil}(\mathbb{Z}_2 \oplus D)$ is a universal vertex of $\Gamma(\mathbb{Z}_2 \oplus D)$. Thus the previous lemma demonstrates that $\Gamma(\mathbb{Z}_2 \oplus D)$ is a star graph, and consequently, planar as well.

Theorem 2. *Suppose that $x \in r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$ is a universal vertex of $\Gamma(\mathcal{M})$. Then $\Gamma(\mathcal{M})$ is a planar graph if and only if at least one of the listed conditions is satisfied:*

(i) $\Gamma(\mathcal{M}) = K_4$.

(ii) $\Gamma(\mathcal{M}) = K_3 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M})\setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})|-3}$, *where* $|\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})| \leq 5$.

(iii) $\Gamma(\mathcal{M}) = P_2 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M})\setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})|-2}$ *or* $\Gamma(\mathcal{M}) = P_2 \vee (\cup_{\alpha_i \in \mathbb{N}} P_{\alpha_i})$.

(iv) $\Gamma(\mathcal{M}) = P_1 \vee \mathcal{G}$, *where* $\mathcal{G}$ *is a graph with* $\omega(\mathcal{G}) \leq 3$ *and* $|N(y) \cap N(z)| \leq 3$ *for each two optional vertices* y, z *of* $\mathcal{G}$.

Proof. Let $\Gamma(\mathcal{M})$ be planar, $x \in r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$ be a universal vertex of $\Gamma(\mathcal{M})$ and $n \in \mathbb{N}$ be the smallest integer such that $x^n\mathcal{M} = 0$ $(x^{n-1}\mathcal{M} \neq 0)$. Thus $x, x^2, \cdots, x^{n-1}$ are distinct. Since x^i is a universal vertex, for each i such that $1 \leq i \leq n-1$, so the set $\{x, x^2, \cdots, x^{n-1}\}$ has at most four elements, otherwise $\Gamma(\mathcal{M})$ has a subdivision isomorphic to K_5, that is conflicting. If $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) \supseteq \{x, x^2, x^3, x^4\}$, then $\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) = r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))$ and $\Gamma(\mathcal{M}) = K_4$. Suppose that $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) \supseteq \{x, x^2, x^3\}$ and $y, z \in \mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus (\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) \cup \{x, x^2, x^3\})$. If y and z are adjacent, then $\Gamma(\mathcal{M})$ contains a subgraph K_5, that is inconsistent. Therefore $\Gamma(\mathcal{M}) = K_3 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M})\setminus(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})\cup\{x,x^2,x^3\})|}$, where $|\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus (\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) \cup \{x, x^2, x^3\})| \leq 2$, otherwise $\Gamma(\mathcal{M})$ has a subdivision isomorphic to $K_{3,3}$. Let $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) \supseteq \{x, x^2\}$. If there are $y_1, y_2, y_3 \in \mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus (\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) \cup \{x, x^2\})$ such that y_i and y_j are adjacent for $1 \leq i < j \leq 3$, thus $\Gamma(\mathcal{M})$ includes a subgraph K_5 so $\Gamma(\mathcal{M}) = P_2 \vee (\cup_{\alpha_i \in \mathbb{N}} P_{\alpha_i})$ or $\Gamma(\mathcal{M}) = P_2 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M})\setminus(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})\cup\{x,x^2\})|}$. Now, assume that $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) \supseteq \{x\}$. If there are $y_1, y_2, y_3, y_4 \in \mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus (\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) \cup \{x\})$ such that y_i and y_j are adjacent for $1 \leq i < j \leq 4$, then $\Gamma(\mathcal{M})$ includes a subgraph K_5. So $\Gamma(\mathcal{M}) = P_1 \vee \mathcal{G}$, where $\mathcal{G}$ is a subgraph of $\Gamma(\mathcal{M})$ with $\omega(\mathcal{G}) \leq 3$ and $|(N(y) \cap N(z)) \setminus \{x\}| \leq 2$ for each two arbitrary vertices y, z of $\mathcal{G}$. □

It is easy to see that $\Gamma(\mathbb{Z}_{16}) = P_1 \vee \mathcal{G}$, where $\mathcal{G}$ is a subgraph of $\Gamma(\mathbb{Z}_{16})$ with $\omega(\mathcal{G}) \leq 3$ and $|(N(y) \cap N(z)) \setminus \{x = 8\}| \leq 2$ for each two arbitrary vertices y, z of $\mathcal{G}$.

A submodule $\mathcal{N}$ of $\mathcal{M}$ is referred to as essential provided that, for each non-trivial submodule $\mathcal{K}$ of $\mathcal{M}$, $\mathcal{N} \cap \mathcal{K} \neq \{0\}$.

Definition 2. [11, Definition 2.1] *Consider that* $\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$ *as the vertex set for a graph over* $\mathcal{R}$*-module* $\mathcal{M}$. *The essential graph of* $\mathcal{M}$ *denoted by* $EG(\mathcal{M})$, *features of two vertices* x, y *from* $\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$*: they are connected whenever* $\mathrm{Ann}_{\mathcal{M}}(xy)$ *is an essential submodule of* $\mathcal{M}$.

We start by considering the subsequent lemma:

Lemma 2. *Suppose that* $x \in \mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$. *If* $\mathrm{Ann}_{\mathcal{M}}(x^3)$ *is an essential submodule of* $\mathcal{M}$, *then* $\mathrm{Ann}_{\mathcal{M}}(x^n)$ *is too, for each* $n \geq 1$. *Furthermore,* $\mathrm{Ann}_{\mathcal{M}}(x)$ *is an essential submodule of* $\mathcal{M}$ *if and only if* $\mathrm{Ann}_{\mathcal{M}}(x^n)$ *is essential, for all* $n \geq 2$.

Proof. Let $\mathrm{Ann}_{\mathcal{M}}(x^3)$ be essential. We aim to prove that $\mathrm{Ann}_{\mathcal{M}}(x)$ is also essential. Assume that $\mathcal{N}$ is a non-zero submodule of $\mathcal{M}$. Therefore, a non-zero element m exists, satisfying $m \in \mathrm{Ann}_{\mathcal{M}}(x^3) \cap \mathcal{N}$, so $x^3m = x(x^2m) = 0$ If $0 \neq x^2m$, then $x^2m \in \mathrm{Ann}_{\mathcal{M}}(x) \cap \mathcal{N}$ and we are done. Now, suppose that $x^2m = 0$. If $0 \neq xm$, then $xm \in \mathrm{Ann}_{\mathcal{M}}(x) \cap \mathcal{N}$ otherwise $m \in \mathrm{Ann}_{\mathcal{M}}(x) \cap \mathcal{N}$. Thus $\mathrm{Ann}_{\mathcal{M}}(x) \cap \mathcal{N} \neq 0$ and $\mathrm{Ann}_{\mathcal{M}}(x)$ is essential, as we needed.

The second assertion is evident from the previous paragraph. □

Theorem 3. *Suppose that* $\mathrm{Ann}_{\mathcal{M}}(x)$ *is an essential submodule of* $\mathcal{M}$ *for some* $x \in \mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))$. *Then* $EG(\mathcal{M})$ *is a planar graph if and only if at least one of the listed conditions is satisfied:*

(i) $EG(\mathcal{M})$ *is a star graph.*

(ii) $EG(\mathcal{M}) = K_{2,\alpha}$, *where* α *is a cardinal number.*

Proof. Let $EG(\mathcal{M})$ be planar, $x \in \mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))$ and $\mathrm{Ann}_{\mathcal{M}}(x)$ be essential. Thus x is universal and so Lemma 2 implies that $\{x^n : n \in \mathbb{N}\}$ has at most four elements otherwise $EG(\mathcal{M})$ includes a subgraph K_5, which is an inconsistency. Thus, $\mathcal{R}$ has an idempotent element, so $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$ and $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, with $\mathcal{R}_1$ and $\mathcal{R}_2$ are subrings of $\mathcal{R}$, and $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{R}$-submodules of $\mathcal{M}$. Hence, by [11, Lemma 2.3], $(a, 0)$ is universal in $EG(\mathcal{M})$ for every $a \in \mathcal{Z}_{\mathcal{R}_1}(\mathcal{M}_1) \setminus \mathrm{Ann}_{\mathcal{R}_1}(\mathcal{M}_1)$ and $\mathcal{Z}_{\mathcal{R}_2}(\mathcal{M}_2) = 0$. Thus, [11, Lemma 4.1] implies that $|\mathcal{R}_1| \leq 3$ otherwise $(1, 0), (a, 0), (b, 0)$ and $(0, 1), (0, c), (0, d)$ are vertices of a subdivision isomorphic to $K_{3,3}$ of $EG(\mathcal{M})$, which is an inconsistency. If $\mathcal{R}_1 \cong \mathbb{Z}_2$, then $\Gamma(\mathcal{M}) = EG(\mathcal{M}) = K_{1,|\mathcal{R}_2|}$ since for $(0, c), (0, d) \in (\{0\} \oplus \mathcal{R}_2) \setminus \{(0, 0)\}$ we have $\mathrm{Ann}_{\mathcal{M}}((0, c)(0, d)) = \mathrm{Ann}_{\mathcal{M}}((0, cd)) = \mathcal{M}_1 \oplus cd\mathcal{M}_2 = \mathcal{M}_1 \oplus 0$. Now, assume that $\mathcal{R}_1 \cong \mathbb{Z}_3$. Thus $\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) = \{(1, 0), (2, 0)\} \cup (0 \oplus \mathcal{R}_2)$ and so $EG(\mathcal{M}) = K_{2,|\mathcal{R}_2|}$. □

Theorem 4. *Consider* $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \neq \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$. *Then* $EG(\mathcal{M})$ *is a planar graph if and only if at least one of the listed conditions is satisfied:*

(i) $EG(\mathcal{M}) = K_4$.

(ii) $EG(\mathcal{M}) = K_3 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|}$, *where* $|\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))| \leq 2$.

(iii) $EG(\mathcal{M}) = P_2 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|}$ *or* $EG(\mathcal{M}) = P_2 \vee (\cup_{\alpha_i \in \mathbb{N}} P_{\alpha_i})$.

(vi) $EG(\mathcal{M}) = P_1 \vee \mathcal{G}$, *where* $\mathcal{G}$ *is a graph with* $\omega(\mathcal{G}) \leq 3$ *and* $|N(y) \cap N(z)| \leq 3$ *for two arbitrary vertices* y *and* z *of* $\mathcal{G}$.

Proof. The proof which we include for the readers convenience, is similar to that of the proof of Theorem 2. Assume that $EG(\mathcal{M})$ is a planar graph. Let $x \in r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$ and let $n \in \mathbb{N}$ be the smallest integer such that $x^n\mathcal{M} = 0$ ($x^{n-1}\mathcal{M} \neq 0$). Thus $x, x^2, \cdots, x^{n-1}$ are distinct. By [11, Lemma 2.1] for i with $1 \leq i \leq n-1$, x^i is universal in $EG(\mathcal{M})$ so the set $\{x, x^2, \cdots, x^{n-1}\}$ has at most four elements otherwise $EG(\mathcal{M})$ contains a subgraph K_5, which is a discrepancy. If $|r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})| = 4$, then $\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) = r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))$ and $EG(\mathcal{M}) = K_4$. Now, let x, x^2, x^3 be distinct and $y \in r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) \cup \{x, x^2, x^3\}$. Hence, according to [11, Lemma 2.1], $EG(\mathcal{M})$ is equal to K_4. Now, let $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) \supseteq \{x, x^2, x^3\}$. When $y, z \in \mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))$ are present such that y and z are adjacent, then $EG(\mathcal{M})$ includes a subgraph K_5, which is a discrepancy. Therefore, $EG(\mathcal{M}) = K_3 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|}$, where $|\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))| \leq 2$. Let $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) \supseteq \{x, x^2\}$. If there are $y_1, y_2, y_3 \in \mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))$, such that y_i and y_j are adjacent for $1 \leq i < j \leq 3$, then $EG(\mathcal{M})$ includes a subgraph K_5. So $EG(\mathcal{M}) = P_2 \vee (\cup_{\alpha_i \in \mathbb{N}} P_{\alpha_i})$ or $EG(\mathcal{M}) = P_2 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|}$. Now, assume that $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) \supseteq \{x\}$. If there are $y_1, y_2, y_3, y_4 \in \mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))$, such that y_i and y_j are adjacent for $1 \leq i < j \leq$

4, then $EG(\mathcal{M})$ includes a subgraph K_5. So $EG(\mathcal{M}) = P_1 \vee \mathcal{G}$, where $\mathcal{G}$ is an induced subgraph of $EG(\mathcal{M})$ with $\omega(\mathcal{G}) \leq 3$ and $|(N(y) \cap N(z)) \setminus \{x\}| \leq 2$, for each $y, z \in \mathcal{G}$. □

It is easy to see that $E(\mathbb{Z}_{25}) = K_4$ and $EG(\mathbb{Z}_8) = K_3$.

In the subsequent sections, we explore the planarity of the essential graph of $\mathcal{M}$ whenever $\mathcal{M}$ is a Noetherian $\mathcal{R}$-module.

Lemma 3. *Consider* $\mathrm{MinAss}_{\mathcal{R}}(\mathcal{M}) = \{\mathfrak{q}_1, ..., \mathfrak{q}_n\}$. *If* $EG(\mathcal{M})$ *is a planar graph, then* $n \leq 3$.

Proof. Let $EG(\mathcal{M})$ be planar and $\mathrm{MinAss}_{\mathcal{R}}(\mathcal{M}) = \{\mathfrak{q}_1, ..., \mathfrak{q}_n\}$. Then, for each t such that $1 \leq t \leq n$, it holds that $|\cap_{s=1,s\neq t}^{n} \mathfrak{q}_s \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))| \geq 1$. Assume that $x_t \in \cap_{s=1,s\neq t}^{n} \mathfrak{q}_s \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))$. Then in view of [11, Theorem 2.5], x_1, x_2, x_3, x_4 and x_5 are vertices of a subgraph contains K_5 of $EG(\mathcal{M})$, which is an opposition. Hence, $n \leq 4$. Moreover, for $n = 4$, $x \in \mathfrak{q}_1 \cap \mathfrak{q}_2$, $x' \in \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{q}_3$ and $x'' \in \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{q}_4$ with y, y', y'' are vertices of a subdivision isomorphic to $K_{3,3}$ of $EG(\mathcal{M})$ where $y \in \mathfrak{q}_3 \cap \mathfrak{q}_4$, $y' \in \mathfrak{q}_1 \cap \mathfrak{q}_3 \cap \mathfrak{q}_4$ and $y'' \in \mathfrak{q}_2 \cap \mathfrak{q}_3 \cap \mathfrak{q}_4$, that is a contradictions so $n \leq 3$. □

Theorem 5. *Consider* $\mathrm{MinAss}_{\mathcal{R}}(\mathcal{M}) = \{\mathfrak{q}\}$. *Then* $EG(\mathcal{M})$ *is a planar graph if and only if at least one of the listed conditions is satisfied:*

(i) $EG(\mathcal{M})$ *is a null graph or* $EG(\mathcal{M}) = K_4$.

(ii) $EG(\mathcal{M}) = K_3 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M})\setminus\mathfrak{q}|}$, *where* $|\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathfrak{q}| \leq 2$.

(iii) $EG(\mathcal{M}) = K_2 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M})\setminus\mathfrak{q}|}$ *or* $EG(\mathcal{M}) = K_1 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M})\setminus\mathfrak{q}|}$, *where* $|\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathfrak{q}| \leq 4$.

Proof. Let $EG(\mathcal{M})$ be planar. If $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) = \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$, then [11, Theorem 4.6] and [8, Lemma 2.1], imply that $EG(\mathcal{M})$ is a null graph. Let $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \neq \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$ and $\mathrm{MinAss}_{\mathcal{R}}(\mathcal{M}) = \{\mathfrak{q}\}$, where $\mathfrak{q} = \mathrm{Ann}_{\mathcal{R}}(m)$ for some $0 \neq m \in \mathcal{M}$. If $x, y \in \mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$ be adjoin in $EG(\mathcal{M})$, then $\mathrm{Ann}_{\mathcal{M}}(xy)$ is essential. So $\mathrm{Ann}_{\mathcal{M}}(xy) \cap Rm \neq 0$. Now, it is evident that either $x \in \mathfrak{q}$ or $y \in \mathfrak{q}$. Therefore, by [11, Lemma 2.2], $|r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})| = |\mathfrak{q} \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})| \leq 4$. If $|r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})| = 4$, then $\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) = r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))$ which leads to $EG(\mathcal{M}) = K_4$. Otherwise, $EG(\mathcal{M}) = K_{|\mathfrak{q}\setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})|} \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M})\setminus\mathfrak{q}|}$, where $|\mathfrak{q} \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})| \leq 3$. If $|\mathfrak{q} \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})| = 3$, then $EG(\mathcal{M}) = K_3 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M})\setminus\mathfrak{q}|}$, where $|\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathfrak{q}| \leq 2$. Now, assume that $|\mathfrak{q} \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})| \leq 2$. Thus $EG(\mathcal{M}) = K_2 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M})\setminus\mathfrak{q}|}$ or $EG(\mathcal{M}) = K_1 \vee \overline{K}_{|\mathcal{Z}_{\mathcal{R}}(\mathcal{M})\setminus\mathfrak{q}|}$, where $|\mathcal{Z}_{\mathcal{R}}(\mathcal{M}) \setminus \mathfrak{q}| \leq 4$ since in view of [12, Theorem 2.9], $EG(\mathcal{M})$ is perfect. □

Theorem 6. *Consider* $\mathrm{MinAss}_{\mathcal{R}}(\mathcal{M}) = \{\mathfrak{q}_1, \mathfrak{q}_2\}$. *Then* $EG(\mathcal{M})$ *is a planar graph if and only if at least one of the listed conditions is satisfied:*

(i) $EG(\mathcal{M}) = K_4$ *or* $EG(\mathcal{M}) = K_2 \vee K_{1,2}$.

(ii) $EG(\mathcal{M}) = P_1 \vee K_{2,2}$ *or* $EG(\mathcal{M}) = P_1 \vee K_{1,2}$ *or* $EG(\mathcal{M}) = P_2 \vee \overline{K}_{|\mathfrak{q}_t\setminus\mathfrak{q}_s|}$, *where* $1 \leq t, s \leq 2$.

(iii) $EG(\mathcal{M}) = K_{|\mathfrak{q}_t\setminus\mathfrak{q}_s|,|\mathfrak{q}_s\setminus\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})|}$, *where* $|\mathfrak{q}_t \setminus \mathfrak{q}_s| \leq 2$ *and* $1 \leq t, s \leq 2$.

Proof. Let $EG(\mathcal{M})$ be planar and $\mathrm{MinAss}_{\mathcal{R}}(\mathcal{M}) = \{\mathfrak{q}_1, \mathfrak{q}_2\}$. Then, as demonstrated by the argument of Theorem 2, we have $|r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})| \leq 4$. If $|r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})| \geq 3$, then from $|\mathfrak{q}_1 \setminus \mathfrak{q}_2| \geq 1$ and $|\mathfrak{q}_2 \setminus \mathfrak{q}_1| \geq 1$ it follow that the graph $EG(\mathcal{M})$ includes a subgraph

K_5, that is a contradictions. Thus $|r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})| \leq 2$. The following circumstances are present:
Case 1: Let $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) = \{x, y\}$ and let $\mathfrak{q}_1 \setminus \mathfrak{q}_2 = \{x_1, x_2\}$ and $\mathfrak{q}_2 \setminus \mathfrak{q}_1 = \{x_3, x_4\}$. Then $EG(\mathcal{M})$ has a subdivision isomorphic to $K_{3,3}$ with vertices x, x_1, x_2 and y, x_3, x_4. Thus $EG(\mathcal{M}) = K_4$ whenever $|\mathfrak{q}_1 \setminus \mathfrak{q}_2| = |\mathfrak{q}_2 \setminus \mathfrak{q}_1| = 1$ and $EG(\mathcal{M}) = K_2 \vee K_{1,2}$ whenever $|\mathfrak{q}_1 \setminus \mathfrak{q}_2| = 2$ and $|\mathfrak{q}_2 \setminus \mathfrak{q}_1| = 1$.
Case 2: Let $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) = \{x\}$. If $|\mathfrak{q}_1 \setminus \mathfrak{q}_2| \geq 2$ and $|\mathfrak{q}_2 \setminus \mathfrak{q}_1| \geq 3$, then $EG(\mathcal{M})$ includes a subgraph $K_{3,3}$ with vertices x, x_1, x_2 and x_3, x_4, x_5, where $x_1, x_2 \in \mathfrak{q}_1 \setminus \mathfrak{q}_2$ and $x_3, x_4, x_5 \in \mathfrak{q}_2 \setminus \mathfrak{q}_1$, that is a contradictions. Thus we may assume that $|\mathfrak{q}_1 \setminus \mathfrak{q}_2| \leq 2$ and $|\mathfrak{q}_2 \setminus \mathfrak{q}_1| \leq 2$. For $|\mathfrak{q}_1 \setminus \mathfrak{q}_2| = 2$ we have $EG(\mathcal{M}) = P_1 \vee K_{2,2}$ or $EG(\mathcal{M}) = P_1 \vee K_{1,2}$. For $|\mathfrak{q}_1 \setminus \mathfrak{q}_2| = 1$ we have $EG(\mathcal{M}) = P_2 \vee \overline{K}_{|\mathfrak{q}_2 \setminus \mathfrak{q}_1|}$.
Case 3: Let $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) = \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$. If $|\mathfrak{q}_1 \setminus \mathfrak{q}_2| \geq 3$ and $|\mathfrak{q}_2 \setminus \mathfrak{q}_1| \geq 3$, then $EG(\mathcal{M})$ contains a subgraph $K_{3,3}$ with vertices x_1, x_2, x_3 and x_4, x_5, x_6, where $x_1, x_2, x_3 \in \mathfrak{q}_1 \setminus \mathfrak{q}_2$ and $x_4, x_5, x_6 \in \mathfrak{q}_2 \setminus \mathfrak{q}_1$, that is a contradictions. Thus either $|\mathfrak{q}_1 \setminus \mathfrak{q}_2| \leq 2$ or $|\mathfrak{q}_2 \setminus \mathfrak{q}_1| \leq 2$. For $|\mathfrak{q}_1 \setminus \mathfrak{q}_2| = 2$ we have $EG(\mathcal{M}) = K_{2,|\mathfrak{q}_2 \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})|}$ and for $|\mathfrak{q}_1 \setminus \mathfrak{q}_2| = 1$, $EG(\mathcal{M}) = K_{1,|\mathfrak{q}_2 \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})|}$. □

Theorem 7. *Consider* $\mathrm{MinAss}_{\mathcal{R}}(\mathcal{M}) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3\}$. *Then* $EG(\mathcal{M})$ *is a planar graph if and only if at least one of the listed conditions is satisfied:*

(i) $EG(\mathcal{M}) = P_1 \vee (K_{1,|\mathfrak{q}_1 \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|} \cup K_{1,|\mathfrak{q}_2 \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|} \cup K_{1,|\mathfrak{q}_3 \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|})$.

(ii) $EG(\mathcal{M}) = K_3 \circ \overline{K}_{|\mathfrak{q}_t \setminus \mathfrak{q}_s \cup \mathfrak{q}_r|}$ *for each* $1 \leq r, s, t \leq 3$.

(iii) $EG(\mathcal{M}) = (K_3 \circ \overline{K}_{|\mathfrak{q}_t \setminus \mathfrak{q}_s \cup \mathfrak{q}_r|}) \cup (K_3 \circ \overline{K}_{|\mathfrak{q}_t \setminus \mathfrak{q}_s \cup \mathfrak{q}_r|})$, *where* $1 \leq r, s, t \leq 3$

Proof. Let $EG(\mathcal{M})$ be planar and $\mathrm{MinAss}_{\mathcal{R}}(\mathcal{M}) = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3\}$. By a similar argument to that of Theorem 6 we have $|r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})| \leq 2$. If $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) = \{x, y\}$ and $x_t \in \cap_{s=1, s \neq t}^{n} \mathfrak{q}_s \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))$. Then x, y, x_1, x_2 and x_3 are vertices of a subgraph that includes K_5 of $EG(\mathcal{M})$, that is a contradictions. Thus the following circumstances are present:
Case 1: Let $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) \setminus \mathrm{Ann}_{\mathcal{R}}(\mathcal{M}) = \{x\}$. Then there are $y \in \mathfrak{q}_1 \cap \mathfrak{q}_2 \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))$, $z \in \mathfrak{q}_2 \cap \mathfrak{q}_3 \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))$ and $w \in \mathfrak{q}_1 \cap \mathfrak{q}_3 \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))$. If $|\mathfrak{q}_1 \cap \mathfrak{q}_2 \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))| \geq 2$, then $EG(\mathcal{M})$ contains a subdivision isomorphic to $K_{3,3}$ with vertices x, y, y' and z, w, w', where $y' \in \mathfrak{q}_1 \cap \mathfrak{q}_2 \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))$ and $w' \in \mathfrak{q}_3 \setminus \mathfrak{q}_1 \cup \mathfrak{q}_2$, that is a contradiction. Thus $|\mathfrak{q}_t \cap \mathfrak{q}_s \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))| = 1$ for each t, s such that $1 \leq t, s \leq 3$ and $t \neq s$. Hence, $EG(\mathcal{M}) = P_1 \vee (K_{1,|\mathfrak{q}_1 \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|} \cup K_{1,|\mathfrak{q}_2 \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|} \cup K_{1,|\mathfrak{q}_3 \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|})$, see the Figure 1, where $\mathfrak{q}_1 \setminus \mathfrak{q}_2 \cup \mathfrak{q}_3 = \{u_0, u_1, u_2, \cdots\}$, $\mathfrak{q}_2 \setminus \mathfrak{q}_1 \cup \mathfrak{q}_3 = \{s_0, s_1, s_2, \cdots\}$ and $\mathfrak{q}_3 \setminus \mathfrak{q}_1 \cup \mathfrak{q}_2 = \{t_0, t_1, t_2, \cdots\}$.

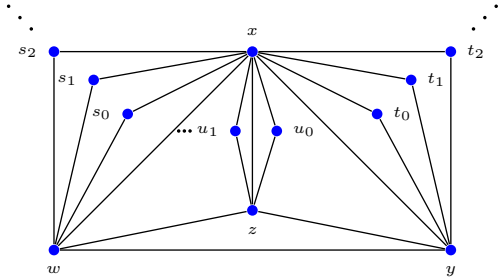


Figure 1: The planar graph $EG(\mathcal{M}) = P_1 \vee (K_{1,|\mathfrak{q}_1 \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|} \cup K_{1,|\mathfrak{q}_2 \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|} \cup K_{1,|\mathfrak{q}_3 \setminus r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M}))|})$.

Case 2: Let $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) = \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$. If $|\mathfrak{q}_1 \cap \mathfrak{q}_2| \geq 3$, then $EG(\mathcal{M})$ contains a subgraph $K_{3,3}$ with vertices x, x', x'' and y, y', y'' where $x, x', x'' \in \mathfrak{q}_1 \cap \mathfrak{q}_2$, $y \in \mathfrak{q}_2 \cap \mathfrak{q}_3$, $y' \in \mathfrak{q}_1 \cap \mathfrak{q}_3$ and $y'' \in \mathfrak{q}_3 \setminus \mathfrak{q}_1 \cup \mathfrak{q}_2$. Thus $|\mathfrak{q}_t \cap \mathfrak{q}_s| \leq 2$ for $1 \leq t, s \leq 3$ and $t \neq s$. If $|\mathfrak{q}_1 \cap \mathfrak{q}_2| = |\mathfrak{q}_2 \cap \mathfrak{q}_3| = 2$, then $EG(\mathcal{M})$ has a subdivision isomorphic to K_5 with vertices $x, x' \in \mathfrak{q}_1 \cap \mathfrak{q}_2$, $y, y' \in \mathfrak{q}_2 \cap \mathfrak{q}_3$, $z \in \mathfrak{q}_1 \cap \mathfrak{q}_3$, $x_1 \in \mathfrak{q}_1 \setminus \mathfrak{q}_2 \cup \mathfrak{q}_3$, $x_2 \in \mathfrak{q}_2 \setminus \mathfrak{q}_1 \cup \mathfrak{q}_3$ and $x_3 \in \mathfrak{q}_3 \setminus \mathfrak{q}_1 \cup \mathfrak{q}_2$, see the Figure 2.

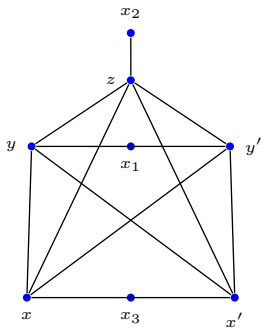


Figure 2: The non-planar graph $EG(\mathcal{M})$ with $|\mathfrak{q}_1 \cap \mathfrak{q}_3| = |\mathfrak{q}_2 \cap \mathfrak{q}_3| = 2$.

Suppose that $|\mathfrak{q}_1 \cap \mathfrak{q}_2| = |\mathfrak{q}_2 \cap \mathfrak{q}_3| = |\mathfrak{q}_1 \cap \mathfrak{q}_3| = 1$. Thus $EG(\mathcal{M})$ is the following planar Corona product graph $K_3 \circ \overline{K}_{|\mathfrak{q}_t \setminus \mathfrak{q}_s \cup \mathfrak{q}_r|}$ for each $t, s, r \in \{1, 2, 3\}$. See the Figure 3, where $\mathfrak{q}_1 \setminus \mathfrak{q}_2 \cup \mathfrak{q}_3 = \{u_0, u_1, u_2, \cdots\}$, $\mathfrak{q}_2 \setminus \mathfrak{q}_1 \cup \mathfrak{q}_3 = \{s_0, s_1, s_2, \cdots\}$ and $\mathfrak{q}_3 \setminus \mathfrak{q}_1 \cup \mathfrak{q}_2 = \{t_0, t_1, t_2, \cdots\}$.

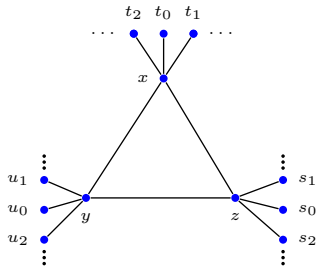


Figure 3: The planar Corona product graph $K_3 \circ \overline{K}_{|\mathfrak{q}_t \setminus \mathfrak{q}_s \cup \mathfrak{q}_r|}$ for each $1 \leq r, s, t \leq 3$.

At this point, we can safely assume that $\mathfrak{q}_1 \cap \mathfrak{q}_2 = \{x, x'\}$, $\mathfrak{q}_2 \cap \mathfrak{q}_3 = \{y\}$ and $\mathfrak{q}_1 \cap \mathfrak{q}_3 = \{z\}$. Then $EG(\mathcal{M})$ is union of two Corona product graphs, that is,

$$EG(\mathcal{M}) = (K_3 \circ \overline{K}_{|\mathfrak{q}_t \setminus \mathfrak{q}_s \cup \mathfrak{q}_r|}) \cup (K_3 \circ \overline{K}_{|\mathfrak{q}_t \setminus \mathfrak{q}_s \cup \mathfrak{q}_r|}).$$

See the Figure 4, where $\mathfrak{q}_1 \setminus \mathfrak{q}_2 \cup \mathfrak{q}_2 = \{u_0, u_1, u_2, \cdots\}$, $\mathfrak{q}_2 \setminus \mathfrak{q}_1 \cup \mathfrak{q}_3 = \{s_0, s_1, s_2, \cdots\}$ and $\mathfrak{q}_3 \setminus \mathfrak{q}_1 \cup \mathfrak{q}_2 = \{t_0, t_1, t_2, \cdots\}$.

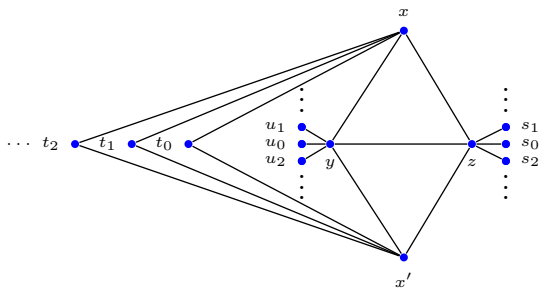


Figure 4: The planar graph $EG(\mathcal{M}) = (K_3 \circ \overline{K}_{|\mathfrak{q}_t \setminus \mathfrak{q}_s \cup \mathfrak{q}_r|}) \cup (K_3 \circ \overline{K}_{|\mathfrak{q}_t \setminus \mathfrak{q}_s \cup \mathfrak{q}_r|})$.

□

Corollary 1. *Consider* $\mathrm{MinAss}_{\mathcal{R}}(\mathcal{M}) = \{\mathfrak{q}_1, ..., \mathfrak{q}_n\}$ $(n \leq 3)$ *and* $r(\mathrm{Ann}_{\mathcal{R}}(\mathcal{M})) = \mathrm{Ann}_{\mathcal{R}}(\mathcal{M})$. *Then* $\Gamma(\mathcal{M})$ *is a planar graph if and only if at least one of the listed conditions is satisfied:*

(i) $\Gamma(\mathcal{M})$ *is a null graph.*

(ii) $\Gamma(\mathcal{M}) = K_{2,|\mathfrak{q}_t \setminus \mathrm{Ann}(\mathcal{M})|}$ *or* $\Gamma(\mathcal{M}) = K_{1,|\mathfrak{q}_t \setminus \mathrm{Ann}(\mathcal{M})|}$, *where* $1 \leq t \leq 2$.

(iii) $\Gamma(\mathcal{M}) = K_3 \circ \overline{K}_{|\mathfrak{q}_t \setminus \mathfrak{q}_s \cup \mathfrak{q}_r|}$ *or* $\Gamma(\mathcal{M}) = (K_3 \circ \overline{K}_{|\mathfrak{q}_t \setminus \mathfrak{q}_s \cup \mathfrak{q}_r|}) \cup (K_3 \circ \overline{K}_{|\mathfrak{q}_t \setminus \mathfrak{q}_s \cup \mathfrak{q}_r|})$, *where* $1 \leq r, s, t \leq 3$.

Proof. This result is derived directly via [11, Theorem 4.6] and Theorems 5, 6, 7. □

3 When $\Gamma(\mathbb{Z}_n)$ and $EG(\mathbb{Z}_n)$ are Planar?

This portion of the text examines the zero-divisor and the essential graphs of the ring $\mathbb{Z}_n$ and explores the conditions under which they exhibit planar or non-planar characteristics. For prime number p, $\mathcal{Z}(\mathbb{Z}_p)^* = \emptyset$ so $\Gamma(\mathbb{Z}_p)$ and $EG(\mathbb{Z}_p)$ are null graphs and inherently planar.

Theorem 8. *Consider prime number* p *and* $n = p^\alpha$, *where* $2 \leq \alpha$. *Then* $\Gamma(\mathbb{Z}_n)$ *is a planar graph if and only if* $n = 2^\alpha$ $(\alpha \leq 4)$, $n = 3^\alpha$ $(\alpha \leq 3)$ *and* $n = 5^\alpha$ $(\alpha \leq 2)$. *Moreover,* $EG(\mathbb{Z}_n)$ *is a planar graph if and only if* $n = 2^\alpha$ $(\alpha \leq 3)$ *and* $n = 3^\alpha, 5^\alpha$ $(\alpha \leq 2)$.

Proof. It is apparent that $kp^{\alpha-1}$ is a universal vertex of $\Gamma(\mathbb{Z}_{p^\alpha})$ for $1 \leq k \leq p-1$. Thus, as demonstrated by the argument of Theorem 2, $\Gamma(\mathbb{Z}_{p^\alpha})$ is non-planar for all $p \geq 7$. Let $p = 2$. Then $2^{\alpha-3}$ and $2 \times 2^{\alpha-3}, 3 \times 2^{\alpha-3}, 4 \times 2^{\alpha-3}, 5 \times 2^{\alpha-3}$ are vertices of a subgraph of $\Gamma(\mathbb{Z}_{2^\alpha})$ isomorphic to K_5. Moreover, $4, 12, 28$ and $8, 16, 24$ are vertices of a subgraph of $\Gamma(\mathbb{Z}_{32})$ isomorphic to $K_{3,3}$ so $\Gamma(\mathbb{Z}_{2^\alpha})$ is non-planar for $\alpha \geq 5$. Also, it is easy to see that $\Gamma(\mathbb{Z}_{2^\alpha})$ is planar for $\alpha \leq 4$. If $p = 3$, then 3^2, $3^{\alpha-2}, 2 \times 3^{\alpha-2}, 3^{\alpha-1}$ and $2 \times 3^{\alpha-1}$ are vertices of a subgraph of $\Gamma(\mathbb{Z}_{3^\alpha})$ isomorphic to K_5. Moreover, $9, 18, 27$ and $36, 45, 54$ are vertices of a subdivision isomorphic to $K_{3,3}$ of $\Gamma(\mathbb{Z}_{81})$ so $\Gamma(\mathbb{Z}_{3^\alpha})$ is non-planar for $\alpha \geq 4$. Now, it is clear that $\Gamma(\mathbb{Z}_{3^\alpha})$ is planar whenever $\alpha \leq 3$. A similar argument shows that for $p = 5$, $\Gamma(\mathbb{Z}_{5^\alpha})$ is planar whenever $\alpha \leq 2$.

For the second assertion, it is well known that $\mathcal{Z}(\mathbb{Z}_{p^\alpha}) = \mathrm{Nil}(\mathbb{Z}_{p^\alpha}) = p\mathbb{Z}_{p^\alpha}$. So by Theorem 5, $EG(\mathbb{Z}_{p^\alpha})$ is a planar graph if and only if

$$|\mathcal{Z}(\mathbb{Z}_{p^\alpha})^*| = p^\alpha - \phi(p^\alpha) - 1 = p^{\alpha-1} - 1 \leq 4.$$

This implies that $p = 2$ for $\alpha \leq 3$ and $p = 3, 5$ for $\alpha \leq 2$. □

In what follows, we illustrate some special cases of Theorem 8.

Example 1. Consider the ring $\mathbb{Z}_{27}$. Theorem 8 shows that $\Gamma(\mathbb{Z}_{27}) = P_2 \vee \overline{K}_6$ is a planar graph while the essential graph of $\mathbb{Z}_{27}$ contains a subgraph K_5, hence it is non-planar, see Figure 5.

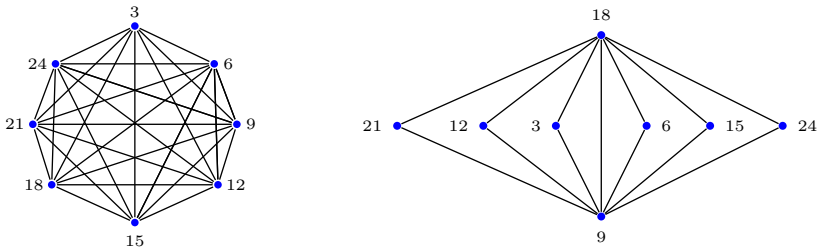


Figure 5: The non-planar graph $EG(\mathbb{Z}_{27})$ and planar graph $\Gamma(\mathbb{Z}_{27})$

Theorem 9. *Consider an integer $n \geq 2$. Then $\Gamma(\mathbb{Z}_n)$ is a planar graph if and only if $n = 12$ or $n = 18$ or $n = 2p$ or $n = 3p$, where p is a prime number.*

Proof. It is easy to see that $\Gamma(\mathbb{Z}_{12})$ and $\Gamma(\mathbb{Z}_{18})$ are planar. For $n = 2p$ and $n = 3p$, Corollary 1(ii) and [11, Theorem 4.6] show that $\Gamma(\mathbb{Z}_{2p}) = K_{1,p-1}$ and $\Gamma(\mathbb{Z}_{3p}) = K_{2,p-1}$.

For the converse we show that if $n \neq 12, 18, 2p, 3p$, then $\Gamma(\mathbb{Z}_n)$ has a subdivision isomorphic to K_5 or $K_{3,3}$ so it stands as a non-planar graph.
(i) If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where $p_1 < p_2 < \cdots < p_k$ are prime numbers, $k \geq 3$ and $\alpha_i \in \mathbb{N}$, then $p_1^{\alpha_1} p_2^{\alpha_2} p_4^{\alpha_4} \cdots p_k^{\alpha_k}, 2p_1^{\alpha_1} p_2^{\alpha_2} p_4^{\alpha_4} \cdots p_k^{\alpha_k}, 4p_1^{\alpha_1} p_2^{\alpha_2} p_4^{\alpha_4} \cdots p_k^{\alpha_k}$ and $p_3^{\alpha_3}, 2p_3^{\alpha_3}, 4p_3^{\alpha_3}$ are vertices of a subgraph isomorphic to $K_{3,3}$ of $\Gamma(\mathbb{Z}_n)$ so it stands as a non-planar graph. So $k \leq 2$.
(ii) If $n = p_1^{\alpha_1} p_2^{\alpha_2}$, where $5 \leq p_1 < p_2$ are prime numbers and $\alpha_i \in \mathbb{N}$, then $p_1^{\alpha_1}, 2p_1^{\alpha_1}, 3p_1^{\alpha_1}$ and $p_2^{\alpha_2}, 2p_2^{\alpha_2}, 3p_2^{\alpha_2}$ are vertices of a subgraph isomorphic to $K_{3,3}$ of $\Gamma(\mathbb{Z}_n)$ so it is a non-planar graph.
(iii) If $n = 3^\alpha p^\beta$, where $p \geq 5$ and $\alpha \geq 1, \beta \geq 2$, then $p^{\beta-1}, 3^\alpha p^{\beta-1}, 2 \times 3^\alpha p^{\beta-1}, 3 \times 3^\alpha p^{\beta-1}$ and $4 \times 3^\alpha p^{\beta-1}$ are vertices of a subdivision includes of K_5 of $\Gamma(\mathbb{Z}_n)$ so it remains a non-planar graph.
(iv) If $n = 3^\alpha p$, where $p \geq 5$ and $\alpha \geq 2$, then $3^\alpha, 2 \times 3^\alpha, 3 \times 3^\alpha$ and $p, 2p, 4p$ are vertices of a subgraph isomorphic to $K_{3,3}$ of $\Gamma(\mathbb{Z}_n)$ so it stands as a non-planar graph.
(v) If $n = 2^\alpha p^\beta$, where $p \geq 3$ and $\alpha, \beta \geq 2$, then $2^\alpha, 2 \times 2^\alpha, 3 \times 2^\alpha$ and $p^\beta, 2p^\beta, 3p^\beta$ are vertices of a subgraph included of $K_{3,3}$ of $\Gamma(\mathbb{Z}_n)$ so it is a non-planar graph.
(vi) If $n = 2^\alpha p$, where $p \geq 5$ and $\alpha \geq 2$, then $2^\alpha, 2^{\alpha+1}, 2^{\alpha+2}$ and $p, 2p, 3p$ are vertices of a subgraph isomorphic to $K_{3,3}$ of $\Gamma(\mathbb{Z}_n)$ so it is a non-planar graph.
(vii) If $n = 2p^\beta$, where $p \geq 5$ and $\beta \geq 2$, then $p^{\beta-1}, 2p^{\beta-1}, 4p^{\beta-1}, 6p^{\beta-1}, 8p^{\beta-1}$ are vertices of a subgraph included of K_5 of $\Gamma(\mathbb{Z}_n)$ so it remains a non-planar graph.
(viii) If $n = 2^\alpha \times 3$, where $\alpha \geq 3$, then $2^{\alpha-1}, 2^\alpha, 2^{\alpha+1}$ and $6, 12, 18$ are vertices of a subgraph included of $K_{3,3}$ of $\Gamma(\mathbb{Z}_n)$ so it stands as a non-planar graph.
(viiii) If $n = 2 \times 3^\beta$, where $\beta \geq 3$, then $6, 12, 24$ and $3^{\beta-1}, 3 \times 3^{\beta-1}, 5 \times 3^{\beta-1}$ are vertices of a subgraph contain of $K_{3,3}$ of $\Gamma(\mathbb{Z}_n)$ so it exists as a non-planar graph. □

Theorem 10. *Consider an integer $n \geq 2$. Then $EG(\mathbb{Z}_n)$ is a planar graph if and only if $n = 2p$ or $n = 3p$, where p is a prime number.*

Proof. From Theorems 6, 9, and considering that $\Gamma(\mathbb{Z}_n)$ is a subgraph of $EG(\mathbb{Z}_n)$, we derive the result. □

The next example shows that Theorems 9 and 10 are satisfied.

Example 2. (i) Consider the rings $\mathbb{Z}_{14}$ and $\mathbb{Z}_{15}$. It is apparent that their zero-divisor and essential graphs are $K_{1,6}$ and $K_{2,4}$, respectively. So they are planar, see Figure 6.

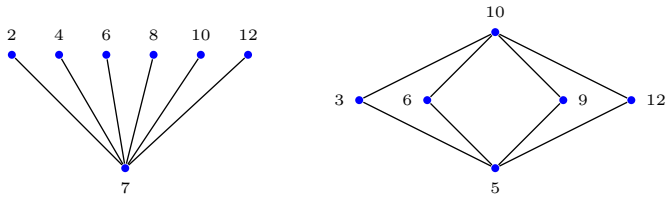


Figure 6: The planar zero-divisor and the essential graphs $\mathbb{Z}_{14}$ and $\mathbb{Z}_{15}$.

(ii) Consider the ring $\mathbb{Z}_{12}$. It is apparent that its essential graph is $P_3 \vee \overline{K}_4$ included subgraph $K_{3,3}$ so it is a non-planar graph but $\Gamma(\mathbb{Z}_{12})$ is planar, see Figure 7.

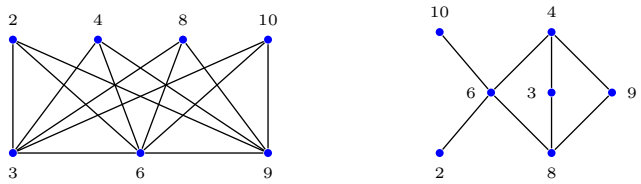


Figure 7: The non-planar graph $EG(\mathbb{Z}_{12})$ and the planar graph $\Gamma(\mathbb{Z}_{12})$.

Acknowledgments

The authors would like to thank the referee for careful reading and Dr. Ali Behtoei for his invaluable comments.

References

[1] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, **217** (1999), 434-447.

[2] D. F. Anderson and S. B. Mulay, *On the diameter and girth of a zero-divisor graph*, J. Pure Appl. Algebra, **210** (2007), 543-550.

[3] D. F. Anderson and D. Weber, *The zero-divisor graph of a commutative ring without identity*, Int. J. Algebra Comput., **23** (2018), 176-202.

[4] I. Beck, *Coloring of commutative rings*, J. Algebra, **116** (1988), 208-226.

[5] M. Behboodi, *Zero-divisor graphs for modules over commutative rings*, J. Comm. Algebra, **4** (2012), 175-197.

[6] Ch. Godsil and G. Royle, *Algebraic graph theory*, Springer-Verlag, New York, (2001).

[7] M. J. Nikmehr, R. Nikandish and M. Bakhtyiari, *On the essential graph of a commutative ring*, J. Algebra Appl., **16** (2017), 1750132.

[8] K. Nozari and Sh. Payrovi, *A generalization of zero-divisor graph for modules*, Publ. Inst. Math., **160** (2019), 39-46.

[9] S. Safaeeyan, M. Baziar and E. Momtahan, *A generalization of the zero-divisor graph for modules*, J. Korean Math. Soc., **51** (2014), 87-98.

[10] R. Y. Sharp, *Steps in commutative algebra*, Second edition, Cambridge University Press, Cambridge, (2000).

[11] F. Soheilnia, Sh. Payrovi and A. Behtoei, *A generalization of the essential graph for modules over commutative rings*, Int. Elec. J. Algebra, **29** (2021), 211-222.

[12] F. Soheilnia, Sh. Payrovi and A. Behtoei, *Perfectness of the essential graph for modules over commutative rings*, AUT J. Math. Comput., **6** (2025) 163-170.

[13] D. B. West, *Introduction to graph theory*, 2nd edition, Prentice Hall, Upper Saddle River, (2001).