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Some results on the sign domination number of the subdivision of a graph

Ahmad Abbasi[†], **Kamran Mirasheh**[‡], **Ebrahim Vatandoost**[§*]

[†] [‡] *Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran*
[§] *Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran*
Emails: aabbasi@guilan.ac.ir, K.mir.1185@gmail.com, vatandoost@sci.ikiu.ac.ir

Abstract. We present some new bounds for signed domination numbers. Let $G = (V, E)$ be a simple and undirected graph. For a function $f : V \longrightarrow \{-1, 1\}$, the weight of is f defined by $w(f) = \sum_{v\in V} f(v)$. For a vertex v in V, we define $f[v] = \sum_{u\in N[v]} f(u)$. A signed domination function of G is a function $f : V \longrightarrow \{-1, 1\}$ such that $f[v] \geq 1$ for all $v \in V$. The signed domination number $\gamma_s(G)$ of G is the minimum weight among all signed domination functions of G. In this paper, we study the signed domination problem of the general graph, and obtain some bounds of the signed domination number of G. We also establish upper and lower bounds of the signed domination number of subdivision construction $S(G)$.

Keywords: Signed domination, Dominating set, Subdivision.
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1 Introduction

In this paper, all graphs are assumed to be finite, simple, and undirected. A graph with no edges (but with at least one vertex) is called *empty*. We will often use the notation $G = (V, E)$ to denote a graph with non-empty vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of a graph is the number of its vertices and the *size* of a graph is the number of its edges. The degree of a vertex $v \in V(G)$ is denoted $deg_G(v)$. The minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. We sometimes write δ, Δ particularly when the graph is clear in the context. A graph G is called *k-regular* if every vertex of G has degree k. A graph is said to be regular if it is k-regular for some nonnegative integer k. In particular, a 3-regular

*Corresponding author
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graph is called a *cubic graph.* A connected graph G is a *cycle* if G is 2-regular. An edge of G with end vertices u and v is denoted by uv. For every vertex $x \in V(G)$, the *open neighborhood* of x is denoted by $N_G(x)$ and defined as $N_G(x) = \{y \in V(G) \ : \ xy \in E(G)\}$, and the *closed neighborhood* of $x \in V(G)$, $N_G[x]$, is $N_G[x] = N_G(x) \cup \{x\}$. For a set $T \subseteq V(G)$, the *open neighborhood* of T is $N_G(T) = \cup_{x\in T} N_G(x)$. For a set $S \subseteq V(G)$, the *induced subgraph* on S is denoted by $G[S]$. The symmetric difference of two sets is the set of all elements that are in either set, but not in both of them. Symmetric difference of the sets A, B is denoted by $A \ominus B$. A graph is *bipartite* if its vertex set can be partitioned into two nonempty disjoint subsets X and Y such that each edge of G has one end in X and the other in Y. In this circumstaces, the pair (X, Y) is called a bipartition of the bipartite graph. The adjacency matrix of G, is the $n \times n$ matrix $A = A(G)$ whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

Given a matrix A, a vector $x \neq 0$ is an eigenvector of A if and only if there exists $\lambda \in \mathbb{C}$ such that $Ax = \lambda x$. In this case, λ is called an eigenvalue of A. The *subdivision graph* of G is the graph obtained by inserting an additional vertex in the middle of each edge of G, and denoted by $S(G)$. A *matching* M in a graph G is a set of edges such that no two have a vertex in common. The size of a matching is the number of edges in it. A vertex incident to an edge of M is said to be covered by M. A matching that covers every vertex of G is called a *perfect matching* or a *1-factor.* Clearly, a graph that contains a perfect matching has an even number of vertices. A maximum matching is a matching with the maximum possible number of edges. A subset D of $V(G)$ is a *dominating set* of G, if every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D. The *domination number* of a graph G, denoted by $\gamma(G)$, is the minimum size of a dominating set of G. For more terminology we refer to [2, 13].
For a function $f : V(G) \longrightarrow \{-1, 1\}$ and a vertex $v \in V$, we define $f[v] = \sum_{u\in N[v]} f(u)$. A *signed dominating function* of G is a function $f : V(G) \longrightarrow \{-1, 1\}$ such that $f[v] \geq 1$ for all $v \in V(G)$. The *weight* of a function f is $\omega(f) = \sum_{v\in V(G)} f(v)$. The *signed domination number* $\gamma_s(G)$ is the minimum weight of all signed dominating functions of G. A signed dominating function of G with minimum weight is called a γ_s- function. This parameter has been studied by several authors, namely [1, 6, 9–12, 14, 15]. The authors in [4, 5] obtained a lower bound for $\gamma_s(G)$ w.r.t. n, δ, Δ, and for $\gamma_s(G) + \gamma_s(\bar{G})$ w.r.t. n, respectively. Moreover, in [7], the authors obtained a lower and an upper bound of $\gamma_s(G)$ w.r.t. $\gamma(G)$ and the packing number, respectively. The nonnegative signed domination number was introduced by Huang et al. [8].

Our main results are stated in the following theorems.
Theorem A:(Theorem 1) Let G be a connected graph of order n and size m. If λ_{max} and λ_{min} are the largest and the smallest of adjacency matrix eigenvalues of G, respectively, then

$$\gamma_s(G) \geq max\left\{ \left\lceil \frac{4m - n\lambda_{min}}{\lambda_{min}} \right\rceil , \left\lceil \frac{2m + n(1 - 2\lambda_{max})}{1 + 2\lambda_{max}} \right\rceil \right\}.$$

Theorem B:(Theorem 2) Let G be a connected graph of order n and size m. Then the following statements hold:

(i) $\left\lceil \dfrac{2m+n(1-\Delta)}{1+\Delta} \right\rceil \leq \gamma_s(G) \leq \left\lfloor \dfrac{4m-n\delta}{\delta} \right\rfloor$,

(ii) $\gamma_s(G) \geq max\left\{ \left\lceil \gamma(G) - \dfrac{n\Delta - m}{1+\Delta} \right\rceil, \left\lceil \gamma(G) - \dfrac{m+\delta}{\delta} \right\rceil \right\}$.

Theorem C:(Theorem 5) Let G be a k-regular and r-factorable graph of order n. Then

$$\gamma_s(S(G)) \leq \begin{cases} \dfrac{3}{2}n, & if\ k\ is\ an\ odd\ integer\ with\ \frac{k-1}{2} \equiv 0\ (mod\ r), \\ n, & if\ k\ is\ an\ even\ integer\ with\ \frac{k}{2} \equiv 0\ (mod\ r). \end{cases}$$

Theorem D:(Theorem 6) Let G be a connected 3−regular graph of order n. Then $\gamma_s(S(G)) = \dfrac{3}{2}n$.

Theorem E:(Theorem 7) Let G be a graph of order $n \geq 4$ and size m with $\delta(G) \geq n-2$. Then $\gamma_s(S(G)) \geq n-m$.

2 Bounds for Signed domination number of G

In this section, we establish the results in Theorems A and B by giving two bounds the signed dominating number the domination number and eigenvalues of the adjacency matrix of the graph.

Definition 1. *Let G be a graph of order n and $f : V(G) \longrightarrow \{-1, 1\}$ be a signed dominating function of G. The following sets are defined.*

$E_0(f) = \{uv \in E(G) : f(v) = f(u) = -1\}$ *and* $m_0^f = |E_0(f)|$,
$E_1(f) = \{uv \in E(G) : f(v)f(u) = -1\}$ *and* $m_1^f = |E_1(f)|$,
$E_2(f) = \{uv \in E(G) : f(v) = f(u) = 1\}$ *and* $m_2^f = |E_2(f)|$,
$V_f^+ = \{v \in V : f(v) = 1\}, V_f^- = \{v \in V : f(v) = -1\}$.

Lemma 1. *Let G be a graph of order n and $f : V(G) \longrightarrow \{-1, 1\}$ be a signed dominating function of G. Then $\gamma(G) \leq |\ V_f^+\ |$.*

Proof. Let $x \in V(G) \setminus V_f^+$. Since f is a signed dominating function of G, there exists $y \in V(G)$ such that $y \in (N_G[x] \cap V_f^+)$. So, V_f^+ is a dominating set for G. Hence, $\gamma(G) \leq |V_f^+|$. □

Lemma 2. *If G is a non-empty graph, then for every signed dominating function f, $2m_0^f + 2\ell \leq m_1^f \leq 2m_2^f$, where $\ell = |V_f^-|$.*

Proof. Let f be a signed dominating function for G.
If for every $v \in V_f^-$ and every $u \in N_G(v)$, $f(u) = 1$, then $m_0^f = 0$. Since f is a signed dominating function for G, there exist at least two vertices x and y belong to V_f^+ such that $\{x, y\} \subseteq N_G(v)$. Hence the edges $x \sim v, y \sim v$ are in $E_1(f)$ (see Fig 1). Thus $2\ell \leq m_1^f$ and so $2m_0^f + 2\ell \leq m_1^f$.

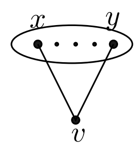

Figure 1

Let $v \in V_f^-$ and there exists $u \in N_G(v)$, such that $f(u) = -1$. Then $u \sim v \in E_0(f)$. Since f is a signed dominating function of G, there exist sets $L_v, L_u \subseteq V_f^+$ such that $\mid L_v \cap N_G(v) \mid \geq 3$ and $\mid L_u \cap N_G(u) \mid \geq 3$ (see Fig 2). Thus $2m_0^f + 2\ell \leq m_1^f$.

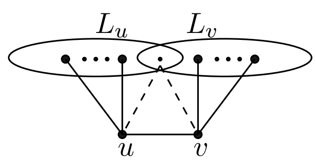

Figure 2

Now we show that $m_1^f \leq 2m_2^f$.
Let for every $a \in V_f^-$ we have $N_G(a) \subseteq V_f^+$. Let $a', a'' \in V_f^+$ such that $a', a'' \in N_G(a)$. Thus $m_1^f \leq 2m_2^f$ (see Fig 3).

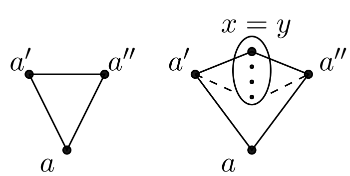

Figure 3

Let $a, b \in V_f^-$ such that $a \sim b$. Also let $a \sim a'$ and $b \sim b'$ be two edge in $E_1(f)$, where $\{a, b\} \subseteq V_f^+$. Then $a' \sim b'$ or there are $x, y \in V_f^+$ such that $a' \sim x$ and $b' \sim y$. Thus $m_1^f \leq 2m_2^f$.

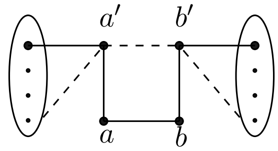

Figure 4

□

Remark 1. *If A is the adjacency matrix of graph G, and $\lambda_1 \geq, \lambda_n \geq, ..., \geq \lambda_n$ eigenvalues of* *matrix A, then since $tr\ A = 0$ and the trace of a square matrix is also equal to the sum of its eigenvalues, we have $\lambda_1 > 0$ and $\lambda_n < 0$.*

In view of this argument, we give the next theorem.

Theorem 1. *Let G be a connected graph of order n and size m. If λ_{max} and λ_{min} are the largest and the smallest of adjacency matrix eigenvalues of G, respectively, then*

$$\gamma_s(G) \geq max\Big\{ \Big\lceil \frac{4m - n\lambda_{min}}{\lambda_{min}} \Big\rceil , \Big\lceil \frac{2m + n(1 - 2\lambda_{max})}{1 + 2\lambda_{max}} \Big\rceil \Big\}.$$

Proof. Let $V = \{v_1, v_2, \cdots, v_n\}$ and let A be the adjacency matrix of G. Also let f be a γ_s-function of G. For $x_i \in \{0, 1\}$, let $X_f = (x_i)$ be a matrix $n \times 1$ such that $x_i = 1$ if and only if $v_i \in V_f^+$. Since $|V_f^+| > 0$, so $X_f^t X_f = |V_f^+| > 0$. By Rayleigh quotient, $\lambda_{min} \leq \frac{X_f^t A X_f}{X_f^t X_f} \leq \lambda_{max}$. We have

$$X_f^t A X_f = \sum_{i=1}^{n} x_i |\ N_G(v_i) \cap V_f^+\ |.$$

Since $x_i = 0$ for $v_i \in V_f^-$ and $x_i = 1$ for $v_i \in V_f^+$, so

$$X_f^t A X_f = \sum_{v_i \in V_f^+} x_i |\ N_G(v_i) \cap V_f^+\ | = \sum_{v_i \in V_f^+} |\ N_G(v_i) \cap V_f^+\ |.$$

Let H be induced subgraph on V_f^+. Then $|\ N_H(v_i) \cap V_f^+\ | = deg_H(v_i)$ and so

$$X_f^t A X_f = \sum_{v_i \in V_f^+} |\ N_G(v_i) \cap V_f^+\ | = \sum_{v_i \in H} deg_H(v_i) = 2|E(H)\ | = 2m_2^f.$$

It is clear that $X_f^t X_f = |\ V_f^+\ | = n - \ell$, where $|\ V_f^-\ | = \ell$. Hence $\lambda_{min} \leq \frac{2m_2^f}{n - \ell} \leq \lambda_{max}$, and so $4m_2^f \leq 2(n - \ell)\lambda_{max}$. By Lemma 2, $(\ell + m_0^f) + m_1^f + m_2^f \leq m_2^f + 2m_2^f + m_2^f \leq 2\lambda_{max}(n - \ell)$. Thus $\ell + m \leq 2\lambda_{max}(n - \ell)$. Hence

$$\frac{m - 2n\lambda_{max}}{1 + 2\lambda_{max}} \leq -\ell.$$

Also we have $\lambda_{min}(n - \ell) \leq 2m_2^f \leq 2m$. Since $\lambda_{min} < 0$, so

$$\frac{2m - n\lambda_{min}}{\lambda_{min}} \leq -\ell.$$

From $\gamma_s(G) = n - 2\ell$, we have

$$\gamma_s(G) = n + 2(-\ell) \geq max\Big\{ \Big\lceil \frac{4m - n\lambda_{min}}{\lambda_{min}} \Big\rceil, \Big\lceil \frac{2m + n(1 - 2\lambda_{max})}{1 + 2\lambda_{max}} \Big\rceil \Big\},$$

which completes the proof of theorem. □

In the following theorem we obtain some new bounds for the signed dominating number.

Theorem 2. *Let G be a connected graph of order n and size m. Then the following statements hold:*

(i) $\left\lceil \dfrac{2m+n(1-\Delta)}{1+\Delta} \right\rceil \leq \gamma_s(G) \leq \left\lfloor \dfrac{4m-n\delta}{\delta} \right\rfloor$;

(ii) $\gamma_s(G) \geq max\left\{ \left\lceil \gamma(G) - \dfrac{n\Delta - m}{1+\Delta} \right\rceil, \left\lceil \gamma(G) - \dfrac{m+\delta}{\delta} \right\rceil \right\}$.

Proof. (i) Let D be the diagonal matrix whose diagonal entry $(D)_{ii}$ is $deg_G(v_i)$. Also, let f be a γ_s- function for G and for $x_i \in \{0,1\}$, $X_f = (x_i)$ is a matrix $n \times 1$ such that $x_i = 1$ if and only if $v_i \in V_f^+$. Then

$$X_f^t D X_f = \sum_{i=1}^{n} x_i^2 (deg_G(v_i)) = \sum_{v_i \in V_f^+} x_i^2 (deg_G(v_i)) = m_1^f + 2m_2^f.$$

We know that $\lambda_{max}(G) \leq \Delta(G)$. By Rayleigh quotient, $\delta \leq \dfrac{X_f^t D X_f}{X_f^t X_f} \leq \Delta$. Since $X_f^t X_f = n - \ell$, $(\ell = \mid V_f^- \mid)$ so

$$\delta \leq \frac{m_1^f + 2m_2^f}{n-\ell} \leq \Delta.$$

By Lemma 2, $m_1^f + 2m_2^f = m_1^f + m_2^f + m_2^f \geq m_1^f + m_2^f + (m_0^f + \ell) = m + \ell$. Thus $m + \ell \leq (n-\ell)\Delta$. Also, $\delta(n-\ell) \leq m_1^f + 2m_2^f \leq 2m$. Hence,

$$\frac{n\delta - 2m}{\delta} \leq \ell \leq \frac{n\Delta - m}{1+\Delta}. \qquad (*)$$

Since $\gamma_s(G) = n - 2\ell$, an easy computation shows that

$$\frac{2m+n(1-\Delta)}{1+\Delta} \leq \gamma_s(G) \leq \frac{4m-n\delta}{\delta}.$$

(ii) Since $\mid V_f^+ \mid = n - \ell$, by Lemma 1, $\gamma(G) \leq n - \ell$. Thus

$$\gamma(G) - \gamma_s(G) \leq \ell. \qquad (**)$$

By $(*)$ and $(**)$ we have $\gamma(G) - \gamma_s(G) \leq \ \ell \ \leq \dfrac{n\Delta - m}{1+\Delta}$.
Now, for $y_i \in \{0,1\}$, let $Y_f = (y_i)$ be a matrix $n \times 1$ such that $y_i = 1$ if and only if $v_i \in V_f^-$. Then

$$Y_f^t D Y_f = \sum_{i=1}^{n} y_i^2 (deg_G(v_i)) = \sum_{v_i \in V_f^-} y_i^2 (deg_G(v_i)) = m_1^f + 2m_0^f.$$

By Rayleigh quotient, $\delta \leq \dfrac{Y_f^t D Y_f}{Y_f^t Y_f} \leq \Delta$. Since $Y_f^t Y_f = \ell$, by Lemma 2, we have

$$\ell \leq \frac{m_1^f + 2m_0^f}{\delta} < \frac{m}{\delta}.$$

Therefore, by $(**)$ we have $\gamma(G) - \gamma_s(G) < \dfrac{m}{\delta}$. □

Corollary 1. *If G is a regular graph, then $\gamma_s(G) \geq 1$ and this bound is sharp.*

Proof. This is a straightforward result of Theorem 2(i). For the sharpness, consider the complete graph K_n, where n is an odd integer. □

3 Signed dominating number of $S(G)$

Recall that for any graph G, the subdivision graph is obtained from G by inserting a vertex on any edge of G. A factor of a graph G is a spanning subgraph of G. A k-factor of G is a factor of G that is k-regular. Thus, a 1-factor of G is a matching that saturates all the vertices of G. For this reason, a 1-factor of G is called a perfect matching of G. A 2-factor of G is a factor of G that is a disjoint union of cycles of G. A graph G is k-factorable if G is an edge-disjoint union of k-factors of G.

In this Section, we study the signed domination number of the subdivision of G. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$. Note that $V(S(G)) = V(G) \cup B$, where

$$B = \Big\{ v_{ij} \ : \ 1 \leq i < j \leq n, \ N_{S(G)}(v_{ij}) = \{v_i, v_j\} \Big\}.$$

Let $\mathcal{F}$ be a family of functions $\varphi : B \longrightarrow \{-1, 1\}$. We define following sets

$$\mathcal{F}_r = \Big\{ \varphi : B \rightarrow \{-1, 1\} \ : \ |N_{S(G)}(v_i) \cap V_\varphi^-| = r, \ \textit{for all } v_i \in V(G) \Big\},$$

$$B_r = \Big\{ v_{ij} \ : \ \varphi(v_{ij}) = -1, \textit{for some } \varphi \in \mathcal{F}_r \Big\}.$$

Remark 2. *Assume that G is a k-regular r-factorable graph and $M = \{M_1, M_2, \cdots, M_\theta\}$ is an edge-disjoint union of r-factors of G. Since $E(G) = \cup_{t=1}^{\theta} E(M_t)$ and $M_{t'} \cap M_t = \emptyset$ $(1 \leq t' \neq t \leq \theta)$, one has $r\theta = k$.*

Lemma 3. *Let G be a k-regular and r-factorable graph. Then $\mathcal{F}_{r\theta} \neq \emptyset$, for all $1 \leq \theta \leq \frac{k}{r}$.*

Proof. Suppose that $M = \{M_1, M_2, \cdots, M_\theta\}$ is an edge-disjoint union of r-factors of G, also $U_t = \{v_{ij} | v_i v_j \in \cup_{t=1}^{\theta} E(M_t)\}$. First, let $t = 1$. We define the function $\phi_1 : B \rightarrow \{1, -1\}$ to be -1 on U_1 and 1 elsewhere. Hence for each $v_i \in V(G)$ we have $|N_{S(G)}(v_i) \cap V_{\varphi_1}^-| = r$, thus $\varphi_1 \in \mathcal{F}_r$. Now if $r \geq 2$, then by continuing this process for function $\varphi_t : B \rightarrow \{-1, 1\}$ with the value -1 only on U_t, since $M_{t'} \cap M_t = \emptyset$ $(1 \leq t' \neq t \leq \theta)$, we have $|N_{S(G)}(v_i) \cap V_{\varphi_t}^-| = rt$, for all $v_i \in V(G)$. Hence $\varphi_t \in \mathcal{F}_{r\theta}$ by Remark 2. This process can be continued up to step $\frac{k}{r}$. Therefore, $\mathcal{F}_{r\theta} \neq \emptyset$, for all $1 \leq \theta \leq \frac{k}{r}$. □

Theorem 3. *[3] For* $n \geq 2$, $\gamma_s(P_n) = n - 2\left\lfloor \frac{n-2}{3} \right\rfloor$.

Theorem 4. *[3] For* $n \geq 3$, $\gamma_s(C_n) = n - 2\left\lfloor \frac{n}{3} \right\rfloor$.

Theorem 5. *Let* G *be a* k*-regular and* r*-factorable and graph of order* n*. Then*

$$\gamma_s(S(G)) \leq \begin{cases} \frac{3}{2}n, & if\ k\ is\ an\ odd\ integer\ and\ \frac{k-1}{2} \equiv 0\ (mod\ r), \\ n, & if\ k\ is\ even\ integer\ and\ \frac{k}{2} \equiv 0\ (mod\ r). \end{cases}$$

Proof. It is clear that, if $n \geq 2$, then $S(P_n) = P_{2n-1}$ and for $n \geq 3$, we have $C_{2n} = S(C_n)$. By Theorems 3 and 4, for $k = 1$ and $k = 2$, the proof is straightforward. So we can assume that $k \geq 3$ is an odd integer with $\frac{k-1}{2} \equiv 0\ (mod\ r)$, by Lemma 3, there exist the function $\varphi \in \mathcal{F}_{\frac{k-1}{2}}$, hence $B_{\frac{k-1}{2}} \neq \emptyset$. Now let $f : V(S(G)) \rightarrow \{-1, 1\}$ is defined, as follows

$$f(x) = \begin{cases} 1, & x \in V(G), \\ 1, & x \in B \backslash B_{\frac{k-1}{2}}, \\ -1, & x \in B_{\frac{k-1}{2}}. \end{cases}$$

We have

$$f[x] = \sum_{y \in N_{S(G)}[x]} f(y) = 2,\ for\ all\ x \in V(G),$$

$$f[x] = \sum_{y \in N_{S(G)}[x]} f(y) = 3,\ for\ all\ x \in B \backslash B_{\frac{k-1}{2}},$$

$$f[x] = \sum_{y \in N_{S(G)}[x]} f(y) = 1,\ for\ all\ x \in B_{\frac{k-1}{2}}.$$

So, f is a signed domination function of $S(G)$ and $|V_f^-| = \dfrac{n(k-1)}{4}$. Therefore, $\gamma_s(S(G)) \leq \dfrac{3}{2}n$. Now, let k is an even integer and $\frac{k}{2} \equiv 0\ (mod\ r)$. by Lemma 3, there exist the function $\varphi_1 \in \mathcal{F}_{\frac{k}{2}}$, hence $B_{\frac{k}{2}} \neq \emptyset$. Consider $f_1 : V(S(G)) \rightarrow \{-1, 1\}$ as defined by

$$f_1(x) = \begin{cases} 1, & x \in V(G); \\ 1, & x \in B \backslash B_{\frac{k}{2}}; \\ -1, & x \in B_{\frac{k}{2}}. \end{cases}$$

We have

$$f_1[x] = \sum_{y \in N_{S(G)}[x]} f_1(y) = 1,\ for\ all\ x \in V(G),$$

$$f_1[x] = \sum_{y \in N_{S(G)}[x]} f_1(y) = 3,\ for\ all\ x \in B \backslash B_{\frac{k}{2}},$$

$$f_1[x] = \sum_{y \in N_{S(G)}[x]} f_1(y) = 1,\ for\ all\ x \in B_{\frac{k}{2}}.$$

So, f_1 is a signed domination function corresponding to $S(G)$ and $|V^-_{f_1}| = \frac{nk}{4}$. Therefore, $\gamma_s(S(G)) \leq n$. □

Theorem 6. *Let G be a connected $3-$regular graph of order n. Then $\gamma_s(S(G)) = \frac{3}{2}n$.*

Proof. We define $T = \{|\ B \cap V^+_f\ | \ :\ f\ is\ a\ \gamma_s - function\ of\ S(G)\}$. Let f_0 be a γ_s- function of $S(G)$, such that $Min(T) = |\ B \cap V^+_{f_0}\ |$. Also, let $D^+_{f_0} = V \cap V^+_{f_0}$, $D^-_{f_0} = V \cap V^-_{f_0}$, $C^+_{f_0} = B \cap V^+_{f_0}$, $C^-_{f_0} = B \cap V^-_{f_0}$. It is clear that $C^+_{f_0} \neq \emptyset$ and $D^+_{f_0} \neq \emptyset$. We now prove the following claims.

Claim 1) If induced subgraph on $(C^-_{f_0} \cup D^+_{f_0})$ in $S(G)$ is H_1, then H_1 is exactly equal to the union of $\frac{1}{2}|\ D^+_{f_0}\ |$ paths of length two.
Since f_0 is $\gamma_s(S(G))-$ function, if $v_{ij} \in C^-_{f_0}$, then $v_i, v_j \in D^+_{f_0}$. So $deg_{S(G)}(v_{ij}) = deg_{H_1}(v_{ij}) = 2$. If $v_i \in D^+_{f_0}$ and $N_{S(G)}(v_i) \cap C^-_{f_0} \neq \emptyset$, then $|N_{S(G)}(v_i) \cap C^-_{f_0}| = 1$. We show that $N_{S(G)}(v_i) \cap C^-_{f_0} \neq \emptyset$ for every v_i in $D^+_{f_0}$. Assume, to the contrary, that there exist $v_i \in D^+_{f_0}$ such that $|N_{S(G)}(v_i) \cap C^+_{f_0}| = 3$. We consider the following two cases

Case i) Let for all $v_j \in N_G(v_i)$ we have $v_j \in D^+_{f_0}$. Then we define

$$f_1(x) = \begin{cases} f_0(x), & x \in V(S(G)) \backslash \{v_i\}; \\ -1, & x = v_i. \end{cases}$$

It is easy to see that f_1 is a signed domination function, which is a contradiction. Thus there exists $v_j \in D^-_{f_0}$ such that $v_j \in N_G(v_i)$.

Case ii) Let $v_j \in D^-_{f_0}$ such that $v_j \in N_G(v_i)$. We define

$$f_2(x) = \begin{cases} f_0(x), & x \in V(S(G)) \backslash \{v_j, v_{ij}\}; \\ -f_0(x), & x \in \{v_j, v_{ij}\}. \end{cases}$$

It is clear that f_2 is a γ_s- function for $S(G)$ such that $|\ C^+_{f_0}\ | > |\ C^+_{f_2}\ |$. This is a contradiction to the fact that $|\ C^+_{f_0}\ |$ is minimum of T. The claim is proved.

Claim 2) If H_2 is induced subgraph on $(C^+_{f_0} \cup D^+_{f_0})$ in $S(G)$, then H_2 is a cycle of length $2|\ D^+_{f_0}|$.
By Claim 1, $deg_{H_2}(x) = 2$, where $x \in D^+_{f_0}$. Now, if $v_{ij} \in C^+_{f_0}$, then $v_i \in D^+_{f_0}$ or $v_j \in D^+_{f_0}$. Without loss of generality, let $v_i \in D^+_{f_0}$ and $v_j \notin D^+_{f_0}$. Obviously, f_2 in the case (ii) of claim (1) is a γ_s- function of $S(G)$, which is a contradiction. Thus, for every $v_{ij} \in C^+_{f_0}$ we have $deg_{H_2}(v_{ij}) = 2$. Since H_2 is a bipartite graph, it is a cycle of length $2|\ D^+_{f_0}|$.

Claim 3) Induced subgraphs on $(C^-_{f_0} \cup D^-_{f_0})$ and $(C^+_{f_0} \cup D^-_{f_0})$ in $S(G)$ are empty graph, respectively. Obviously, induced subgraph on $(C^-_{f_0} \cup D^-_{f_0})$ is empty graph.
If v_jv_{ij} is an edge in induced subgraph $S(G)[C^+_{f_0} \cup D^-_{f_0}]$, then f_2 in the case (ii) of claim (1) is a γ_s- function of $S(G)$, which is impossible. So the induced subgraph on $(C^+_{f_0} \cup D^-_{f_0})$ is empty graph.

However, we have $2| C^-_{f_0} | = | D^+_{f_0} |$, $| C^+_{f_0} | = | D^+_{f_0} |$ and $| D^-_{f_0} | = 0$. Hence, $\frac{5n}{2} = |V(S(G))| = | C^-_{f_0} | + | C^+_{f_0} | + | D^+_{f_0} | + | D^-_{f_0}| = 5\ell$, where $| V^-_{f_0} | = \ell$. Since $\gamma_s(S(G)) = \frac{5}{2}n - 2\ell$. Hence $\gamma_s(S(G)) = \frac{3}{2}n$. □

Theorem 7. *[7] For any graph G of order n with minimum degree δ and maximum degree Δ, one has*

$$\gamma_s(G) \geq -n + 2max\left\{ \left\lceil \frac{\Delta + 2}{2} \right\rceil , \left\lceil \frac{\delta + 2\gamma(G)}{2} \right\rceil \right\}.$$

Theorem 8. *Let G be a graph of order $n \geq 4$ and size m with $\delta(G) \geq n - 2$. Then $\gamma_s(S(G)) \geq n - m$.*

Proof. Without loss of generality, we may assume that v_1 is adjacent to v_2 in G. Also let, $D' = \{v_3, v_4, \cdots, v_n\} \cup \{v_{12}\}$. Then the vertices v_1 and v_2 are dominated by v_{12}. Also, each vertex v_{ij} with $i \neq j$ is dominated by v_i or v_j for $i \neq 1$ and $j \neq 2$. So, $\gamma(S(G)) \leq n - 1$. On the contrary, we assume that $\gamma(S(G)) = n - 2$ and D is a dominating set for $S(G)$ with cardinality $n - 2$. We define

$$L = \left\{ | D \cap V(G) | \; : \; | D | = n - 2, \; N_{S(G)}[D] = V(S(G)) \right\}.$$

Let D_0 be a dominating set of $S(G)$ such that $|D_0| = n - 2$ and $|D_0 \cap V(G)| = Max(L)$.
If $D_0 \cap B = \emptyset$, then $n = | V(G) | = | D_0 | = n - 2$, which is false.
If $D_0 \cap V(G) = \emptyset$, then $B = D_0$. Since G is a connected graph, so $n - 1 \leq | B | = | D_0 | = n - 2$. Which is a contradiction. Thus $D_0 \cap B \neq \emptyset$ and $D_0 \cap V(G) \neq \emptyset$. Let $v_{ij} \in D_0$, for some i, j. Then we have

i) $\{v_i, v_j\} \cap D_0 = \emptyset$.
If $v_i, v_j \in D_0$, then $D_0 \setminus \{v_{ij}\}$ is a dominating set of $S(G)$, which is not true. Now, without loss of generality, suppose that $v_i \in D_0$ and $v_j \notin D_0$. Then $D_1 = (D_0 \cup \{v_j\}) \setminus \{v_{ij}\}$ is a dominating set and $| D_0 \cap V(G) | < | D_1 \cap V(G) |$, which is contradiction to the fact that $| D_0 |$ is $Max(L)$.

ii) $\{v_{ij}\} = D_0 \cap (N_{S(G)}[v_i] \cup N_{S(G)}[v_j])$.
It is clear that $\{v_{ij}\} \subseteq D_0 \cap (N_{S(G)}[v_i] \cup N_{S(G)}[v_j])$. Suppose that, $r \neq j$ and $\{v_{ir}\} \in D_0 \cap (N_{S(G)}[v_i] \cup N_{S(G)}[v_j])$. Without loss of generality, suppose that v_r is adjacent to v_i in G, where $i < r$. Then $D_2 = (D_0 \setminus \{v_{ir}\}) \cup \{v_r\}$ is a dominating set of $S(G)$. Since $| D_2 \cap V(G) | \in L$, so $| D_2 \cap V(G) | > | D_0 \cap V(G) |$, which is not true.

iii) If $v_t \in N_G(v_i) \cup N_G(v_j)$, then $v_t \in D_0$.
First one let, $v_t \in N_G(v_i) \cap N_G(v_j)$, by (i) and (ii), $\{v_i, v_j, v_{it}, v_{jt}\} \cap D_0 = \emptyset$ and $\{v_i, v_t, v_{it}\} \cap D_0 = N_{S(G)}[v_{it}] \cap D_0 \neq \emptyset$, hence v_t belongs to D_0.
Now, suppose that $v_t \in N_G(v_i) \ominus N_G(v_j)$, we may assume that v_t is in $N_G(v_i)$. Since $\{v_i, v_{it}\} \cap D_0 = \emptyset$ and $N_{S(G)}[v_{it}] \cap D_0 \neq \emptyset$. Thus $v_t \in D_0$.

Now, if $v_{ij} \in D_0$ for some i, j and $deg_G(v_i) = n-1$, then by (iii), for every $v_t \in V(G)$ $(i \neq t \neq j)$ we have $v_t \in D_0$. So $(V(G) \backslash \{v_i, v_j\}) \cup \{v_{ij}\} \subseteq D_0$, which is not true.
Similary, if $v_{ij} \in D_0$ for some i, j and $N_G(v_i) \cup N_G(v_j) = V(G)$, then $(V(G) \backslash \{v_i, v_j\}) \cup \{v_{ij}\} \subseteq D_0$, which is impossible. Hence, we may assume that, there exists a vertex v_t such that $v_t \notin D_0$. Since $\delta(G) \geq n-2$, so $v_s \in N_G(v_i) \cup N_G(v_j)$. By (iii), $v_s \in D_0$, which is a contradiction. Thus $\gamma(S(G)) = n-1$, consider the following expressions:

$$\left\lceil \frac{\Delta(S(G)) + 2}{2} \right\rceil \quad and \quad \left\lceil \frac{\delta(S(G)) + 2\gamma(S(G))}{2} \right\rceil.$$

Since $S(G)$ is the subdivision graph of a connected graph with $\delta(G) \geq n-2$, it follows that $\delta(S(G)) = 2$ and $\Delta(S(G)) \leq n-1$. Using $\gamma(S(G)) = n-1$, we obtain

$$\left\lceil \frac{\delta(S(G)) + 2\gamma(S(G))}{2} \right\rceil = n > \left\lceil \frac{\Delta(S(G)) + 2}{2} \right\rceil.$$

By Theorem 7, it follows that

$$\gamma(S(G)) \geq -|V(S(G))| + 2max\left\{ \left\lceil \frac{\Delta(S(G)) + 2}{2} \right\rceil, \left\lceil \frac{\delta(S(G)) + 2\gamma(S(G))}{2} \right\rceil \right\}.$$

We know that $|V(S(G))| = n + m$. Therefore

$$\gamma(S(G)) \geq -(n+m) + 2n = n - m.$$

□

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