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Ordered right (left) quasi-adequate semigroups

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Abstract. In a previous paper, the author investigated the structure of a right (left) quasi-adequate semigroup with a particular normal medial idempotent. The subject of the current paper is an order analogue. Namely, we provide a structure theorem for a naturally ordered quasi-adequate semigroup $(S, \leq)$ with a maximum idempotent u in which uSu is an adequate subsemigroup of S with the property that the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant. We achieve this result as a combination of those in the one-sided case.

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1 Introduction

The main theme of this paper is the subject of naturally ordered (one-sided) abundant semigroups. The study of such semigroups relies heavily on the basic facts of the relation $\mathcal{L}^*$ or its dual, the relation $\mathcal{R}^*$. As it is stated in [23] and elsewhere later, the relation $\mathcal{L}^*$ is defined on any semigroup S by the rule that any two elements a, b in S are $\mathcal{L}^*$-related if and only if they are related by the Green's relation $\mathcal{L}$ in an over-semigroup of S. Fountain [12] investigated the fundamental properties of the equivalence relations $\mathcal{L}^*$ and $\mathcal{R}^*$, where he initiated the study of abundant semigroups. A semigroup S is *abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent. In contrast to regular semigroups, we distinguish between semigroups in which each $\mathcal{L}^*$-class contains an idempotent called *right abundant semigroups* and their dual *left abundant semigroups*. The basic results of the abundant semigroup theory emerged from corresponding results in the regular semigroup theory. There are now well developed theorems on the structure of classes of (one or two-sided) abundant semigroups. For more information we refer the reader to [3, 5, 7, 22]. As in the regular semigroups, classes of (one or two-sided) abundant semigroups can be recognized by imposing certain conditions on the set of idempotents of a semigroup S.

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To furnish for particular classification of (one or two-sided) abundant semigroups, we introduce a B-*semigroup* to be that in which its set of idempotents is a band B. The B-semigroups are the subject of [9] where certain structure theorems for classes of B-semigroups were established. Any B-semigroup is a *right* (resp. *left*) *quasi-adequate semigroup* if it is right (resp. left) abundant. In this paper we extend some results of ordered orthodox semigroups to the corresponding classes of ordered right (resp. left) quasi-adequate semigroups. Any B-semigroup which is (two-sided) abundant is called *quasi-adequate*. The class of quasi-adequate semigroups contains properly the class of orthodox semigroups. The study of the class of quasi-adequate semigroups is initiated in [8]. A right (resp. left) quasi-adequate semigroup with band E of idempotents is called *right* (*resp. left*) *adequate* if E is a semilattice. The subject of [10] is a one-sided adequate semigroups while [11] is the startup of a study of (two-sided) adequate semigroup theory. We refer the reader to the survey paper [19] for more details on the relationship between most of the cited classes of semigroups.

The basic idea in our discussion is the order relations on the abundant and related semigroups. We introduce this idea by recalling the well-known partial order relation ω on any set E of idempotents of a semigroup S which is defined for any $e, f \in E$ by

$$e \ \omega \ f \text{ if and only if } ef = e = fe. \tag{1}$$

A partial order $\leq$ on the semigroups S is *natural partial order* if it extends ω in the sense that for any $e, f \in E$

$$e \ \omega \ f \quad \text{implies} \quad e \leq f. \tag{2}$$

A specific natural partial order on S is formulated in [24]. That order is not -in general- compatible with the binary operation of S (see [26]). For a naturally partially ordered semigroup $(S, \leq)$, we say the order $(\leq)$ is a *right* (*resp. left*) *natural order* if;

1. The order $(\leq)$ is right (resp. left) compatible with respect to the binary operation of S.

2. For any $a, b \in S; e, f \in E(S)$; the following two conditions are satisfied;

   (i) $a \leq b$ implies $ea \leq eb$ (resp. $af \leq bf$)

   (ii) $a \leq b$ and $e \leq f$ imply $ae \leq bf$ (resp. $ea \leq fb$).

A *natural order* on a semigroup S is a natural partial order which is compatible -on both sides- with respect to the binary operation of S. Obviously, a natural partial order on S is a natural order if and only if it is both right and left natural order. In this case, S is said to be naturally ordered.

A naturally ordered abundant semigroup S is first investigated by Lawson [21]. There are now remarkable structure theorems for some classes of naturally ordered abundant semigroups. A sample of such theorems can be found in [14–18, 25, 27]. Particularly, Guo and Shun investigate the naturally ordered abundant semigroups for which each idempotent has a maximum inverse (see [16] and [17]) where a structure theorem for a class of such abundant semigroups is established. A class of naturally ordered abundant semigroups S which satisfy the regularity condition ($\langle E(S) \rangle$ is regular) and contains a maximum idempotent is studied in [27]. As the maximum idempotent u in a naturally ordered semigroup is medial idempotent [2], so we can utilize the structure obtained in [5] for a class of right (resp. left) quasi-adequate semigroups with normal medial idempotents to extend -in this paper- the structure established in [2] for naturally ordered orthodox semigroups with a maximum idempotent that is a left (resp. right) identity to the right (resp. left) quasi-adequate case.

After reviewing basic ideas in Section 2 concerning abundant and related semigroups, the concept of the abundancy of $\mathcal{L}^*$ (resp. $\mathcal{R}^*$) on a partially ordered right (resp. left) abundant semigroup S is introduced in Section 3 to relate the algebraic properties of S to its order constraints, provided that S contains a maximum idempotent. The subject of Section 4 is the structure of right naturally ordered right quasi-adequate semigroups on which $\mathcal{L}^*$ is abundant and containing a left identity u, where Su is a right adequate subsemigroup. This structure is established via a right quasi-direct product of a specific naturally ordered band and a right naturally ordered right adequate semigroup containing a left identity and in which $\mathcal{L}^*$ is abundant. A dual (the left side case) of the established structure in Section 4 is demonstrated in Section 5. Section 6 is devoted to the structure of a naturally ordered quasi-adequate semigroup on which $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant and contains a maximum idempotent u such that uSu is adequate.

We use the notations and terminologies of Howie's book [20], other undefined terms can be found in Fountain's papers [11] and [12].

2 Right (left) abundant semigroups

We review in this section certain results concerning classes of one or two-sided abundant semigroups. These results will be used frequently in the sequel. The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are the key concepts for studying such semigroups. So we have to start with the basic properties of these relations.

Let S be a semigroup. *The relation* $\mathcal{L}^*$ on S is defined as follows;

$$\mathcal{L}^* = \{(a, b) \in S \times S;\ \text{for any } s, t \in S^1;\ as = at \Leftrightarrow bs = bt\}. \tag{3}$$

The relation $\mathcal{R}^*$ is defined dually. To avoid confusion we may write $\mathcal{L}^*(S)$ and $\mathcal{R}^*(S)$ to specify S in the discussion. This definition coincides with that mentioned in the introduction [11]. It is evident that $\mathcal{L}^*$ is right congruence and $\mathcal{R}^*$ is left congruence. The $\mathcal{L}^*$-class ($\mathcal{R}^*$-class) containing the element a in S will be denoted by $L^*_a(S)$ ($R^*_a(S)$) in case of ambiguity, otherwise we just write L^*_a (R^*_a).

From [11] we have the following characterization;

Lemma 1. *If e is an idempotent of a semigroup S, then for any $a \in S$, the following two statements are equivalent.*

(i) $(e, a) \in \mathcal{L}^*$ $((e, a) \in \mathcal{R}^*)$.

(ii) $ae = a$ $(ea = a)$ *and for any* $s, t \in S^1$;

$$as = at\ (sa = ta)\ \textit{implies}\ es = et\ (se = te). \tag{4}$$

It is clear from the definition of $\mathcal{L}^*$ ($\mathcal{R}^*$) -see [11], [12] or [23]- that on any semigroup S we have the Green's relation $\mathcal{L}$ ($\mathcal{R}$) is included in $\mathcal{L}^*$ ($\mathcal{R}^*$), and for any regular elements a, b in S;

$$(a,b) \in \mathcal{L}^* \ ((a,b) \in \mathcal{R}^*) \text{ implies } (a,b) \in \mathcal{L}((a,b) \in \mathcal{R}). \tag{5}$$

In particular, on any regular semigroup, we have;

$$\mathcal{L}^* = \mathcal{L} \quad \text{and} \quad \mathcal{R}^* = \mathcal{R}. \tag{6}$$

As it has been said in [5] and elsewhere that the two conditions;

(i) each $\mathcal{L}^*$-class contains an idempotent.

(ii) each $\mathcal{R}^*$-class contains an idempotent.

give rise to three classes of semigroups, namely -as we recall- those which satisfy condition $(i)$and are known as *right abundant semigroups*, those which satisfy condition (ii) and are known as *left abundant semigroups* and those which satisfy both conditions (i) and (ii) which are known as *abundant semigroups*. We refer the reader to [12] and [23] for further details.

For any right (left) abundant semigroup S and any element a in S, a typical idempotent in the $\mathcal{L}^*$-class ($\mathcal{R}^*$-class) containing a will be denoted by a^* ($a^\dagger$) where a^* ($a^\dagger$) may not be unique when the idempotents of S do not commute.

Recall that a right (left) abundant semigroup in which the idempotents commute is called a *right (left) adequate semigroup*. A right (left) abundant semigroup in which the set of idempotents is a subsemigroup is called a *right (left) quasi-adequate semigroup*. Accordingly, adequate and quasi-adequate semigroups are defined. From [11] we conclude the following proposition;

Proposition 1. *Let S be a right (left) adequate semigroup. Then for any $a, b \in S$,*

(i) *$a \ \mathcal{L}^* b$ ($a \ \mathcal{R}^* b$) if and only if $a^* = b^*$ ($a^\dagger = b^\dagger$);*

(ii) *$(ab)^* = (a^*b)^*$ ($(ab)^\dagger = (ab^\dagger)^\dagger$);*

(iii) *$(ab)^* \ \omega \ b^*$ ($(ab)^\dagger \ \omega \ a^\dagger$).*

Let S be a right (left) abundant semigroup. A subsemigroup U of S is *right (left) $*$-subsemigroup* if for any $a \in U$ there exists an idempotent e in U such that $a\mathcal{L}^*(S)e$ ($a\mathcal{R}^*(S)e$). A -two-sided- *$*$-subsemigroup* of an abundant semigroup is defined as usual.

The following lemma could be easily concluded from [7];

Lemma 2. *If S is a right (left) abundant semigroup and e is an idempotent in S, then Se (eS) is a right (left) $*$-subsemigroup of S. If S is abundant, then eSe is a $*$-subsemigroup of S.*

The following corollary is a direct consequence of Lemma 2.

Corollary 1. *If S is a right (left) adequate semigroup, then for any idempotent e in S, Se (eS) is a right (left) adequate $*$-subsemigroup of S. In particular, if S is adequate, then eSe is an adequate $*$-subsemigroup of S.*

We consider -particularly- the abundant semigroups S that contain *medial idempotents* u, that is $eue = e$ for any $e \in E(S)$. The following properties of u (see [4, 5, 25] will be in use frequently.

Proposition 2. *Let S be an abundant semigroup that contains a medial idempotent u and let a be an element of S. Then;*

(i) $a\ \mathcal{L}^* ua\ \mathcal{L}^* ua^*$,

(ii) $a\ \mathcal{R}^* au\ \mathcal{R}^* a^\dagger u$,

(iii) $ua^*u\ \mathcal{L}^* uau\ \mathcal{R}^* ua^\dagger u$,

for any choice of a^* and $a^\dagger$.

3 The abundancy of $\mathcal{L}^*$ and $\mathcal{R}^*$

The relation $\mathcal{L}^*$ (resp. $\mathcal{R}^*$) is abundant on a partially ordered right (resp. left) abundant semigroup S if $\mathcal{L}^*$ (resp. $\mathcal{R}^*$) preserves in a way the order of S. To introduce this concept we start with some observations on naturally partially ordered semigroups.

Proposition 3. *Let $(S, \leq)$ be a naturally partially ordered semigroup with set of idempotents E in which for any elements $a, b \in S$;*

$$a \leq b \text{ implies } ea \leq eb \text{ and } ae \leq be \text{ for any } e \in E. \tag{7}$$

If $(S, \leq)$ contains a maximum idempotent u, then;

(i) *u is a medial idempotent.*

(ii) *For any $e, f \in E$;*

  (a) *ue (fu) is the maximum idempotent in L_e (R_f).*

  (b) *$e\mathcal{L}f$ ($e\mathcal{R}f$) if and only if $ue = uf$ ($eu = fu$).*

Proof. Notice that for any $e \in E$; $e \leq u$, we have

$$e \leq u \;\Rightarrow\; ue \leq u \cdot u = u \;\Rightarrow\; ueu \leq u \cdot u = u \;\Rightarrow\; ueue \leq ue, \tag{8}$$

and

$$e \leq u \;\Rightarrow\; e = e \cdot e \leq eu \;\Rightarrow\; ue \leq ueu \;\Rightarrow\; ue = ue \cdot e \leq ueue. \tag{9}$$

That is;

$$ueue \leq ue \leq ueue. \tag{10}$$

Therefore; $ue = ueue$ and thus $eue = eueue$ so that $eue \in E$. Since $e \cdot eue = eue = eue \cdot e$, then $eue\ \omega\ e$ and $eue \leq e$ ($\leq$ is natural). But;

$$e \leq u \;\Rightarrow\; e \leq eu \;\Rightarrow\; e \leq eue. \tag{11}$$

Hence $eue = e$ and (i) holds.

Consequently, $ue \in E$ and $e\mathcal{L}ue$. If $e, f \in E$ such that $e\mathcal{L}f$, then as $f \leq u$ we get $f = fe \leq ue$. Similarly, $fu \in E$ and $f\mathcal{R}fu$ so that $e\mathcal{R}f$ implies $e \leq fu$ since $e \leq u$ and thus $fe \leq fu$. Therefore, part (iia) holds and hence part (iib) follows immediately. □

Corollary 2. *Let S be a right (resp. left) abundant semigroup with set of idempotents E such that $(S, \leq)$ is naturally partially ordered in which for any elements $a, b \in S$;*

$$a \leq b \text{ implies } ea \leq eb \text{ and } ae \leq be \text{ for any } e \in E. \tag{12}$$

If $(S, \leq)$ contains a maximum idempotent u. Then for any elements $a, b \in S$; the following statements are equivalent;

- (i) *$a \leq b$ implies $a^* \leq b^*$ (resp. $a^\dagger \leq b^\dagger$) for some a^*, b^* (resp. $a^\dagger, b^\dagger$).*
- (ii) *$a \leq b$ implies $ua^* \leq ub^*$ (resp. $a^\dagger u \leq b^\dagger u$) for any a^*, b^* (resp. $a^\dagger, b^\dagger$).*

Proof. We prove a half of the corollary, the other will follow similarly. If (i) holds, two idempotents e, f in S can be chosen so that;

$$e\mathcal{L}^*a \quad , \quad f\mathcal{L}^*b \quad \text{and} \quad e \leq f. \tag{13}$$

Then $ue \leq uf$. It follows from Proposition 3(ii) that;

$$ue = ua^* \quad \text{and} \quad uf = ub^* \quad \text{for any } a^*, b^*, \tag{14}$$

and (ii) holds by Proposition 2(i).

Since u is a maximum idempotent, then by Proposition 3(i) u is medial and then $ua^*\mathcal{L}a^*$ so that $ua^*\mathcal{L}^*a$ and (i) follows from (ii) is evident. □

Let $(S, \leq)$ be a partially ordered right (resp. left) abundant semigroup. As in [4], the relation $\mathcal{L}^*$ (resp. $\mathcal{R}^*$) is said to be *abundant* on $(S, \leq)$ if for any $a, b \in S$;

$$a \leq b \text{ implies } a^* \leq b^* \text{ (resp. } a^\dagger \leq b^\dagger) \text{ for some } a^*, b^* \text{ (resp. } a^\dagger, b^\dagger). \tag{15}$$

When S is regular, then $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{R} = \mathcal{R}^*$ on S so that $\mathcal{L}^*$ (resp. $\mathcal{R}^*$) is abundant coincides with $\mathcal{L}$ (resp. $\mathcal{R}$) is regular in the sense of [].

Corollary 2 gives an alternative statement for the abundancy of $\mathcal{L}^*$ (resp. $\mathcal{R}^*$) on any naturally ordered right (resp. left) abundant semigroup containing a maximum idempotent. For the rest of the section, let $(S, \leq)$ be a naturally ordered quasi-adequate semigroup containing a maximum idempotent u. As in [5] we have;

Proposition 4. *The Cartesian product $Su \times uS$ with a binary operation defined by;*

$$(xu, ua)(yu, ub) = (xua^*yu, uayub^*); \ x, a, y, b \in S; \tag{16}$$

is a semigroup.

Proof. Since for any $a, b \in S$;

$$\begin{aligned} ua = ub &\Rightarrow ua^*\mathcal{L}ub^* && \text{(Proposition 3}(i)\text{ and Proposition 2)} \\ &\Rightarrow ua^* = ub^* && \text{(Proposition 3}(ii)), \end{aligned} \tag{17}$$

then clearly the binary operation is well-defined. To verify the associativity, let x, a, y, b, z and c be elements in S. Since $ayu\mathcal{L}^*a^*yu$ ($a\mathcal{L}^*a^*$ and $\mathcal{L}^*$ is right congruence), then $ayub^*\mathcal{L}^*(a^*yu)^*b^*$, $(a^*yu)^*b^*$ is an idempotent (S is quasi-adequate) and $(ayub^*)^*\mathcal{L}(a^*yu)^*b^*$ so that $u(ayub^*)^* = u(a^*yu)^*b^*$ (Proposition 3(ii)).

Hence;

$$xua^*yu(ayub^*)^*zu = xua^*yu(a^*yu)^*b^*zu = xua^*yub^*zu \qquad \text{(Lemma 1)}, \tag{18}$$

and the first components of the two triple products;

$$[(xu, ua)(yu, ub)](zu, uc) = (xua^*yu, uayub^*)(zu, uc) = (xua^*yu(ayub^*)^*zu, u(ayub^*)zuc^*), \tag{19}$$

and

$$(xu, ua)[(yu, ub)(zu, uc)] = (xu, ua)(yub^*zu, ubzuc^*) = (xua^*yub^*zu, uayub^*zu(bzuc^*)^*) \tag{20}$$

coincide.

By a similar argument, we can see that; $bzuc^*\ \mathcal{L}^*(b^*zu)^*c^*$ and thus; $u(bzuc^*)^* = u(b^*zu)c^*$. Therefore;

$$uayub^*zu(bzuc^*)^* = uayub^*zu(b^*zu)^*c^* = uayub^*zuc^*. \tag{21}$$

Hence, the second components coincide and the associativity holds. □

Recall from [5] that when $Su \times uS$ is equipped with the binary operation described in Proposition 4, it is called $\mathcal{L}^*$-*overlap product of* Su *and* uS. The dual is *the* $\mathcal{R}^*$-*overlap product of* Su *and* uS whose binary operation is defined by;

$$(xu, ua)(yu, ub) = (x^\dagger uayu, uay^\dagger ub); \quad x, a, y, b \in S. \tag{22}$$

From the context, it should not raise any confusion by indicating the Cartesian order on $Su \times uS$ by $\leq$ as that for the order of S.

Proposition 5. *The following two statements are equivalent;*

(i) *The* $\mathcal{L}^*$ *(resp.* $\mathcal{R}^*$*)-overlap product of* Su *and* uS *is an ordered semigroup with respect to the Cartesian order* $\leq$.

(ii) $\mathcal{L}^*$ *(resp.* $\mathcal{R}^*$*) is abundant on* $(S, \leq)$.

Proof. Since a half of the proposition is a dual of the other, it suffices to prove the $\mathcal{L}^*$ case. If (i) holds, then for any $a, b \in S$ such that $a \leq b$ we get $(u, ua) \leq (u, ub)$ and;

$$(u, ua^*) = (u, u)(u, ua) \leq (u, u)(u, ub) = (u, ub^*), \tag{23}$$

which implies $ua^* \leq ub^*$ and (ii) holds (see Corollary 2). Conversely, it is readily the Cartesian order on $Su \times uS$ is partial order. Let $(xu, ua), (yu, ub)$ and (zu, uc) be elements in $Su \times uS$ such that $(xu, ua) \leq (yu, ub)$. Then $xu \leq yu$ and $ua \leq ub$. Since $ua\mathcal{L}^*ua^*$ and $ub\mathcal{L}^*ub^*$ (Proposition 3(i) and Proposition 2(i)) so then by (ii) -see Corollary 2- $ua^* \leq ub^*$ and from the compatibility of the order on S, we obtain;

$$(xua^*zu, uazuc^*) \leq (yub^*zu, ubzuc^*). \tag{24}$$

Therefore;

$$(xu, ua)(zu, uc) \leq (yu, ub)(zu, uc). \tag{25}$$

Also, by (ii);

$$(zuc^*xu, ucxua^*) \leq (zuc^*yu, ucyub^*), \tag{26}$$

and

$$(zu, uc)(xu, ua) \leq (zu, uc)(uy, ub). \tag{27}$$

Hence $Su \times uS$ is an ordered semigroup with respect to the Cartesian order. □

Consider the subset of $Su \times uS$ given by;

$$Su| \times |uS = \{(xu, ux) : x \in S\}. \tag{28}$$

It follows as in [5] that $Su| \times |uS$ is a subsemigroup of $\mathcal{L}^*$ $(\mathcal{R}^*)$-overlap product of Su and uS with a binary operation defined by;

$$(xu, ux)(yu, uy) = (xyu, uxy); \quad x, y \in S. \tag{29}$$

This can be reduced from either $\mathcal{L}^*$- or $\mathcal{R}^*$-overlap product of Su and uS. Clearly we can obtain from Proposition 5 the following result;

Proposition 6. *If $\mathcal{L}^*$ (resp. $\mathcal{R}^*$) is abundant on S, then $Su| \times |uS$ is an order subsemigroup of the $\mathcal{L}^*$- (resp. $\mathcal{R}^*$-) overlap product of Su and uS.*

This proposition is a direct generalisation of [1, Theorem 4.5]. We can see also that [1, Theorem 4.6] is a special case of the following proposition.

Proposition 7. *If $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant on S, then the map $\theta : S \to Su| \times |uS$ defined by $x\theta = (xu, ux)$; $x \in S$ is an order isomorphism.*

Proof. The map θ is an isomorphism follows as in the proof of [5, Proposition 3.6]. Since for any $x, y \in S$;

$$\begin{aligned} x \leq y &\Rightarrow xu \leq yu,\ ux \leq uy \\ &\Rightarrow (xu, ux) \leq (yu, uy) \\ &\Rightarrow x\theta \leq y\theta, \end{aligned} \tag{30}$$

and

$$\begin{aligned} x\theta \leq y\theta &\Rightarrow xu \leq yu \quad &&\text{and} \quad ux \leq uy \\ &\Rightarrow x^\dagger u \leq y^\dagger u \quad &&\text{and} \quad ux^* \leq uy^* \qquad (\mathcal{L}^* \text{ and } \mathcal{R}^* \text{ are abundant on } S). \end{aligned} \tag{31}$$

It follows that;

$$x = x^\dagger uxux^* \leq y^\dagger uyuy^* = y. \tag{32}$$

Therefore; $x \leq y$ if and only if $x\theta \leq y\theta$. Hence the result holds. □

4 Ordered right quasi-adequate semigroups

Let $(S, \leq)$ be a naturally partially ordered right quasi-adequate semigroup whose band of idempotents is B where $\leq$ is a right natural order as defined in Section 1. Moreover, consider the following three conditions;

1. S possesses a left identity u.
2. Bu is a semilattice.
3. $\mathcal{L}^*$ is abundant on $(S, \leq)$.

The objective is to establish a structure theorem for the right naturally ordered right quasi-adequate semigroup $(S, \leq)$ that satisfies the three conditions: $(1), (2)$ and (3). The cornerstone for such structure is the following proposition, which can be compared with [1, Theorem 4.7].

Proposition 8. *Let B be a naturally ordered band with a left identity u such that Bu is a semilattice. Then B is the semilattice Y of rectangular bands B_α $(\alpha \in Y)$ where;*

(i) *The idempotent u is the maximum element of Bu.*

(ii) *B_α $(\alpha \in Y)$ are right zero semigroups.*

(iii) *Bu consists of representative elements of B_α $(\alpha \in Y)$.*

(iv) *The semilattice Y is isomorphic to Bu.*

Proof. Consider $\underset{B}{\leq}$ to be the given natural order on B. Since for any $e \in B$; $eu \cdot u = eu = u \cdot eu$, then $eu \ \omega \ u$ so that $eu \underset{B}{\leq} u$ and u is the maximum element of Bu. It is well known (see [20, Chapter IV]) that B is a semilattice Y of rectangular bands B_α's $(\alpha \in Y)$ and for any $\alpha \in Y$, if $e \in B_\alpha$ then also $eu \in B_\alpha$. Let $e, f \in B_\alpha$ for some $\alpha \in Y$. Then $eu \underset{B}{\leq} u$ and $euf \underset{B}{\leq} uf$ so that $ef \underset{B}{\leq} f, efe \underset{B}{\leq} fe$ and $e \underset{B}{\leq} fe$ (e, f are in the same rectangular band). Also $fu \underset{B}{\leq} u$ and $fe \underset{B}{\leq} e$. Hence $fe = e$ and B_α is a right zero semigroup. Since $eue = e$ for any $e \in B$, then $eu\mathcal{R}e$. For any $e, f \in B_\alpha$ $(\alpha \in Y)$, $eu\mathcal{R}fu$. As Bu is a semilattice, then $eu = fu$. This is the idempotent representing B_α in Bu. Finally, as $\alpha \mapsto eu$ $(e \in B_\alpha)$ describes an isomorphism from Y onto Bu, hence the verification of the statements of the proposition is completed. □

Considering the hypothesis and notations of Proposition 8 and recalling that Y and Bu are isomorphic semilattices, so we may identify Bu by Y (for notation simplification). Notice that $B = \bigcup_{\alpha \in Y} B_\alpha$ and for any $\alpha \in Y$ there exists $eu \in B_\alpha$ for some $e \in B$. Moreover, $|B_\alpha \cap Bu| = 1$. Hence Y is a skeleton of B (see [6] or [13]). In fact, for any α in Y -with the previous convention- can be identified by eu where $e \in B_\alpha$ and thus B_α coincides with the R-class R_{eu} in B.

To proceed towards establishing a right naturally ordered right quasi-adequate semigroup Q on which $\mathcal{L}^*$ is abundant and possesses a left identity $\bar{u}$ such that $Q\bar{u}$ is a right adequate subsemigroup, let $(B, \underset{B}{\leq})$ be a naturally ordered band with a left identity u such that Bu is a semilattice.

Let H be the set of all mappings of B into Bu. Clearly, H is a semigroup under the composition of mappings. It is a routine matter to check that the relation $\underset{H}{\leq}$ defined for any two mappings ϕ, ψ in H by the rule;

$$\phi \underset{H}{\leq} \psi \text{ if and only if } e\phi \underset{B}{\leq} e\psi \text{ for any } e \in B, \tag{33}$$

is a partial order relation on H.

Let $(T, \underset{T}{\leq})$ be a right naturally ordered right adequate semigroup whose semilattice of idempotents is (isomorphic to and identified with) Bu and u be the left identity of T, provided that $\underset{T}{\leq}|_{Bu} = \underset{B}{\leq}|_{Bu}$ $(= \underset{Bu}{\leq})$ and $\mathcal{L}^*$ is abundant on $(T, \underset{T}{\leq})$.

Inspired by [1, Theorem 4.8], we have:

Proposition 9. *There is a unique homomorphism* $\zeta : T \to H$ *described by* $y\zeta = \zeta_y$ $(y \in T)$ *that satisfies condition A, where;*

(A) *For any* $y \in T, f \in B$; $f\zeta_y = (fy)^*$; $(fy)^* \in Bu$.

Moreover; ζ *is isotone.*

Proof. Let $\theta : B \to Bu$ be defined for any $f \in B$ by $f\theta = fu$. Clearly θ is an isotone homomorphism, and for any $f \in B$, $f \in R_\alpha$ for some $\alpha \in Y$ (in notation of Proposition 8) and $f\theta$ is the representative of R_α in Bu so then $fu \in T$ and $fuy = fy$ for any $y \in T$ (u is the left identity of T). Therefore, for any $f, h \in B$;

$$fu = hu \;\Rightarrow\; fuy = huy \;\Rightarrow\; (fy)^* = (hy)^* \quad (y \in T). \tag{34}$$

Hence, for any $y \in T$;

$$\Pi_y : Bu \to Bu \text{ defined by } (fu)\Pi_y = (fy)^*;\; (fu \in Bu) \text{ is a map.} \tag{35}$$

Put $\zeta_y = \theta\Pi_y$ $(y \in T)$. It is readily $\zeta_y \in H$. For any $f \in B$; $f\zeta_y = (fy)^*$, where $(fy)^* \in Bu$ and ζ_y $(y \in T)$ satisfies condition (A).

Since for any $x, y \in T, f \in B$;

$$f\zeta_x\zeta_y = (f\theta\Pi_x)\theta\Pi_y = (fx)^*\theta\Pi_y = ((fx)^*y)^* = (fxy)^* = f\zeta_{xy}, \tag{36}$$

then $\zeta : T \to H$ described by $x\zeta = \zeta_x$ $(x \in T)$ is a homomorphism.

The uniqueness of ζ follows from the fact that if $\phi : T \to H$ is a homomorphism described by $x\phi = \phi_x$ $(x \in T)$ where ϕ_x satisfies condition (A), then for any $y \in T, e \in B$;

$$e\phi_y = (ey)^* = e\zeta_y. \tag{37}$$

Hence $\phi = \zeta$.

Now let $x, y \in T$ and $e \in B$. Then;

$$\begin{aligned} x \underset{T}{\leq} y \Rightarrow eux \underset{T}{\leq} euy \qquad & (T \text{ is right naturally ordered}) \\ \Rightarrow (ex)^* \underset{Bu}{\leq} (ey)^* \qquad & (\mathcal{L}^* \text{ is abundant on } (T, \underset{T}{\leq}), \underset{T}{\leq}|_B = \underset{Bu}{\leq}) \end{aligned} \tag{38}$$

and;

$$e\zeta_x = (ex)^* \underset{B}{\leq} (ey)^* = e\zeta_y. \tag{39}$$

Therefore; $\zeta_x \underset{H}{\leq} \zeta_y$ and ζ is isotone.

Hence the result holds. □

Put

$$Q = Q(T, B) = \{(x, f) \in T \times B; f \in R_{x^*}\}. \tag{40}$$

Endow Q with law of composition;

$$(x, f)(y, h) = (xy, f\zeta_y h); \qquad (x, f), (y, h) \in Q, \tag{41}$$

where ζ_y $(y \in T)$ as defined in the proof of Proposition 9. Notice that

$$f\zeta_y = (fy)^* = (xy)^*. \tag{42}$$

For any $x, y \in T$; $(xy)^* y^* = (xy)^*$ (Proposition 1(iii)). Notice also that for any (x, f) and (y, h) in Q;

$$(xy)^* h \in R_{(xy)^*} R_{y^*} \subseteq R_{(xy)^* y^*} = R_{(xy)^*}. \tag{43}$$

Hence the binary operation of Q is well-defined. Directly from Proposition 1(iii), it is easy to verify that Q is a semigroup. We call -as in [2] and [5]- $Q = Q(T, B)$ is the right quasi-direct product of T and B. The following sequence of lemmas provide more information on Q.

Lemma 3. *The set of idempotents of Q is;*

$$E(Q) = \{(x, f) \in Q; x \in Bu\}, \tag{44}$$

which forms a band isomorphic to B.

Proof. The same argument as in [5, Lemma 4.2]. □

Similar to [5, Lemma 4.3], we have;

Lemma 4. *The semigroup Q is right quasi-adequate.*

Proof. Let (x, f) be an element in Q. Consider the idempotent (x^*, f) in Q. Notice that;

$$fu = x^* u, \quad f\zeta_{x^*} = (fux^*)^* = (x^* x^*)^* = x^* \text{ and } f\zeta_{x^*} f = x^* f = f \quad (f \in R_{x^*}). \tag{45}$$

Therefore;

$$(x, f)(x^*, f) = (xx^*, f\zeta_{x^*} f) = (x, f). \tag{46}$$

For any elements $(y, h), (z, j)$ in Q we have;

$$\begin{aligned}(x, f)(y, h) = (x, f)(z, j) &\Rightarrow (xy, f\zeta_y h) = (xz, f\zeta_z j)\\ &\Rightarrow (x^* y, f\zeta_y h) = (x^* z, f\zeta_z j)\\ &\Rightarrow (x^*, f)(y, h) = (x^*, f)(z, j),\end{aligned} \tag{47}$$

so that $(x^*, f)\mathcal{L}^*(x, f)$ -Lemma 1- and every $\mathcal{L}^*$-class in Q contains an idempotent. Now the result follows by Lemma 3. □

Lemma 5. *The Cartesian order $\leq$ on Q is right natural.*

Proof. Recall that $(T, \underset{T}{\leq})$ is a right naturally ordered right adequate semigroup and $(B, \underset{B}{\leq})$ is a naturally ordered band. The Cartesian order $\leq$ on Q; which is defined for any (x, f) and (y, h) in Q by;

$$(x, f) \leq (y, h) \text{ if and only if } x \underset{T}{\leq} y \text{ and } f \underset{B}{\leq} h, \tag{48}$$

clearly is a natural partial order. In fact it is right compatible, for, let $(x, f), (y, h)$ and (z, j) be elements in Q such that;

$$(x, f) \leq (y, h) \text{ so then } x \underset{T}{\leq} y \text{ and } f \underset{B}{\leq} h. \tag{49}$$

Since $\underset{T}{\leq}$ is right compatible -on T- then $xz \underset{T}{\leq} yz$. Also;

$$f \underset{B}{\leq} h \text{ implies } fu \underset{Bu}{\leq} hu \text{ and } fz \underset{T}{\leq} hz. \tag{50}$$

As $\mathcal{L}^*$ is abundant on T, it follows that $(fz)^* \underset{Bu}{\leq} (hz)^*$. Therefore, $f\zeta_z \underset{B}{\leq} h\zeta_z$ and $f\zeta_z j \underset{B}{\leq} h\zeta_z j$. Hence, we obtain;

$$(x, f)(z, j) = (xz, f\zeta_z j) \leq (yz, h\zeta_z j) = (y, h)(z, j), \tag{51}$$

and $\leq$ is right compatible on Q.

Now let $(x, f), (y, h)$ be two elements and $(z, i), (w, j)$ be two idempotents in Q -$z, w \in Bu$ (Lemma 3)- such that $(x, f) \leq (y, h)$ and $(z, i) \leq (w, j)$, that is;

$$x \underset{T}{\leq} y, \ f \underset{B}{\leq} h, \ z \underset{Bu}{\leq} w, \ i \underset{B}{\leq} j. \tag{52}$$

Then,

$$xz \underset{T}{\leq} yw \qquad (T \text{ is right naturally ordered}), \tag{53}$$

and

$$(xz)^* \underset{Bu}{\leq} (yw)^* \qquad (\mathcal{L}^* \text{ is abundant on } (T, \underset{T}{\leq})). \tag{54}$$

So we have; $(wz)^* i \leq (yw)^* j$. Therefore;

$$(x, f)(z, i) = (xz, (xz)^* i) \leq (yw, (yw)^* j) = (y, h)(w, j). \tag{55}$$

Moreover;

$$\begin{aligned} x \underset{T}{\leq} y \Rightarrow zx \underset{T}{\leq} zy & \qquad (T \text{ is right naturally ordered}, z \in Bu) \\ \Rightarrow (xz)^* \underset{Bu}{\leq} (xy)^* & \qquad (\mathcal{L}^* \text{ is abundant on } (T, \underset{T}{\leq})) \\ \Rightarrow (zx)^* f \underset{B}{\leq} (zy)^* h & \qquad (\underset{B}{\leq} \text{ is compatible}), \end{aligned} \tag{56}$$

and conclude;

$$(z, i)(x, f) = (xz, (xz)^* f) \leq (zy, (zy)^* h) = (z, i)(y, h). \tag{57}$$

Hence the Cartesian order $\leq$ on Q is right natural. $\square$

Lemma 6.

(*i*) *The idempotent* $\bar{u} = (u, u)$ *is a left identity of* Q.

(*ii*) $Q\bar{u}$ *is a right adequate semigroup orderly isomorphic to* $(T, \underset{T}{\leq})$.

(*iii*) *The relation* $\mathcal{L}^*$ *is abundant on* $(Q, \leq)$.

Proof. (*i*) Since for any $(x, f) \in Q$;

$$(u, u)(x, f) = (ux, u\zeta_x f) = (x, x^* f) = (x, f) \qquad (f \in R_{x^*}), \tag{58}$$

then $\bar{u} = (u, u)$ is a left identity of Q.
(*ii*) Notice that for any $(x, f) \in Q$;

$$(x, f)(u, u) = (xu, f\zeta_u u) = (x, (fu)^* u) = (x, x^* u) = (x, x^*), \tag{59}$$

and;

$$Q\bar{u} = \{(x, x^*) : x \in T\}. \tag{60}$$

Define $\psi : Q\bar{u} \to T$ by $(x, x^*)\psi = x$; $(x, x*) \in Q\bar{u}$. It is readily ψ is a bijection. For any $x, y \in T$;

$$(x, x^*)(y, y^*) = (xy, x^*\zeta_y y^*) = (xy, (x^* y)^* y^*) = (xy, (xy)^*), \tag{61}$$

so clearly ψ is a homomorphism. Therefore, ψ is an isomorphism.

For any $x, y \in T$ notice that;

$$(x, x^*) \leq (y, y^*) \Rightarrow x \underset{T}{\leq} y. \tag{62}$$

Also;

$$\begin{aligned} x \underset{T}{\leq} y &\Rightarrow x \underset{T}{\leq} y \text{ and } x^* \underset{Bu}{\leq} y^* \quad (\mathcal{L}^* \text{ is abundant on } (T, \underset{T}{\leq})). \\ &\Rightarrow (x, x^*) \leq (y, y^*). \end{aligned} \tag{63}$$

Hence ψ is an isotone isomorphism and $Q\bar{u}$ is orderly isomorphic to T. In particular, $Q\bar{u}$ is a right adequate semigroup.
(*iii*) Recall that for any $(x, f), (y, h)$ in Q;

$$\begin{aligned} (x, f) \leq (y, h) &\Rightarrow x \underset{T}{\leq} y,\ f \underset{B}{\leq} h \\ &\Rightarrow x^* \underset{T}{\leq} y^*,\ f \underset{B}{\leq} h \\ &\Rightarrow (x^*, f) \leq (y^*, h), \end{aligned} \tag{64}$$

where (x^*, f) and (y^*, h) are idempotents in $Q, \mathcal{L}^*$ -related to (x, f) and (y, h) respectively (see proof of Lemma 4). Hence $\mathcal{L}^*$ is abundant on Q. □

Summing up we have the necessary part of the following theorem;

Theorem 1. *Let $(B, \underset{B}{\leq})$ be a naturally ordered band with a left identity u in which Bu is a semilattice. Let $(T, \underset{T}{\leq})$ be a right naturally ordered right adequate semigroup whose semilattice of idempotents is (isomorphic to and identified with) Bu, the relation $\mathcal{L}^*$ is abundant on $(T, \underset{T}{\leq})$ and u is a left identity of T where $\underset{T}{\leq} |_{Bu} = \underset{B}{\leq} |_{Bu}$ $(= \underset{Bu}{\leq})$. Then the right quasi-direct product $Q = Q(T, B)$ of T and B is a right naturally ordered right quasi-adequate semigroup on which $\mathcal{L}^*$ is abundant and contains a left identity $\bar{u}$ such that $Q\bar{u}$ is a right adequate subsemigroup of Q (orderly isomorphic to T).*

Conversely, any right naturally ordered right quasi-adequate semigroup $(S, \leq)$ on which $\mathcal{L}^*$ is abundant and contains a left identity u such that Su is a right adequate subsemigroup of S is orderly isomorphic to a semigroup so constructed.

To demonstrate the sufficient part of Theorem 1, let $(S, \underset{S}{\leq})$ be a right naturally ordered right quasi-adequate semigroup on which $\mathcal{L}^*$ is abundant and contains a left identity u such that Su is a right adequate subsemigroup of S. Let B be the band of idempotents of S. The relation $\underset{S}{\leq} |_B$ $(= \underset{B}{\leq})$ is a natural order on B where $E(Su) = Bu$, so then Bu is a semilattice. The relation $\underset{Su}{\leq}$ $(= \underset{S}{\leq} |_{Su})$ is a right natural order on Su and $\underset{Su}{\leq} |_{Bu} = \underset{S}{\leq} |_{Bu}$ $(= \underset{Bu}{\leq})$.

For any $e \in Bu$, put $R_e = \{x \in B; x\mathcal{R}e\}$. R_e is a right zero semigroup, $xu = e$ for any $x \in R_e$. Moreover, for any $x \in R_e, y \in R_f$ where $e, f \in Bu$, it follows that;

$$\begin{aligned} &xyef = xyfe = xfe = xef = ef, \\ &efxy = fexy = fxy = fxuy = xufy = xy. \end{aligned} \tag{65}$$

Therefore; $xy \in R_{ef}$ and $R_e R_f \subseteq R_{ef}$. Hence B is a semilattice Bu of the right zero semigroups R_e's. Consider the right quasi-direct product $Q = Q(Su, B)$ of Su and B. Define $\alpha : Q \to S$ by $(xu, f)\alpha = xf$, $((xu, f) \in Q)$. Clearly, α is well-defined.

For any $y \in S$, $y^*\mathcal{L}^*y$, $y^*u \in Bu$, $y^*u\mathcal{R}y^*$ and $y^*u = (yu)^*$, $y^* \in R_{(yu)^*}$. Therefore $(yu, y^*) \in Q$ and $(yu, y^*)\alpha = yuy^* = y$ so then α is surjective.

Let (xu, f) and (yu, h) be elements in Q such that $xf = yh$; $f, h \in B$, $f \in R_{(xu)^*}$, $h \in R_{(yu)^*}$. Then;

$$f = x^*f\mathcal{L}^*xf = yh\mathcal{L}^*y^*h = h, \tag{66}$$

and $fu\mathcal{L}hu$. As $fu, hu \in Bu$ -Bu is a semilattice- then $fu = hu$. Since $f\mathcal{R}fu, hu\mathcal{R}h$, then $f\mathcal{R}h$. But also $f\mathcal{L}h$. Hence $f\mathcal{H}h$ and $f = h$. It follows that $x^*u = y^*u$. But also;

$$xf = yh \Rightarrow xfx^*u = yhy^*u \Rightarrow xx^*u = yy^*u \Rightarrow xu = yu. \tag{67}$$

Then -in this case- $(xu, f) = (yu, h)$ so that α is injective.

In fact it is also a homomorphism, for, it (xu, f) and (yu, h) are elements in Q, then;

$$\begin{aligned}[(xu,f)(yu,h)]\,\alpha &= (xyu,(fyu)^*h)\alpha = xyu(fyu)^*h \\ &= xyu(xyu)^*h \\ &= xyuh \\ &= xx^*yh \\ &= xfx^*yh \\ &= xfux^*uyh \\ &= xx^*ufuyh \\ &= xfyh \\ &= (xu,f)\alpha(yu,h)\alpha. \end{aligned} \tag{68}$$

Hence α is an isomorphism.

Also, referring to the Cartesian order $\leq$ on Q;

$$\begin{aligned}(xu,f) \leq (yu,h) &\Rightarrow xu \underset{Su}{\leq} yu, f \underset{B}{\leq} h \\ &\Rightarrow xu \underset{S}{\leq} yu, f \underset{B}{\leq} h \\ &\Rightarrow xuf \underset{S}{\leq} yuh \qquad\qquad (\underset{S}{\leq} \text{ is right natural}) \\ &\Rightarrow (xu,f)\alpha \underset{S}{\leq} (yu,h)\alpha. \end{aligned} \tag{69}$$

On the other hand; let $(xu, f)\alpha \underset{S}{\leq} (yu, h)\alpha$. Recall that $x^*\mathcal{L}^*x$ so then $x^*f\mathcal{L}^*xf$ and

$$x^*f = x^*uf = f \qquad (x^*u\mathcal{R}f). \tag{70}$$

Therefore, $f\mathcal{L}^*xf$. Likewise $h\mathcal{L}^*yh$ so that;

$$\begin{aligned}(xu,f)\alpha \underset{S}{\leq} (yu,h)\alpha &\Rightarrow xf \underset{S}{\leq} yh \\ &\Rightarrow uf \underset{B}{\leq} uh \qquad (\mathcal{L}^* \text{ is abundant on } (S,\underset{S}{\leq}) \text{ and Corollary 2}) \\ &\Rightarrow f \underset{B}{\leq} h \qquad (u \text{ is left identity}). \end{aligned} \tag{71}$$

But $f \underset{B}{\leq} h$ implies $fu \underset{Bu}{\leq} hu$ and $x^*u \underset{Bu}{\leq} y^*u$, for, $x^*u\mathcal{R}f, y^*u\mathcal{R}h$ so then by Proposition 3;

$$x^*u = fu, y^*u = hu. \tag{72}$$

Since $(S, \underset{S}{\leq})$ is right natural, then;

$$\begin{aligned}(xu,f)\alpha \underset{S}{\leq} (yu,h)\alpha &\Rightarrow xfx^*u \underset{S}{\leq} yhy^*u \\ &\Rightarrow xx^*u \underset{S}{\leq} yy^*u \qquad (x^*u\mathcal{R}f, y^*u(\mathcal{R}h)) \\ &\Rightarrow xu \underset{S}{\leq} yu. \end{aligned} \tag{73}$$

In conclusion, we have;

$$\begin{aligned}(xu,f)\alpha \underset{S}{\leq} (yu,h)\alpha &\Rightarrow xu \underset{S}{\leq} yu \text{ and } f \underset{B}{\leq} h \\ &\Rightarrow (xu,f) \leq (yu,h).\end{aligned} \tag{74}$$

Hence α is an isotone isomorphism so then S and Q are isomorphic as ordered semigroups. □

The construction presented in Theorem 4.7 is established in corresponding to [1, Theorem 4.9] in the regular case.

5 Ordered left quasi-adequate semigroups

This section is a dual of Section 4 and it is for adjustment the notations involved in the discussion. The arguments presented here are similar to that demonstrated in Section 4. However, statements may be repeated for the sake of completeness.

Let $(S, \underset{S}{\leq})$ be a left naturally ordered left quasi-adequate semigroup on which $\mathcal{R}^*$ is abundant and whose band of idempotents B contains a right identity u (for S) where uS is a left adequate subsemigroup of S. The objective is to clarify the structure of the semigroup S. For this structure -as in the right side case- the following proposition is essential where its proof is similar to that of Proposition 8.

Proposition 10. *Let $(E, \underset{E}{\leq})$ be a naturally ordered band with a right identity u such that uE is a semilattice. Then E is the semilattice $\mathcal{Y}$ of rectangular bands E_α's ($\alpha \in \mathcal{Y}$) where;*

- (i) *The idempotent u is the maximum element of uE.*
- (ii) *E_α's ($\alpha \in \mathcal{Y}$) are left zero semigroups.*
- (iii) *uE consists of representative elements of E_α's ($\alpha \in \mathcal{Y}$).*
- (iv) *The semilattice $\mathcal{Y}$ is isomorphic to uE.*

Retain the hypothesis and notations of Proposition 10. The semilattices $\mathcal{Y}$ and uE are isomorphic, so we may identify uE by $\mathcal{Y}$ and we can see immediately that $\mathcal{Y}$ is a skeleton of E. As in Section 4, any $\alpha \in \mathcal{Y}$ can be identified by ue for any $e \in E_\alpha$ and thus E_α coincides with the $\mathcal{L}$-class L_{ue} in E. The set of representative elements of E_α's ($\alpha \in \mathcal{Y}$) is uE.

Let $(N, \underset{N}{\leq})$ be a left naturally ordered left adequate semigroup whose semilattice of idempotents is (isomorphic to and identified with) uE and u be the right identity of N provided that $\underset{N}{\leq}|_{uE} = \underset{E}{\leq}|_{uE}$ $(= \underset{uE}{\leq})$.

Recall that the relation $\mathcal{R}^*$ is said to be abundant on $(N, \underset{N}{\leq})$ if for any $x, y \in N$;

$$x \underset{N}{\leq} y \text{ implies } x^\dagger \underset{uE}{\leq} y^\dagger. \tag{75}$$

Assume this is the case. As in the proof of Proposition 9, define $\sigma : E \to uE$ by $e\sigma = ue$ for any $e \in E$. The idempotent ue is the representative element of L_e. The map σ is a homomorphism.

Let $\tau_x : uE \to uE$ be defined by $(ue)\tau_x = (xe)^\dagger$; $(x \in N)$. Put $\gamma_x = \sigma\tau_x$. Since for any $x, y \in N$, $e \in E$;

$$\begin{aligned} e\gamma_y\gamma_x &= (e\sigma\tau_y)\sigma\tau_x = (yue)^\dagger\sigma\tau_x = (xu(yue)^\dagger)^\dagger = (xuyue)^\dagger \\ &\Rightarrow (xyue)^\dagger = e\gamma_{xy}. \end{aligned} \tag{76}$$

So then $\gamma_y\gamma_x = \gamma_{xy}$ and $x \mapsto \gamma_x$ $(x \in N)$ describes an anti-homomorphism of N into K, the set of mappings of E into uE. Similar to Section 4, K is a semigroup partially ordered by $\underset{H}{\leq}$, where;

$$\phi \underset{H}{\leq} \psi \text{ if and only if } e\phi \underset{E}{\leq} e\psi \text{ for any } e \in E; (\phi, \psi \in K). \tag{77}$$

If $x \in N, e \in E$, then;

$$e\gamma_x = e\sigma\tau_x = (ue)\tau_x = (xe)^\dagger. \tag{78}$$

Hence, we have a dual of Proposition 9 as follows;

Proposition 11. *There is a unique anti-homomorphism γ from $(N, \underset{N}{\leq})$ into $(K, \underset{H}{\leq})$ described by $x \mapsto \gamma_x$ $(x \in N)$ satisfying the condition B where;*

(B) *For any $x \in N, e \in E$; $e\gamma_x = (xe)^\dagger$; $(xe)^\dagger \in uE$.*

Moreover, γ is isotone.

To proceed to construct a left naturally ordered left quasi-adequate semigroup P on which $\mathcal{R}^*$ is abundant and a right identity $\bar{u}$ (for P) such that $P\bar{u}$ is a left adequate subsemigroup of P. Let E be as in the hypothesis of Proposition 10 and let N be as above. Put;

$$P = P(E, N) = \{(e, x) \in E \times N; e \in L_{x^\dagger}\}. \tag{79}$$

Endow P with law of multiplication;

$$(e, x)(g, y) = (e(g\gamma_x), xy), \qquad (e, x), (g, y) \in P, \tag{80}$$

where $\gamma_x \in K$; $(x \in N)$ as defined earlier.

Since for any x and y in N; $x^\dagger(xy)^\dagger = (xy)^\dagger$, then by condition (B) -of Proposition 11- we have for any elements (e, x) and (g, y) in P, that;

$$e(g\gamma_x) = e(xy)^\dagger \in L_{x^\dagger}L_{(xy)^\dagger} \subseteq L_{x^\dagger(xy)^\dagger} = L_{(xy)^\dagger}, \tag{81}$$

and the multiplication of P is well-defined. Due to the fact that each γ_x is a homomorphism and;

$$\gamma_y \cdot \gamma_x = \gamma_{xy};\ \gamma_x, \gamma_y \in K;\ x, y \in N. \tag{82}$$

It is a routine matter to verify that P is a semigroup. We may call -as in [2] or [5]- that $P = P(E, N)$ is the left quasi-direct product of E and N.

As in the proof of Lemma 3, we have

Lemma 7.

(i) $E(P) = \{(e,x) \in P; x \in uE\}$.

(ii) $E(P)$ *is a band.*

(iii) $E(P)$ *is isomorphic to* E.

For any $(e,x) \in P; (e,x^\dagger) \in E(P)$ and $(e,x^\dagger)\mathcal{R}^*(e,x)$, (see the proof of Lemma 4). Hence we get;

Lemma 8. *The semigroup* P *is left quasi-adequate.*

It follows from the order of E and that on N, the Cartesian order, $\leq$, say, on P is a natural partial order. As γ is isotone (Proposition 11), then;

$$e \underset{E}{\leq} g \Rightarrow e\gamma_z \underset{E}{\leq} g\gamma_z \text{ for any } e,g \in E, z \in N. \tag{83}$$

So we can see the Cartesian order $\leq$ on P is left compatible. In fact it is left natural (as defined in Section 1) by a similar argument as in Lemma 5. The idempotent $\bar{u} = (u,u)$ is the right identity of P, for, if $(e,x) \in P$, then as u is the right identity of N and $e \in L_{x^\dagger}$ we get;

$$(e,x)(u,u) = (e(u\gamma_x), xu) = (e(xu)^\dagger, xu) = (ex^\dagger, x) = (e,x). \tag{84}$$

Furthermore (see Proposition 3(*iib*));

$$(u,u)(e,x) = (u(e\gamma_u), ux) = (u(ue)^\dagger, x) = (ue,x) = (ux^\dagger, x) = (x^\dagger, x), \qquad (ux = x \Rightarrow ux^\dagger = x^\dagger) \tag{85}$$

and;

$$\bar{u}P = \{(x^\dagger, x); x \in N\}, \tag{86}$$

so that $(x^\dagger, x) \mapsto x$ describes an isomorphism from $\bar{u}P$ onto N. It is an order isomorphism (see Lemma 6(*ii*)). Accordingly $\bar{u}P$ is a left adequate semigroup. It follows also -as in Lemma 6(*iii*)- that $\mathcal{R}^*$ is abundant on $(P,\leq)$.

In conclusion, we capture the direct part of the following theorem;

Theorem 2 (A dual of Theorem 1)**.** *Let* $(E, \underset{E}{\leq})$ *be a naturally ordered band with a right identity* u *such that* uE *is a semilattice. Let* $(N, \underset{N}{\leq})$ *be a left naturally ordered left adequate semigroup on which* $\mathcal{R}^*$ *is abundant and whose semilattice of idempotents coincides with* uE *such that* u *is the right identity for* N *and* $\underset{E}{\leq}|_{uE} = \underset{N}{\leq}|_{uE}$ $(= \underset{uE}{\leq})$. *Then the left quasi-direct product* $P = P(E,N)$ *of* E *and* N *is a left naturally ordered left quasi-adequate semigroup on which* $\mathcal{R}^*$ *is abundant and with a right identity* $\bar{u}$ *such that* $\bar{u}P$ *is a left adequate subsemigroup of* P *(orderly isomorphic to* N*).*
Conversely, any left naturally ordered left quasi-adequate semigroup $(S, \underset{S}{\leq})$ *on which* $\mathcal{R}^*$ *is abundant and with a right identity* u *such that* uS *is a left adequate subsemigroup of* S *is orderly isomorphic to a semigroup so constructed.*

To clarify the converse part of Theorem 2, let $(S, \underset{S}{\leq})$ be a left naturally ordered left quasi-adequate semigroup on which $\mathcal{R}^*$ is abundant and with a right identity u such that uS is a left adequate subsemigroup of S. Put $E(S) = E$.

It follows that E is a naturally ordered band and uS is a left naturally ordered left adequate semigroup. $E(uS) = uE$ which is a semilattice. As in Section 4, we have the left quasi-direct product $P = P(E, uS)$ of E and uS. The map $\beta : P \to S$ defined by $(e, x)\beta = ex$ is an isomorphism from P onto S which is isotone. Hence P and S are isomorphic as ordered semigroups.

6 Ordered quasi-adequate semigroups

Let $(S, \underset{S}{\leq})$ be a naturally ordered quasi-adequate semigroup containing a maximum idempotent u such that uSu is an adequate subsemigroup of S where $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant on S. The adequate semigroup uSu is naturally ordered with respect to the order $\underset{S}{\leq}$ of S. Recall that u is a medial idempotent (Proposition 3), and that uSu is a $*$-subsemigroup of S (Lemma 2) and $uxu\mathcal{L}^*ux^*u$ $(x \in S)$ for any x^* (Proposition 2). Let uxu and uyu be two elements in uSu $(x, y \in S)$. Then;

$$uxu \underset{S}{\leq} uyu \text{ implies } ux^*u \underset{S}{\leq} uy^*u, \qquad (\mathcal{L}^* \text{ is abundant on } S) \tag{87}$$

so then $\mathcal{L}^*$ (similarly $\mathcal{R}^*$) is abundant on uSu. The objective is to establish a structure for the semigroup S (Theorem 3). As S is (two-sided) quasi-adequate, the required structure will be a combination of that for the right quasi-adequate case which is described in Section 4 and that for the left quasi-adequate case which is described in Section 5. So we should think of two-sided quasi-direct product of an adequate semigroup and two specific bands. To establish such product, consider L and R to be two naturally ordered bands related to the order relations; $\underset{L}{\leq}$ and $\underset{R}{\leq}$ respectively with a common maximum element u, that is, a right identity for L and a left identity for R, where both uL and Ru are semilattices. Assume that the semilattice uL coincides with the semilattice Ru and put $uL = Ru$ $(= E$, say,) provided that

$$\underset{L}{\leq} |_E = \underset{R}{\leq} |_E \ (= \underset{E}{\leq}). \tag{88}$$

Let $(N, \underset{N}{\leq})$ be a naturally ordered adequate semigroup on which $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant and whose semilattice of idempotents coincides with the semilattice E and that $\underset{N}{\leq} |_E = \underset{E}{\leq}$.

From Proposition 8, it follows that R is a semilattice E of right zero semigroups R_α's $(\alpha \in E)$ and for any $f \in R$, the idempotent fu is the representative element in E of the right zero semigroup R_f. Then fu is the maximum element in R_f. Similarly, the band L is a semilattice E of left zero semigroups L_α's $(\alpha \in E)$ -Proposition 10- and for any $e \in L$ the idempotent ue is the representative element in E of the left zero semigroup L_e. The idempotent ue is the maximum element in L_e.

By Proposition 9, there is a unique homomorphism ζ from N into the semigroup H of mappings of R into E, described by $x \mapsto \zeta_x$ $(x \in N)$ satisfying the condition (A) which is also isotone. By Proposition 11, there is a unique anti-homomorphism γ from N into the semigroup K of mappings of L into E, described by $x \mapsto \gamma_x$ $(x \in N)$ satisfying condition (B) which is -also- isotone.

Put;

$$V = V(L, N, R) = \{(e, x, f) \in L \times N \times R; e\mathcal{L}x^{\dagger}, f\mathcal{R}x^{*}\}. \tag{89}$$

Recall for any $x \in N, x^{\dagger}, x^{*} \in E$ and that $ux = x = xu$. Endow V with the following law of multiplication;

$$\begin{aligned}(e, x, f)(g, y, h) &= (e(g\gamma_x), xy, (f\zeta_y)h) \\ &= (e(xy)^{\dagger}, xy, (xy)^{*}h).\end{aligned} \tag{90}$$

It is a routine matter to verify that V is a semigroup by applying Proposition 1(iii). We call $V = V(L, N, R)$ the quasi-direct product of L, N and R. The following results provide more information.

Proposition 12.

(i) $E(V) = \{(e, x, f) \in V; x \in E\}$ *which is a band.*

(ii) *The semigroup* $V = V(L, N, R)$ *is quasi-adequate.*

Proof. (i) Let $(e, x, f) \in V$ and notice that;

$$(e, x, f)(e, x, f) = (e(xx)^{\dagger}, xx, (xx)^{*}f). \tag{91}$$

Then clearly (e, x, f) is an idempotent if and only if x is an idempotent in N. Hence the result holds.

(ii) For any element (e, x, f) in V, the element $(e, x^{\dagger}, x^{\dagger})$ is an idempotent in V. Since;

$$(e, x^{\dagger}, x^{\dagger})(e, x, f) = (e(x^{\dagger}x)^{\dagger}, x^{\dagger}x, (x^{\dagger}x)^{*}f) = (ex^{\dagger}, x, x^{*}f) = (e, x, f) \tag{92}$$

and for any elements $(g, y, h), (i, z, j)$ in V, one can notice that;

$$\begin{aligned}(g, y, h)(e, x, f) = (i, z, j)(e, x, f) &\Rightarrow (g(yx)^{\dagger}, yx, (yx)^{*}f) = (i(zx)^{\dagger}, zx, (zx)^{*}f) \\ &\Rightarrow (g(yx^{\dagger})^{\dagger}, yx^{\dagger}, (yx)^{*}f) = (i(zx^{\dagger})^{\dagger}, zx^{\dagger}, (zx)^{*}f) \\ &\Rightarrow (g(yx^{\dagger})^{\dagger}, yx^{\dagger}, (yx^{\dagger})^{*}x^{\dagger}) = (i(zx^{\dagger})^{\dagger}, zx^{\dagger}, (zx^{\dagger})^{*}x^{\dagger}) \\ &\Rightarrow (g, y, h)(e, x^{\dagger}, x^{\dagger}) = (i, z, j)(e, x^{\dagger}, x^{\dagger}).\end{aligned} \tag{93}$$

Hence $(e, x, f)\mathcal{R}^{*}(e, x^{\dagger}, x^{\dagger})$ -Lemma 1- and every $\mathcal{R}^{*}$-class in V contains an idempotent. Similarly, (x^{*}, x^{*}, f) is an idempotent in V that is $\mathcal{L}^{*}$-related to (e, x, f). Thus every $\mathcal{L}^{*}$-class in V contains an idempotent. Now the result follows by part (i). □

Consider the Cartesian order $\leq$ on $V = V(L, N, R)$. The basic properties of $\leq$ are in the following proposition;

Proposition 13.

(i) *The Cartesian order* $\leq$ *on* V *is a natural order.*

(ii) *The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant on $(V, \leq)$.*

(iii) *The idempotent $\bar{u} = (u, u, u)$ is the maximum idempotent in $(V, \leq)$.*

(iv) *N is orderly isomorphic to $\bar{u}V\bar{u}$.*

Proof. (i) Recall that the semigroups: $(L, \underset{L}{\leq}), (N, \underset{N}{\leq})$ and $(R, \underset{R}{\leq})$ are naturally ordered. It is evident that the Cartesian order $\leq$ on V is partial order. Let $(e, x, f), (g, y, h)$ and (i, z, j) be elements in V such that;

$$(e, x, f) \leq (g, y, h), \text{ that is, } e \underset{L}{\leq} g, x \underset{N}{\leq} y \text{ and } f \underset{R}{\leq} h. \tag{94}$$

Since $\underset{N}{\leq}$ is right compatible, then

$$x \underset{N}{\leq} y \Rightarrow xz \underset{N}{\leq} yz. \tag{95}$$

As $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant on N and $\underset{N}{\leq}|_E = \underset{E}{\leq}$, then;

$$xz \underset{N}{\leq} yz \Rightarrow (xz)^\dagger \underset{E}{\leq} (yz)^\dagger \text{ and } (xz)^* \underset{E}{\leq} (yz)^*, \tag{96}$$

but also

$$e \underset{L}{\leq} g \Rightarrow e(xz)^\dagger \underset{L}{\leq} g(yz)^\dagger. \tag{97}$$

Also, we have; $(xz)^*j \underset{R}{\leq} (yz)^*j$. Therefore;

$$(e(xz)^\dagger, xz, (xz)^*j) \leq (g(yz)^\dagger, yz, (yz)^*j). \tag{98}$$

Hence;

$$(e, x, f)(i, z, j) \leq (g, y, h)(i, z, j) \tag{99}$$

and the Cartesian order of V is right compatible. By a similar argument, it is left compatible. So then the order $\leq$ on V is compatible. In fact it is natural, for, let (e, x, f) and (g, y, h) be two idempotents in V $(x, y \in E)$ such that;

$$(e, x, f)(g, y, h) = (e, x, f) = (g, y, h)(e, x, f). \tag{100}$$

That is; $(e(xy), xy, (xy)h) = (e, x, f) = (g(yx), yx, (yx)f)$ and $(ey, xy, xh) = (e, x, f) = (gx, yx, yf)$, so then;

$$ey = e = gx, \ xy = x, \ xh = f = yf. \tag{101}$$

Therefore,

$$ge = e, \ eg = gxg = gxug = gugx = gx = e \text{ and } e \ \omega \ g. \tag{102}$$

Hence ewg and thus $e \underset{L}{\leq} g$. Simlarly;

$$fh = f, \ hf = hyf = huyf = yhuf = yhf = yhxh = yhuxh = yxhuh = yxh = yf = f, \tag{103}$$

that is, $f\ \omega\ h$ so then $f \underset{R}{\leq} h$. Clearly $x\ \omega\ y$ so then $x \underset{E}{\leq} y$. Hence $(e,x,f) \leq (g,y,h)$ and the Cartesian order $\leq$ on V is natural.

(ii) If (e,x,f) and (g,y,h) are elements in V such that $(e,x,f) \leq (g,y,h)$, then, $e \underset{L}{\leq} g, x \underset{N}{\leq} y$ and $f \underset{R}{\leq} h$. As $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant on N, then;

$$x \underset{N}{\leq} y \text{ implies } x^* \underset{E}{\leq} y^* \text{ and } x^\dagger \underset{E}{\leq} y^\dagger. \tag{104}$$

Therefore;

$$(e,x^\dagger,x^\dagger) \leq (g,y^\dagger,y^\dagger) \text{ and } (x^*,x^*,f) \leq (y^*,y^*,h). \tag{105}$$

Now the result follows from the proof of Proposition 12(ii).

(iii) Let (e,x,f) be an idempotent in V $(x \in E)$. Since u is a maximum element for both L and R, then it is also a maximum element for E and;

$$e \underset{L}{\leq} u,\ \ x \underset{N}{\leq} u,\ \ f \underset{R}{\leq} u. \tag{106}$$

Therefore; $(e,x,f) \leq (u,u,u)$. Hence the result holds.
(iv) Let (e,x,f) be an element of V. Notice that;

$$\begin{aligned}(u,u,u)(e,x,f)(u,u,u) &= (u(ux)^\dagger, ux, (ux)^* f)(u,u,u)\\ &= (x^\dagger, x, f)(u,u,u)\\ &= (x^\dagger(xu)^\dagger, xu, (xu)^* u)\\ &= (x^\dagger, x, x^*).\end{aligned} \tag{107}$$

□

Denote (u,u,u) by $\bar{u}$ so that

$$\bar{u}V\bar{u} = \{(x^\dagger,x,x^*) : x \in N\} \tag{108}$$

Define $\psi : \bar{u}V\bar{u} \to N$ by $(x^\dagger,x,x^*)\psi = x$; $(x \in N)$. The map ψ is a bijection. By Proposition 1(iii), for any $x,y \in N$;

$$x^\dagger(xy)^\dagger = (xy)^\dagger \text{ and } (xy)^* y^* = (xy)^*, \tag{109}$$

so then $(x^\dagger,x,x^*)(y^\dagger,y,y^*) = ((xy)^\dagger, xy, (xy)^*)$ and ψ is an isomorphism. Accordingly $\bar{u}V\bar{u}$ is adequate.

For any $x,y \in N$;

$$(x^\dagger,x,x^*) \leq (y^\dagger,y,y^*) \text{ if and only if } x \underset{N}{\leq} y, \tag{110}$$

($\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant on N) and ψ is isotone.

Remark; Retain the hypothesis and notation of Proposition 13;
(1) $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant on $\bar{u}V\bar{u}$, for, if $(x^\dagger,x,x^*) \in \bar{u}V\bar{u}$, then $(x^\dagger,x^\dagger,x^\dagger)$ and (x^*,x^*,x^*) are idempotents in $\bar{u}V\bar{u}$ such that;

$$(x^\dagger,x^\dagger,x^\dagger)\mathcal{R}^*(x^\dagger,x,x^*)\mathcal{L}^*(x^*,x^*,x^*) \tag{111}$$

and

$$\begin{aligned}(x^\dagger, x, x^*) \le (y^\dagger, y, y^*) &\Rightarrow x^\dagger \underset{L}{\le} y^\dagger, x^* \underset{R}{\le} y^* \\ &\Rightarrow x^\dagger \underset{E}{\le} y^\dagger, x^* \underset{E}{\le} y^* \qquad (x^\dagger, x^*, y^\dagger, y^* \in E) \\ &\Rightarrow (x^\dagger, x^\dagger, x^\dagger) \le (y^\dagger, y^\dagger, y^\dagger) \text{ and } (x^*, x^*, x^*) \le (y^*, y^*, y^*).\end{aligned} \tag{112}$$

(2) Define the relation δ on V by the rule;

$$((e, x, f), (g, y, h)) \in \delta \text{ if and only if } x = y. \tag{113}$$

Clearly, δ is a congruence on V.

Moreover; for any $(e, x, f) \in V$:

$$(e, x, f)\delta(x^*, x^*, f)\delta = (e, x, f)\delta, \tag{114}$$

and if (g, y, h) and (i, z, j) are two elements in V such that:

$$(e, x, f)\delta(g, y, h)\delta = (e, x, f)\delta(i, z, j)\delta, \tag{115}$$

then we get $xy = xz$ and thus $x^*y = x^*z$, which implies

$$(x^*x^*, f)\delta(g, y, h)\delta = (x^*, x^*, f)\delta(i, z, j)\delta. \tag{116}$$

Therefore, $(e, x, f)\delta\mathcal{L}^*(V/\delta)(x^*, x^*, f)\delta$, and thus every $\mathcal{L}^*(V/\delta)$-class contains an idempotent. Similarly, every $\mathcal{R}^*(V/\delta)$-class contains an idempotent. Hence V/δ is abundant. Now, if $(e, x, f)\delta$ is an idempotent in V/δ, then

$$(e, x, f)\delta(e, x, f)\delta = (e, x, f)\delta, \tag{117}$$

so that $x \cdot x = x$ and x is an idempotent in N. Therefore, $E(V/\delta) = \{(e, x, f)\delta : x \in E\}$ and this is clearly a semilattice so that V/δ is adequate. If one describes a mapping $V/\delta \to N$ by $(e, x, f) \mapsto x$, then one can immediately conclude that this mapping is an order isomorphism. Therefore, any two of the three semigroups: $\bar{u}V\bar{u}, N$ and V/δ are orderly isomorphic.

In conclusion, the direct part of the following theorem is established.

Theorem 3. *Let L and R be two naturally ordered bands related to the relations $\underset{L}{\le}$ and $\underset{R}{\le}$ respectively with a common maximum element u, that is a right identity for L and a left identity for R. Assume that uL and Ru are semilattices such that $uL = Ru$ ($= E$, say) and $\underset{L}{\le}|_E = \underset{R}{\le}|_E$ ($=\underset{E}{\le}$). Let $(N, \underset{N}{\le})$ be a naturally ordered adequate semigroup on which $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant and whose semilattice of idempotents coincides with E, where $\underset{N}{\le}|_E = \underset{E}{\le}$. Then the quasi-direct product $V = V(L, N, R)$ of L, N and R is a naturally ordered quasi-adequate semigroup on which $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant and containing a maximum idempotent $\bar{u}$ such that $\bar{u}V\bar{u}$ is an adequate semigroup (orderly isomorphic to N).*
Conversely, any naturally ordered quasi-adequate semigroup $(S, \underset{S}{\le})$ on which $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant and containing a maximum idempotent u such that uSu is an adequate subsemigroup of S is orderly isomorphic to a semigroup so constructed.

For the converse part of Theorem 3, let $(S, \underset{S}{\leq})$ be a naturally ordered quasi-adequate semigroup with band of idempotents B containing a maximum idempotent u such that uSu is an adequate subsemigroup of S provided that $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant on $(S, \underset{S}{\leq})$. Consider -in one hand- the subsemigroup uS of S. The idempotent u is a left identity for uS. The semigroup $(uS)u$ is adequate. uB is the band of uS where uBu is the semilattice of uSu.

The band uB is naturally ordered with left identity u and it is a semilattice uBu of right zero semigroups R_α's -see Proposition 8-. Recall that uSu is a naturally ordered adequate semigroup whose semilattice of idempotents is uBu. The relation $\mathcal{R}^*$ (and also $\mathcal{L}^*$) is abundant on uSu. Consider the right quasi-direct product $Q = Q(uSu, uB)$ of uSu and uB, as in Section 4. The semigroup uS is right naturally ordered right quasi-adequate semigroup with band of idempotents uB, the idempotent u is a left identity for uS and $(uS)u$ is a right (and also a left) adequate semigroup. Then the semigroup Q is orderly isomorphic to uS where $\alpha : Q \to uS$ defined by

$$(uxu, f)\alpha = uxf;\ (uxu, f) \in Q \tag{118}$$

is an isomorphism which is isotone (see the proof of Theorem 1).

On the other hand, consider the subsemigroup Su of S. The idempotent u is a right identity for Su, where $u(Su)$ is an adequate semigroup. Bu is the band of Su where -as we recall- uBu is the semilattice of uSu. The band Bu is naturally ordered whose right identity is u and it is a (uBu) semilattice of left zero semigroups L_α's (see Proposition 10). The (left) adequate semigroup uSu is (left) naturally ordered whose semilattice of idempotents is uBu where $\mathcal{R}^*$ is abundant on uSu. Consider the left quasi-direct product $P = P(Bu, uSu)$ of Bu and uSu as in Section 5. The semigroup P is left naturally ordered left quasi-adequate whose band of idempotents is isomorphic to Bu and contains an idempotent $\bar{u}$, that is a right identity of P and $\bar{u}P$ is a left adequate semigroup. The semigroup P is orderly isomorphic to Su where $\beta : P \to Su$ defined by $(e, uxu)\beta = exu$ ($(e, uxu) \in P$) is an isomorphism which is isotone (see the proof of Theorem 2). We remark that all the semigroups: uS, Su, uSu, uB, Bu and uBu inherit the order from $(S, \underset{S}{\leq})$ so then the requirements related to the order on these semigroups are guaranteed.

Summing up, Bu and uB are two naturally ordered bands with a common maximum idempotent u such that u is a right identity for Bu and a left identity for uB where uBu is the common semilattice of Bu and uB and that uSu is a naturally ordered adequate semigroup whose semilattice of idempotents is uBu. The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are abundant on uSu. Hence we obtain the quasi-direct product;

$$V = V(Bu, uSu, uB) = \{(e, uxu, f) \in Bu \times uSu \times uB : e \in L_{(uxu)^\dagger}, f \in R_{(uxu)^*}\} \tag{119}$$

of Bu, uSu and uB.

The verification of the converse part of Theorem 3 will be completed when V and S are orderly isomorphic. This will be accomplished if the map $\theta : V \to S$ defined by $(e, uxu, f)\theta = exf$ ($(e, uxu, f) \in V$) is on order isomorphism. Recall that -as u is a maximum idempotent- u is a medial idempotent (Proposition 3). Moreover uSu is adequate. Therefore, we immediately conclude from the proof of [5, Proposition 6.7] that the mapping θ is an isomorphism. The last step of the required proof is included in the following lemma;

Lemma 9. *The isomorphism θ is isotone.*

Proof. Let (e, uxu, f) and (g, uyu, h) be two elements in V. Notice that;

$$\begin{aligned}(e, uxu, f) \leq (g, uyu, h) &\Rightarrow e \underset{L}{\leq} g, uxu \underset{uSu}{\leq} uyu, f \underset{R}{\leq} h \\ &\Rightarrow euxuf \underset{S}{\leq} guyuh \\ &\Rightarrow exf \underset{S}{\leq} gyf.\end{aligned} \tag{120}$$

For the converse implication, notice that;

$$e\mathcal{R}^*exf\mathcal{L}^*f,\ g\mathcal{R}^*gyh\mathcal{L}^*h \text{ (see the proof of Proposition 6.7 in [5]).} \tag{121}$$

Then;

$$\begin{aligned}exf \underset{S}{\leq} gyh \Rightarrow uf \underset{uB}{\leq} uh \qquad & (\mathcal{L}^* \text{ is abundant on } S \text{ and Corollary 2}) \\ \Rightarrow f \underset{uB}{\leq} h \qquad & (f, h \in uB).\end{aligned} \tag{122}$$

Similarly,

$$exf \underset{S}{\leq} gyh \Rightarrow e \underset{uB}{\leq} g. \qquad (\mathcal{R}^* \text{ is abundant on } S) \tag{123}$$

Since $e\mathcal{L}(uxu)^\dagger$ and $ue\mathcal{L}e$, $e \in Bu$, then;

$$ue\mathcal{L}(uxu)^\dagger \text{ and } ue = (uxu)^\dagger = ux^\dagger u \quad (ue, (uxu)^\dagger \in uBu). \tag{124}$$

Similarly; $ug = (uyu)^\dagger = uy^\dagger u$, $fu = (uxu)^* = ux^*u$, $hu = (uyu)^* = uy^*u$ (see [5]). Therefore;

$$exf \underset{S}{\leq} gyf \Rightarrow ux^*u \underset{uBu}{\leq} uy^*u \text{ and } ux^\dagger u \underset{uBu}{\leq} uy^\dagger u, \tag{125}$$

and thus;

$$\begin{aligned}exf \underset{S}{\leq} gyf &\Rightarrow ux^\dagger u\ exf\ ux^*u \underset{S}{\leq} uy^\dagger u\ gyh\ uy^*u \\ &\Rightarrow uxu \underset{S}{\leq} uyu.\end{aligned} \tag{126}$$

Hence,

$$\begin{aligned}exf \underset{S}{\leq} gyf &\Rightarrow e \underset{L}{\leq} g,\ uxu \underset{uSu}{\leq} uyu,\ f \underset{R}{\leq} h \\ &\Rightarrow (e, uxu, f) \leq (g, uyu, h),\end{aligned} \tag{127}$$

and the result holds. □

As the structure theorem [2, Theorem 6.3] for quasi-adequate semigroups with normal idempotents can be specialized to produce a structure theorem for orthodox semigroups with normal idempotents, we can establish the order analogue by coordinating the main concepts of the one sided case of ordered quasi-adequate semigroups with corresponding ideas in the regular semigroups (see Section 4) and write the analogue of Theorem 3 in the regular case as follows:

Theorem 4. *Let L and R be two naturally ordered bands related to the relations $\underset{L}{\leq}$ and $\underset{R}{\leq}$ respectively with a common maximum element u, that is a right identity for L and a left identity for R. Assume that uL and Ru are semilattices such that $uL = Ru$ $(= E$, say$)$ and $\underset{L}{\leq}|_E = \underset{R}{\leq}_E$ $(= \underset{E}{\leq})$.*

Let $(N, \nleq)$ be a naturally ordered inverse semigroup whose semilattice of idempotents coincides with E, where $\nleq|_E = \underset{E}{\leq}$. Then the quasi-direct product $V = V(L, N, R)$ of L, N, and R is a naturally ordered orthodox semigroup on which $\mathcal{L}$ and $\mathcal{R}$ are regular and contains a maximum idempotent $\overline{u}$ such that $\overline{u}V\overline{u}$ is an inverse semigroup (orderly isomorphic to N).

Conversely, any naturally ordered orthodox semigroup $(S, \leq)$ on which $\mathcal{L}$ and $\mathcal{R}$ are regular and contain a maximum idempotent u such that uSu is an inverse subsemigroup of S is orderly isomorphic to a semigroup so constructed.

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