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Join maximal element graph of lattice module

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Abstract. Let $\pounds$ be a C-lattice and M be a lattice module over $\pounds$. The join maximal element graph $\mathbb{G}(M)$ is a simple, undirected graph with all proper non-zero elements of M as vertices, and two distinct vertices, N and K, are adjacent if and only if $N \vee K \in Max(M)$, where $Max(M)$ is the collection of all maximal elements of M. In this paper, some properties of the graph $\mathbb{G}(M)$ like diameter, girth and clique number are investigated. Also, the interplay between the algebraic properties of M and the properties of those graphs is studied.

Keywords: Maximal element, Jacobson radical $J_{rad}(M)$, Join maximal element graph $\mathbb{G}(M)$.
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1 Introduction

The zero-divisor graphs of a commutative ring and its various generalizations has been studied in ([6–11], [13, 14] and [22]). In 1988, I. Beck defined the concept of zero-divisor graphs with the help of commutative rings (see [12]). This study is further extended by Akbari et al. [3] and introduced the intersection graph of a R-module M. In [5], A. H. Alwan generalized the concept of maximal ideal graph of a ring (see [4]) and defined the new graph called as maximal submodule graph of a module.

Various graph structures associated with module over ring structure can be identified in([1], [23]). In this paper, we used the concept introduced by A. H. Alwan to define the join maximal element graph of $\pounds$-module M, and throughout this paper $\pounds$ is a multiplicative lattice and M is a lattice module over $\pounds$.

A *multiplicative lattice* is denoted as $(\pounds, 0_\pounds, 1_\pounds, *)$, where $\pounds$ is a complete lattice with least element $0_\pounds$, greatest element $1_\pounds$ and $*$ is a binary operation defined on $\pounds$ that satisfies the

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following properties for all $a, b, c \in \pounds$:

1. $a * b \leq a \wedge b$.
2. $a * b = b * a$.
3. $(a * b) * c = a * (b * c)$.
4. $a * (\vee_{\alpha \in I} b_\alpha) = \vee_{\alpha \in I}(a * b_\alpha)$, where $b_\alpha \in \pounds$ and I is an indexing set.
5. $a * 1_\pounds = a$.

Henceforth, we write $a * b = ab$ for convenience only. By a *C-lattice* we mean a multiplicative lattice $(\pounds, 0_\pounds, 1_\pounds, *)$ in which the greatest element $1_\pounds$ is compact as well as multiplicative identity and $(\pounds, 0_\pounds, 1_\pounds, *)$ is generated under join by a multiplicatively closed set C of compact elements. For more reading about multiplicative lattices, one may refer to ([17], [18]- [19]).

Let $\pounds$ be a multiplicative lattice. A *lattice module* or an *£-module* is complete lattice M with least elements 0_M and the greatest element 1_M over $\pounds$ if for $u \in \pounds$ and $N \in M$ a multiplication denoted by $u.N \in M$ satisfies the following properties:

1. $(uv).N = u(v.N)$;
2. $(\bigvee_\alpha u_\alpha).(\bigvee_\beta N_\beta) = (\bigvee_{\alpha\beta} u_\alpha.N_\beta)$;
3. $1_\pounds.N = N$;
4. $0_\pounds.N = 0_M$; for all $u, v, u_\alpha \in \pounds$, and for all $N, N_\beta \in M$.

An element $N \in M$ is called *meet-principal*, if for each $a \in \pounds$ and $A \in M$, we have $A \wedge aN = (a \wedge (A : N))N$. An element $N \in M$ is called *join-principal*, if $((aN \vee A) : N) = (a \vee (A : N))$ for each $a \in \pounds$ and $A \in M$. If N is both meet-principal and join-principal, then N is called *principal* element of M. Note that, if each member of M is the join of principal elements then M is called *principally generated lattice* or *PG-lattice*.
Throughout this paper, $u.N$ will be written as uN. For $N \in M$ and $b \in L$, denote $(N : b) = \vee\{K \in M | bK \leq N\}$ and for $A, B \in M$, $(A : B) = \vee\{x \in \pounds | xB \leq A\}$. Here the operation : called *residual division*.
A proper element $N \in M$ is a *maximal element* if for every element $B \in M$ such that $N \leq B$, implies either $N = B$ or $B = 1_M$. Denote $Max(M)$ for the collection of all maximal elements of a lattice module M over a C-lattice $\pounds$. The meet of all members of $Max(M)$ is called the *Jacobson radical* of M and it is denoted by $J_{rad}(M)$.
For various concepts and related aspects of the lattice module, we refer ([2], [16], [18] and [21]).

Let $G = (V, E)$ be a graph with the set of vertex $V = V(G)$ and the set of edges $E = E(G)$. If $E = \emptyset$, then G is called an *empty graph*. Let $b \in V$, the *degree* of b in a graph G refers to the number of edges incident to that vertex and it is denoted by $deg(b)$. For $a, b \in V$, the length of the shortest path between a and b is denoted by $d(a, b)$. If there is no path between a and b, then $d(a, b) = \infty$. The *diameter* of G is $dim(G) = sup\{d(a, b) | a, b \in V\}$. A graph's greatest complete subgraph is called a *clique*. The number of vertices in largest clique is called the clique number of G, which is represented as $\omega(G)$. The *girth* of graph G, denoted by $gr(G)$, is the

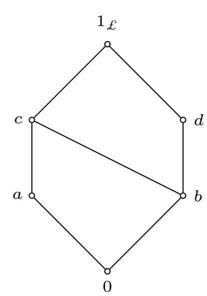
length of the shortest cycle in G. For $a \in V$, the eccentricity $e(a)$ of a is the greatest distance between a and any other vertex b in G, i.e., $e(a) = Max\{d(a,b)|$ for all $b \in V\}$. Let G be a connected graph. The *radius* of G, denoted by $\zeta(G)$, is essentially the minimum eccentricity among all the vertices in G, i.e., $\zeta(G) = Min\{e(u)|u \in V\}$. For $a \in V$, the collection $\{b_i\}$ of all adjacent vertices to a is called *neighborhood* of a, and it is denoted by $\Delta(a)$.
For more information on graph theory, the reader may refer ([15], [20], [24]).

In this paper, we prove that the graph $\mathbb{G}(M)$ is not a complete graph if the lattice module M has more that one maximal elements. Also, we show that the number of maximal elements of M can not be greater than 5 if the graph $\mathbb{G}(M)$ is planar. It is proved that, the graph $\mathbb{G}(M)$ is connected with $dim(\mathbb{G}(M)) \leq 4$, under certain conditions.

2 Connectivity, diameter and girth of $\mathbb{G}(M)$

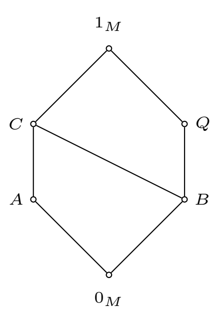
Definition 1. *Let M be a £-module. The join maximal element graph of M, which is denoted as $\mathbb{G}(M)$ is simple, undirected graph with all proper non-zero elements of M as vertices and two distinct vertices N and K are adjacent if and only if $N \vee K \in Max(M)$.*

Example 1. The lattice depicted in Fig.(4.1) is a C-lattice £ and the lattice depicted in Fig.(4.2) is a lattice module M over £. Fig.(4.3) represents the join maximal element graph $\mathbb{G}(M)$ of M as shown in Fig.(4.2) with vertex set $V(\mathbb{G}(M)) = \{A, B, C, Q\}$.



.	0	a	b	c	d	$1_£$
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
$1_£$	0	a	b	c	d	$1_£$

Fig.(4.1) Multiplicative Lattice £



.	0_M	A	B	C	Q	1_M
0	0_M	0_M	0_M	0_M	0_M	0_M
a	0_M	A	0_M	A	0_M	A
b	0_M	0_M	0_M	0_M	B	B
c	0_M	A	0_M	A	B	C
d	0_M	0_M	B	B	Q	Q
$1_£$	0_M	A	B	C	Q	1_M

Fig.(4.2) Lattice Module M over itself

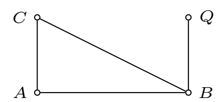


Fig.(4.3) Graph $\mathbb{G}(M)$ of M

Note that, each $N \notin Max(M)$ is adjacent to at least one $K \in Max(M)$. Also, for each $K_i \in Max(M)$, $J_{rad}(M)$ is adjacent to K_i. We know that for two distinct elements $N, K \in M$, if $N \vee K = 1_M$, then N and K are said to be *co-maximal*. It is clear that the co-maximal elements of M are not adjacent in $\mathbb{G}(M)$ and so have the following.

Throughout the paper, we fundamentally need the following Lemma 1.

Lemma 1 ([21])**.** *Let M be a lattice module over a C-lattice L with 1_M is compact. Then for every proper element $X \in M$, there exists a maximal element N of M such that $X \leq N$.*

Theorem 1. *Let M be a £-module with 1_M compact and $1 < |Max(M)| < \infty$. Then there is no vertex in $V(\mathbb{G}(M))$ which is adjacent to the remaining vertices. Furthermore, the graph $\mathbb{G}(M)$ is not a complete graph.*

Theorem 2. *Let M be a £-module with 1_M compact and $N, P, K \in V(\mathbb{G}(M))$ with $K \in Max(M)$. Then the following statements hold:*

1. *$K \in \Delta(N) \cap \Delta(P)$ if and only if $K \in \Delta(N \vee P)$, where $N \vee P \neq K, 1_M$.*
2. *If $N \leq J_{rad}(M)$, then $K \in \Delta(N)$.*
3. *If $N \leq P$ and $P \notin Max(M)$, then $N \notin \Delta(P)$.*

Proof. 1) Suppose that $K \in \Delta(N) \cap \Delta(P)$. Then by definition, K is adjacent to N and P. Since $K \in Max(M)$, we have $N \leq K$ and $P \leq K$, and hence $N \vee P \leq K$. This implies that $K \in \Delta(N \vee P)$. The converse part is clear.
2) Let $N \leq J_{rad}(M)$. Then $N \leq J_{rad}(M) \leq K$. This implies that $N \leq K$ and hence $K \in \Delta(N)$.
3) If $N \leq P$, then $N \vee P = P$. But $P \notin Max(M)$, therefore N is not adjacent to P. □

Theorem 3. *Let M be a £-module with 1_M compact and $\mathbb{G}(M)$ be the join maximal element graph of M. If $P, Q \notin Max(M)$ are adjacent vertices, then there exists only one $K \in Max(M)$ such that $K \in \Delta(P) \cap \Delta(Q)$.*

Proof. Suppose that $K_1, K_2 \in Max(M)$ such that $K_1, K_2 \in \Delta(P) \cap \Delta(Q)$. By definition, P, Q are adjacent to both K_1, K_2 in $\mathbb{G}(M)$. Since P, Q are adjacent vertices, we have $P \vee Q \in Max(M)$, therefore $K_1 = P \vee Q = K_2$. □

Theorem 4. *Let M be a £-module with 1_M compact and $J_{rad}(M) \notin Max(M) \cup \{0_M\}$. Then $\mathbb{G}(M) \cong K_{m,n}$, where $m, n \in \mathbb{Z}^+$ if and only if every member of $M \setminus Max(M)$ (i.e, non maximal element) is less than $J_{rad}(M)$.*

Proof. Suppose that every member of $M \setminus Max(M)$ is less than $J_{rad}(M)$. Let $V_1 = Max(M)$ and $V_2 = \{P \in V(\mathbb{G}(M)) | P \leq J_{rad}(M)\}$. By Remark 2, any two vertices $P, Q \in V_1$ are not adjacent in $\mathbb{G}(M)$. Since $J_{rad}(M) \notin Max(M)$, for any $P, K \in V_2$ we have $P \vee K \notin Max(M)$. This implies that any two vertices of V_2 are not adjacent in $\mathbb{G}(M)$. Also, by Theorem 2(2), it is observed that each $N \in V_1$ is adjacent to each $K \in V_2$ and hence $\mathbb{G}(M) \cong K_{m,n}$, where $m = |V_1|$ and $n = |V_2|$. Conversely, if $\mathbb{G}(M)$ is a complete bipartite graph with two part Y_1 and Y_1. It is very clear that $Y_1 = Max(M)$ and $Y_2 = \{P \in V(\mathbb{G}(M)) | P \leq J_{rad}(M)\}$. □

Corollary 1. *Let M be a £-module with 1_M compact and $J_{rad}(M) \notin Max(M) \cup \{0_M\}$. If $\mathbb{G}(M)$ is not complete bipartite graph and $|V\mathbb{G}(M) \setminus (Max(M) \cup \{P \in V(\mathbb{G}(M)) | P \leq J_{rad}(M)\})| = 1$, then $\mathbb{G}(M)$ is tripartite.*

Proof. Since $\mathbb{G}(M)$ is not complete bipartite graph, by Theorem 4 we have $P \nleq J_{rad}(M)$ for some $P \in M \setminus Max(M)$. Now, let $V_1 = Max(M)$, $V_2 = \{P \in V(\mathbb{G}(M)) | P \leq J_{rad}(M)\}$ and $V_3 = V(\mathbb{G}(M)) - V_1 \cup V_2$. It is simple to demonstrate that each pair of vertices in V_i is non adjacent for $i = 1, 2, 3$. Hence, V_1, V_2 and V_3 are nothing but 3 partition of $\mathbb{G}(M)$. □

A *star* graph is a particular kind of graph wherein nodes are connected to other nodes, referred to as leaves, by edges, starting at the center node. Although the leaf nodes are not connected to one another, every leaf node is connected to the central node.

Proposition 1. *Let M be a £-module with 1_M compact. If $M = C_n$, i.e., M is a chain with n elements, then $\mathbb{G}(M)$ is a star graph.*

Proof. Proof is straightforward. □

Theorem 5. *Let M be a £-module with 1_M compact and $J_{rad}(M) \neq 0_M$. Then $gr(\mathbb{G}(M))$, is either 3 or 4 or ∞.*

Proof. If there is no any cycle in $\mathbb{G}(M)$ then $gr(\mathbb{G}(M)) = \infty$.
Suppose that vertex $P, K \in V(\mathbb{G}(M)) \setminus Max(M)$ are adjacent in $\mathbb{G}(M)$. Then by definition, $P, K \neq P \vee K \in Max(M)$. Therefore we have a cycle $P - K - P \vee K - P$ of length 3 in $\mathbb{G}(M)$. Now suppose that for every adjacent vertices X, Y either $X \in Max(M)$ or $Y \in Max(M)$ and let $N_1 - N_2 - \cdots - N_{n-1} - N_n - N_1$ be a cycle of length n. By Remark 2, members of $Max(M)$ are not adjacent in $\mathbb{G}(M)$, therefore the vertices in cycle $N_1 - N_2 - \cdots - N_{n-1} - N_n - N_1$ are alternatively member of $Max(M)$ and non-member of $Max(M)$ and hence $J_{rad}(M) \notin Max(M)$. Let $N_1 \in Max(M)$. Then $J_{rad}(M)$ is adjacent to maximal elements in cycle $N_1 - N_2 - \cdots - N_{n-1} - N_n - N_1$, say $N_1, N_3, N_5, \cdots$. If $J_{rad}(M) = N_2$, then we have a cycle $J_{rad}(M) = N_2 - N_3 - N_4 - N_5 - N_2$. If $J_{rad}(M) \neq N_2$, then $J_{rad}(M) - N_1 - N_2 - N_3 - J_{rad}(M)$ is a cycle in the graph $\mathbb{G}(M)$. From both the cases, it is observed that, $gr(\mathbb{G}(M))$ is either 3 or 4. □

A graph G in which every pair of vertices is connected by a path is said to be a *connected* graph.

Theorem 6. *Let M be a £-module with 1_M compact and $\mathbb{G}(M)$ be the join maximal element graph of M. If whenever $N \wedge K \neq 0_M$ for any two distinct element N, K of M, then $\mathbb{G}(M)$ is connected graph with $dim(\mathbb{G}(M)) \leq 4$.*

Proof. Suppose that $N, K \in V(\mathbb{G}(M))$ and $N \neq K$. If $N - K$ then we are through. Suppose that N and K are not adjacent. This implies that either $N \vee K = 1_M$ or $N \vee K < P$ for some $P \in Max(M)$. For $N \vee K < P$, we have a path $N - P - K$ of length equal to 2 in $\mathbb{G}(M)$. Suppose that $N \vee K = 1_M$. This implies following cases:
(i) At least one of N and K is a member of $Max(M)$ and they are not comparable.
(ii) Both N and K are not member of $Max(M)$ and they are not comparable.
Suppose that case (i) holds and $N \in Max(M)$. If $K \in Max(M)$, then we have path $N - N \wedge K - K$ between N and K in $\mathbb{G}(M)$. Let $K \notin Max(M)$. Then we have $P \in Max(M)$ such that K and P are adjacent in the graph $\mathbb{G}(M)$. If $P = N$, then we have a path $K - N$ of length equal to 1 in $\mathbb{G}(M)$. If $P \neq N$, then $N - N \wedge P - P - K$ is a path in $\mathbb{G}(M)$. Thus in case (i), it is observed that N and K are connected and $dim(\mathbb{G}(M)) \leq 3$.
If case (ii) holds. Since $N \wedge K \neq 0$, by Lemma 1 we have $X \in Max(M)$ such that $N \wedge K \leq X$. Thus $N - N' - N \wedge K - K' - K$ is a path of length equal to 4 in $\mathbb{G}(M)$. Consequently, from each case, we have any $N, K \in V(\mathbb{G}(M))$ that are connected and $dim(\mathbb{G}(M)) \leq 4$. □

A *cut vertex* is a vertex in a graph G such that if we remove it (along with all incident edges), the graph becomes disconnected or has at least two connected components. In other words, removing a cut vertex from the graph G increases the number of connected components.

Theorem 7. *Let M be a £-module with 1_M compact and $N \wedge K \neq 0_M$ for any two distinct element N, K of M. If Y is a cut vertex of the graph $\mathbb{G}(M)$, then $Y = P \wedge Q$ for $P, Q \in Max(M)$.*

Proof. If $Y \in Max(M)$, then the proof is clear. Suppose that $Y \notin Max(M)$. Since Y is a cut vertex, we have different components in $\mathbb{G}(M) - \{Y\}$. Let X_1 and X_2 be two vertices in different components of $\mathbb{G}(M) - \{Y\}$. Consider the following cases:
Case-1) If $X_1, X_2 \in Max(M)$, then it is clearly observed that $X_1 \wedge X_2 \in \Delta(X_1) \cap \Delta(X_2)$ and hence $Y = X_1 \wedge X_2$.
Case-2) If $X_1 \in Max(M)$ and $X_2 \notin Max(M)$. Then we have $S \in Max(M)$ such that $X_2 \in \Delta(S)$. Since $X_1 \wedge S$ is adjacent to X_1 and S, therefore $Y = X_1 \wedge S$
Case-3) If $X_1, X_2 \notin Max(M)$. Then there is $C, D \in Max(M)$ such that C and D are adjacent to X_1 and X_2 respectively. As Y is cut vertex, we should have $C \neq D$. Thus as like Case-2), we have $Y = C \wedge D$. □

An *induced subgraph* of a graph G is a subgraph that is formed by selecting a subset of vertices from the original graph along with all the edges that are present between those vertices in the original graph. In other words, an induced subgraph retains all edges between the selected vertices and includes no additional edges.

Proposition 2. *Let M be a £-module with 1_M compact and for unique $Z \in Max(M)$, $\Pi(M)$ be the induced subgraph by $\{K \in V(\mathbb{G}(M)) | K \leq Z\}$. Then the clique of $\mathbb{G}(M)$ is contained in $\Pi(M)$.*

Proof. Suppose that Γ is the clique of $\mathbb{G}(M)$. Since any two distinct maximal elements $N, P \in Max(M)$ are not adjacent in $\mathbb{G}(M)$, we have only one maximal element in Γ. Thus by definition of Γ and Proposition 3, we have unique maximal element $Z \in Max(M)$ such that Γ is a subgraph induced by $\{K \in V(\mathbb{G}(M)) | K \leq Z\}$. □

Theorem 8. *Let* M *be a* $\pounds$*-module with* 1_M *compact and* $J_{rad}(M) \neq \emptyset$*. If* $|Max(M)| \geq 2$ *then* $\zeta(\mathbb{G}(M)) = 2$*.*

Proof. Since $J_{rad}(M)$ is adjacent to each $K_i \in Max(M)$, we have $d(J_{rad}(M), T) = 1$ for any $T \in Max(M)$. Also, since each $N \notin Max(M)$ is adjacent to at least one $K \in Max(M)$, we have $d(J_{rad}(M), K) \leq 2$ for any $K \notin Max(M)$. Let $X, Y \in Max(M)$ with $X \neq Y$. If $X \wedge Y$ is adjacent to $J_{rad}(M)$, then we have $P \in Max(M)$ such that $J_{rad}(M) \vee (X \wedge Y) = P$. But $J_{rad}(M) \vee (X \wedge Y) \leq X, Y$, therefore $P = X = Y$, which is contradiction to $X \neq Y$. This implies that $X \wedge Y$ is not adjacent to $J_{rad}(M)$. So the eccentricity e of $J_{rad}(M)$ is 2. Suppose that $F \in V(\mathbb{G}(M))$ with $e(F) = 1$. Then F is adjacent to each $W \in Max(M)$. It is clear that $F \notin Max(M)$. Since for any $X, Y \in Max(M)$ with $X \neq Y$, $X \wedge Y$ is not adjacent to $J_{rad}(M)$, we have neither $F = X \wedge Y$ nor $F = J_{rad}(M)$ and hence $F < J_{rad}(M)$. Thus F is adjacent to each $W \in Max(M)$ and $F < J_{rad}(M)$ which contradicts that $e(F) = 1$, therefore $J_{rad}(M)$ has the minimum eccentricity among all the vertices of $\mathbb{G}(M)$. Consequently, $\zeta(\mathbb{G}(M)) = 2$. □

We will now investigate the planar property of the join maximal element graph $\mathbb{G}(M)$.

Proposition 3. *Let* M *be a* $\pounds$*-module with* 1_M *compact and* $\mathbb{G}(M)$ *be the join maximal element graph of* M*. If for each* $Z \in Max(M)$*, the induced subgraph by* $\{K \in V(\mathbb{G}(M)) | K \leq Z\}$ *is planar, then* $\omega(\mathbb{G}(M)) \in \{2, 3, 4\}$*.*

Proof. Proof follows from Proposition 2 and Koratowski's theorem [15]. □

In the following theorem we prove that the planarity of the join maximal element graph $\mathbb{G}(M)$ put the control on $|Max(M)|$.

Theorem 9. *Let* M *be a* $\pounds$*-module with* 1_M *compact and* $J_{rad}(M) \neq \emptyset$*. If the join maximal element graph* $\mathbb{G}(M)$ *is planar, then* $|Max(M)| \leq 4$*.*

Proof. Suppose that $\mathbb{G}(M)$ is planar graph and the $\pounds$-module M has at least five maximal elements say P, Q, R, S and T. Then it is clear that any one element from the set $\{P \wedge Q \wedge R$, $P \wedge Q \wedge R \wedge S$, $P \wedge Q \wedge R \wedge S \wedge T\}$ is non-zero vertex which is adjacent to each of P, Q and R in $\mathbb{G}(M)$. This implies that the graph $\mathbb{G}(M)$ contains $K_{3,3}$, i.e., complete bipartite graph, which contradicts the fact that $\mathbb{G}(M)$ is planar graph. Consequently, $|Max(M)| \leq 4$. □

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