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Some properties of the edge ideal of a simple graph in the theory of local cohomology modules

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Abstract. In this paper, we have R a commutative Noetherian ring, with nonzero identity, and $\mathfrak{a}$ an ideal of R. Here, we give some results of the theory of modules for local cohomology involving the edge ideal. We introduce the concept of $\mathfrak{a}$-edge minimax R-modules and also the concept of $\mathfrak{a}$-edge cominimax R-modules, together with the edge ideal of a simple and finite graph, with no isolated vertices. We put results involving these new concepts and present relationships that exist between them.

Keywords: Edge Goldie dimension, $\mathfrak{a}$-edge minimax modules, $\mathfrak{a}$-edge cominimax modules, Local cohomology, Edge ideal of a graph.
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1 Introduction

Throughout this paper, R is a commutative Noetherian ring with nonzero identity.

Let $\mathfrak{a}$ be an ideal of R and let $I(G)$ be the edge ideal of a simple graph, finite and with no isolated vertices. By $\mathrm{H}^i_{\mathfrak{a}}(I(G))$ we mean

$$\varinjlim_{t\in\mathbb{N}} \mathrm{Ext}^i_R(R/\mathfrak{a}^t, I(G)),$$

the i-th local cohomology module of $I(G)$ with respect to the ideal $\mathfrak{a}$, for $i \geq 0$. For more details on local cohomology modules, see [3].

Local cohomology was introduced by Grothendieck and many people have worked about the understanding of their structure, (non)-vanishing and finiteness properties. For example, Grothendieck's non-vanishing theorem is one of the important theorems in local cohomology.

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We provide here results for local cohomology modules which involve the theory of graphs, together with the edge ideal of a graph.

In the Section 2, we put some definitions and prerequisites for a better understanding of the theory and results. We introduce preliminaries of the theory of graphs which involving the edge ideal of a graph G; associated to the graph G is a monomial ideal

$$I\left(G\right)=\left(v_iv_j \mid v_iv_j \text{ is an edge of } G\right),$$

with $v_iv_j = v_jv_i$ and with $i \neq j$, in the polynomial ring $R = K\left[v_1, v_2, \ldots, v_s\right]$ over a field K, called the *edge ideal* of G. The preliminaries of the theory of graphs were introduced in this Section 2 together with the concepts suitable for the work.

In the Section 3, we prove some results of local cohomology modules with respect to theory in question, results that involve the edge ideal of a graph G, which is a simple graph and finite, with no isolated vertices. Moreover, we introduce the definition of edge Goldie dimension for the R-module $I(G)$, and the definition of $\mathfrak{a}$-edge minimax module. And this definition will be used in this paper results.

In this Section 4, we introduce the definition of $\mathfrak{a}$-edge cominimax module, and we put results involving this definition. These results also involve local cohomology modules with respect to graph theory, results that involve the edge ideal of a graph G, and provide applications for such theory with respect to this particular R-module.

Throughout the paper, we mean by a graph G, a finite simple graph with the vertex set $V\left(G\right)$ and with no isolated vertices.

Here we use properties of commutative algebra and homological algebra for the development of the results (see [2] and [4]).

2 Prerequisites of the graphs theory

In this section, we present the concepts of graph theory that we will use in the course of this work.

2.1 Edge ideal of a graph

This section is in accordance with [1] and [5].

Let $R = K\left[v_1, \ldots, v_s\right]$ be a polynomial ring over a field K, and let $Z = \{z_1, \ldots, z_q\}$ be a finite set of monomials in R. The *monomial subring* spanned by Z is the K-subalgebra,

$$K\left[Z\right] = K\left[z_1, \ldots, z_q\right] \subset R.$$

In general, it is very difficult to certify whether $K\left[Z\right]$ has a given algebraic property - e.g., Cohen-Macaulay, normal - or to obtain a measure of its numerical invariants - e.g., Hilbert function. This arises because the number q of monomials is usually large.

Thus, we consider any graph G, simple and finite without isolated vertices, with vertex set $V\left(G\right) = \{v_1, \ldots, v_s\}$.

Let Z be the set of all monomials $v_iv_j = v_jv_i$, with $i \neq j$, in $R = K\left[v_1, \ldots, v_s\right]$, such that $\{v_iv_j\}$ is an edge of G, i.e., the graph finite and simple G, with no isolated vertices, is such that the squarefree monomials of degree two are defining the *edges* of the graph G.

Definition 1. A *walk* of length s in G is an alternating sequence of vertices and edges $w = \{v_1, z_1, v_2, \ldots, v_{s-1}, z_h, v_s\}$, where $z_i = \{v_{i-1}v_i\}$ is the edge joining v_{i-1} and v_i.

Now, we have the following definition.

Definition 2. A walk is *closed* if $v_1 = v_s$. A walk may also be denoted by $\{v_1, \ldots, v_s\}$, the edges being evident by context. A *cycle* of length s is a closed walk, in which the points $v_1, \ldots, v_s$ are distinct.

A *path* is a walk with all the points distinct. A *tree* is a connected graph without cycles and a graph is *bipartite* if all its cycles are even. A vertex of degree one will be called an *end point.*

Definition 3. A subgraph $G^{'} \subseteq G$ is called *induced* if $v_iv_j = v_jv_i$, with $i \neq j$, is an edge of $G^{'}$ whenever v_i and v_j are vertices of $G^{'}$ and v_iv_j is an edge of G.

The *complement* of a graph G, for which we write G^c, is the graph on the same vertex set in which $v_iv_j = v_jv_i$, with $j \neq i$, is an edge of G^c if and only if it is not an edge of G.

Finally, let C_k denote the cycle on k vertices; a *chord* is an edge which is not in the edge set of C_k. A cycle is called *minimal* if it has no a chord.

Moreover, if G is a graph without isolated vertices, simple and finite, then let R denote the polynomial ring on the vertices of G over some fixed field K, i.e., we have that $R = K[v_1, \ldots, v_s]$.

We finalize this part with the definition of the edge ideal of a graph, which will be used throughout of the article.

Definition 4 ([1])**.** *According to the previous context, the edge ideal of a finite simple graph G, with no isolated vertices, is defined by*

$$I\left(G\right) = \left(v_iv_j \mid v_iv_j \text{ is an edge of } G\right),$$

with $v_iv_j = v_jv_i$, and with $i \neq j$.

3 $\mathfrak{a}$-edge modules and edge Goldie dimension

In this section, we present some results about local cohomology modules which involve the theory of graphs together with the edge ideal of a graph G, which is simple and finite and with no isolated vertices.

Here, we take K a fixed field and we consider $K[v_1, v_2 \ldots, v_s]$ the polynomial ring over the field K. Since K is a field, we have that K is a Noetherian ring and then $K[v_1, \ldots, v_s]$ is also a Noetherian ring (Theorem of the Hilbert Basis).

Remark 1. In the previous context, $R = K[v_1, v_2 \ldots, v_s]$ is a commutative Noetherian ring, with nonzero identity. Thus, the edge ideal $I\left(G\right)$ is an R-module which is Noetherian since it is finitely generated, and thus we can get characterizations for this module under certain hypothesis.

We denote by $\mathfrak{a}$ an ideal any of $R = K[v_1, \ldots, v_s]$, where we have $I(G)$ a monomial ideal of R which is finitely generated.

We first we put the following definition, which uses the edge ideal of a graph, that we shall use in the paper.

Definition 5. Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. For the R-module $I(G)$, the edge Goldie dimension of $I(G)$ is defined as the cardinal of the set of indecomposable submodules of $\mathrm{E}(I(G))$ which appear in a decomposition of $\mathrm{E}(I(G))$ into a direct sum of indecomposable submodules. We shall use $\operatorname{eg}\dim(I(G))$ for to denote the edge Goldie dimension of $I(G)$.

For a prime ideal $\mathfrak{p}$ of R, let $\mu^0(\mathfrak{p}, I(G))$ denote the 0-th Bass number of $I(G)$ with respect to the prime ideal $\mathfrak{p}$. It is known that $\mu^0(\mathfrak{p}, I(G)) > 0$ if and only if $\mathfrak{p} \in \operatorname{Ass}_R(I(G))$. Thus, by the definition of edge Goldie dimension, it follows that

$$\operatorname{eg}\dim(I(G)) = \sum_{\mathfrak{p} \,\in\, \operatorname{Ass}_R(I(G))} \mu^0(\mathfrak{p}, I(G)).$$

Also, for an ideal $\mathfrak{a}$ of R, the $\mathfrak{a}$-relative edge Goldie dimension of $I(G)$ is defined as

$$\operatorname{eg}\dim_{\mathfrak{a}}(I(G)) = \sum_{\mathfrak{p} \,\in\, \mathrm{V}(\mathfrak{a})} \mu^0(\mathfrak{p}, I(G)).$$

The R-module $I(G)$ (or a module which involves the R-module $I(G)$, i.e., that depends of the R-module $I(G)$) is said to be an edge minimax module if there exists a submodule N of $I(G)$, such that $I(G)/N$ is an Artinian R-module. Note that the submodule N is finitely generated, since $I(G)$ is a Noetherian R-module.

Definition 6. Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{a}$ be an ideal of R. The R-module $I(G)$ (or a module which involves the R-module $I(G)$, i.e., that depends of the R-module $I(G)$) is said to be edge minimax with respect to the ideal $\mathfrak{a}$ or $\mathfrak{a}$-edge minimax if the $\mathfrak{a}$-relative edge Goldie dimension of any quotient module of $I(G)$ is finite; i.e., for any submodule N of $I(G)$ we have that

$$\operatorname{eg}\dim_{\mathfrak{a}}(I(G)/N),$$

is finite, i.e.,

$$\operatorname{eg}\dim_{\mathfrak{a}}(I(G)/N) < \infty.$$

Remark 2. Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{a}$ be an ideal of R.

(1.) If $\mathfrak{a} = 0$, then $I(G)$ is $\mathfrak{a}$-edge minimax if and only if $I(G)$ is edge minimax.

(2.) If $\mathfrak{b}$ is a second ideal of R such that $\mathfrak{a} \subseteq \mathfrak{b}$ and $I(G)$ is $\mathfrak{a}$-edge minimax, then $I(G)$ is $\mathfrak{b}$-edge minimax. In particular, every edge minimax module $I(G)$ is $\mathfrak{a}$-edge minimax module.

The following proposition is needed in the proof of results of this paper. Thus, we presented now this such result.

Proposition 1. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{a}$ be an ideal of R. Let*

$$0 \to I(G^{'}) \to I(G) \to I(G^{''}) \to 0,$$

be an exact sequence of R-modules which are edge ideals of graphs simple, finites and with no isolated vertices. Then $I(G)$ is $\mathfrak{a}$-edge minimax if and only if $I(G^{'})$ and $I(G^{''})$ are both $\mathfrak{a}$-edge minimax modules.

Proof. We may suppose for the proof that $I(G^{'})$ is a submodule of $I(G)$ (and in this case we have that $G^{'} \subseteq G$ is a subgraph induced of G) and that $I(G^{''}) = I(G)/I(G^{'})$. If $I(G)$ is $\mathfrak{a}$-edge minimax, then it easily follows from of the definition that $I(G^{'})$ and $I(G)/I(G^{'})$ are $\mathfrak{a}$-edge minimax. Now, suppose that $I(G^{'})$ and $I(G)/I(G^{'})$ are $\mathfrak{a}$-edge minimax modules. Let N be an arbitrary submodule of $I(G)$, and let $\mathfrak{p} \in \mathrm{Ass}_R(I(G)/N) \cap \mathrm{V}(\mathfrak{a})$. Then the exact sequence

$$0 \to \frac{I(G^{'}) + N}{N} \to \frac{I(G)}{N} \to \frac{I(G)}{I(G^{'}) + N} \to 0,$$

induces the exact sequence

$$0 \to \mathrm{Hom}_{R_{\mathfrak{p}}}\left(k(\mathfrak{p}), \frac{I(G^{'})_{\mathfrak{p}}}{I(G^{'})_{\mathfrak{p}} \cap N_{\mathfrak{p}}}\right) \to \mathrm{Hom}_{R_{\mathfrak{p}}}\left(k(\mathfrak{p}), \frac{I(G)_{\mathfrak{p}}}{N_{\mathfrak{p}}}\right) \to \mathrm{Hom}_{R_{\mathfrak{p}}}\left(k(\mathfrak{p}), \frac{I(G)_{\mathfrak{p}}}{I(G^{'})_{\mathfrak{p}} + N_{\mathfrak{p}}}\right),$$

where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Moreover, since

$$\mathrm{Ass}_R(I(G)/N) \subseteq \mathrm{Ass}_R\left(\frac{I(G^{'}) + N}{N}\right) \cup \mathrm{Ass}_R\left(\frac{I(G)}{I(G^{'}) + N}\right),$$

and the sets

$$\mathrm{Ass}_R\left(\frac{I(G^{'}) + N}{N}\right) \cap \mathrm{V}(\mathfrak{a}) \text{ and } \mathrm{Ass}_R\left(\frac{I(G)}{I(G^{'}) + N}\right) \cap \mathrm{V}(\mathfrak{a}),$$

are finite, it follows that we have

$$\operatorname{eg\,dim}_{\mathfrak{a}}(I(G)/N) < \infty,$$

and so $I(G)$ is an $\mathfrak{a}$-edge minimax R-module. □

We have so the following corollary.

Corollary 1. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{a}$ be an ideal of R. Then any quotient of the $\mathfrak{a}$-edge minimax module $I(G)$, as well as any finite direct sum of $\mathfrak{a}$-edge minimax modules, is an $\mathfrak{a}$-edge minimax module.*

Proof. The assertion it follows from the definition and of the Proposition 1. □

Corollary 2. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{a}$ be an ideal of R and let N be an $\mathfrak{a}$-edge minimax R-submodule of $I(G)$. Then, we have that $\mathrm{Ext}^i_R(I(G), N)$ and $\mathrm{Tor}^R_i(I(G), N)$ are $\mathfrak{a}$-edge minimax modules for all $i \geq 0$.*

Proof. As R is Noetherian and $I(G)$ is finitely generated, it follows that $I(G)$ possesses a free resolution

$$\mathbb{F}_\bullet : \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0,$$

whose free modules have finite ranks.

Thus, $\mathrm{Ext}^i_R(I(G), N) = \mathrm{H}^i(\mathrm{Hom}_R(\mathbb{F}_\bullet, N))$ is a subquotient of a direct sum of finitely many copies of N. Therefore, it follows from Corollary 1 that

$$\mathrm{Ext}^i_R(I(G), N) \text{ is } \mathfrak{a}\text{-edge minimax module for all } i \geq 0.$$

By using a similar proof as above we can deduce that $\mathrm{Tor}_i^R(I(G), N)$ is $\mathfrak{a}$-edge minimax module for all $i \geq 0$. □

We have now the following proposition.

Proposition 2. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{a}$ be an ideal of R. Suppose that the R-module $I(G)$ is an $\mathfrak{a}$-edge minimax R-module such that $\mathrm{Ass}_R(I(G)) \subseteq \mathrm{V}(\mathfrak{a})$. Then, we have that $\mathrm{H}^i_{\mathfrak{a}}(I(G))$ is an $\mathfrak{a}$-edge minimax module, for all $i \geq 0$.*

Proof. If $i = 0$, then $\mathrm{H}^0_{\mathfrak{a}}(I(G)) = \Gamma_{\mathfrak{a}}(I(G))$ is a submodule of $I(G)$, and so by the Proposition 1, we have that $\Gamma_{\mathfrak{a}}(I(G))$ is an $\mathfrak{a}$-edge minimax module. As $\mathrm{Ass}_R(I(G)/\Gamma_{\mathfrak{a}}(I(G))) \subseteq \mathrm{Ass}_R(I(G))$, it follows from $\mathrm{Ass}_R(I(G)) \subseteq \mathrm{V}(\mathfrak{a})$ that we have $I(G) = \Gamma_{\mathfrak{a}}(I(G))$. Consequently, by [3, Corollary 2.1.7(*ii*)], we have that $\mathrm{H}^i_{\mathfrak{a}}(I(G)) = 0$ for all $i > 0$, and so $\mathrm{H}^i_{\mathfrak{a}}(I(G))$ is $\mathfrak{a}$-edge minimax for all $i \geq 0$, as required. □

We are now ready for to state and to prove the following result, which is an important theorem of this section.

Theorem 1. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{a}$ be an ideal of R and let N be a submodule of $I(G)$. Let t be a non-negative integer such that $\mathrm{Ext}^i_R(I(G), N)$ is $\mathfrak{a}$-edge minimax module for all $i \leq t$. Then for any finitely generated R-module L with $\mathrm{Supp}_R(L) \subseteq \mathrm{Supp}_R(I(G))$, we have that $\mathrm{Ext}^i_R(L, N)$ is $\mathfrak{a}$-edge minimax module for all $i \leq t$.*

Proof. Since $\mathrm{Supp}_R(L) \subseteq \mathrm{Supp}_R(I(G))$ we have, according to Gruson s Theorem [6, Theorem 4.1], that there exists a chain

$$0 = L_0 \subset L_1 \subset \ldots \subset L_k = L,$$

such that the factors L_j/L_{j-1} are homomorphic images of a direct sum of finitely many copies of $I(G)$. Now consider the exact sequences

$$0 \to K \to I(G)^n \to L_1 \to 0,$$

$$0 \to L_1 \to L_2 \to L_2/L_1 \to 0,$$

$$\vdots$$

$$0 \to L_{k-1} \to L_k \to L_k/L_{k-1} \to 0,$$

for some positive integer n.

Now from the long exact sequence

$$\cdots \to \operatorname{Ext}_R^{i-1}(L_{j-1}, N) \to \operatorname{Ext}_R^i(L_j/L_{j-1}, N) \to \operatorname{Ext}_R^i(L_j, N) \to \operatorname{Ext}_R^i(L_{j-1}, N) \to \cdots$$

and an easy induction on k, it suffices for to prove the case when $k = 1$.

Thus there exists an exact sequence

$$0 \to K \to I(G)^n \to L \to 0 \qquad (*)$$

for some $n \in \mathbb{N}$ and for some finitely generated R-module K.

Now, we use induction on t. First, $\operatorname{Hom}_R(L, N)$ is an R-submodule of the R-module $\operatorname{Hom}_R(I(G)^n, N)$; hence in view of the assumption and of the Corollary 1, we have that $\operatorname{Ext}_R^0(L, N)$ is $\mathfrak{a}$-edge minimax. So, we assume that $t > 0$ and that $\operatorname{Ext}_R^j(L', N)$ is $\mathfrak{a}$-edge minimax module for every finitely generated R-module L' with

$$\operatorname{Supp}_R(L') \subseteq \operatorname{Supp}_R(I(G)) \text{ and all } j \leq t-1.$$

Now, the exact sequence $(*)$ induces the long exact sequence

$$\cdots \to \operatorname{Ext}_R^{i-1}(K, N) \to \operatorname{Ext}_R^i(L, N) \to \operatorname{Ext}_R^i(I(G)^n, N) \to \cdots,$$

so that, by the inductive hypothesis, we have that

$$\operatorname{Ext}_R^{i-1}(K, N) \text{ is } \mathfrak{a}-\text{edge minimax module for all } i \leq t.$$

On the other hand, according to Corollary 1, we have that

$$\operatorname{Ext}_R^i(I(G)^n, N) \cong \bigoplus^n \operatorname{Ext}_R^i(I(G), N)$$

is $\mathfrak{a}$-edge minimax. Therefore, it follows from Proposition 1 that we have

$$\operatorname{Ext}_R^i(L, N) \text{ is } \mathfrak{a}-\text{edge minimax module for all } i \leq t,$$

and the inductive step is complete. This ends the proof. □

Corollary 3. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{a}$ be an ideal of R, and let t be a non-negative integer. Then, the following conditions are equivalent:*

(1). $\operatorname{Ext}_R^i(R/\mathfrak{a}, I(G))$ *is $\mathfrak{a}$-edge minimax module for all $i \leq t$.*

(2). *For any ideal $\mathfrak{b}$ of R with $\mathfrak{a} \subseteq \mathfrak{b}$, we have that $\operatorname{Ext}_R^i(R/\mathfrak{b}, I(G))$ is $\mathfrak{a}$-edge minimax module for all $i \leq t$.*

(3). *For any R-submodule N of $I(G)$ with $\operatorname{Supp}_R(N) \subseteq \operatorname{V}(\mathfrak{a})$ we have that $\operatorname{Ext}_R^i(N, I(G))$ is $\mathfrak{a}$-edge minimax module for all $i \leq t$.*

(4). *For any minimal prime ideal $\mathfrak{p}$ over $\mathfrak{a}$, we have that $\operatorname{Ext}^i_R(R/\mathfrak{p}, I(G))$ is $\mathfrak{a}$-edge minimax module for all $i \leq t$.*

Proof. In view of Theorem 1, it is enough to show that (4) implies (1). To do this, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes of $\mathfrak{a}$. Then, by assumption, the R-modules

$$\operatorname{Ext}^i_R(R/\mathfrak{p}_j, I(G)) \text{ are } \mathfrak{a} - \text{edge minimax modules for all } j = 1, 2, \ldots, n.$$

Hence, by the Corollary 1, we have that

$$\bigoplus_{j=1}^{n} \operatorname{Ext}^i_R(R/\mathfrak{p}_j, I(G)) \cong \operatorname{Ext}^i_R(\bigoplus_{j=1}^{n} R/\mathfrak{p}_j, I(G)),$$

is $\mathfrak{a}$-edge minimax module. Since

$$\operatorname{Supp}_R\left(\bigoplus_{j=1}^{n} R/\mathfrak{p}_j\right) = \operatorname{Supp}_R(R/\mathfrak{a})$$

it follows from Theorem 1 that $\operatorname{Ext}^i_R(R/\mathfrak{a}, I(G))$ is $\mathfrak{a}$-edge minimax module, for all $i \leq t$, as required. □

4 $\mathfrak{a}$-edge cominimax modules and local cohomology modules

In this section, we continue in the same context as the previous section.

Definition 7. Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{a}$ be an ideal of R. We say that the R-module $I(G)$ (or a module which involves the R-module $I(G)$, i.e., that depends of the R-module $I(G)$), is $\mathfrak{a}$-edge cominimax if the support of $I(G)$ is contained in $\operatorname{V}(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, I(G))$ is a finitely generated R-module and is $\mathfrak{a}$-edge minimax for all $i \geq 0$.

Proposition 3. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{a}$ be an ideal of R. Suppose that for the R-module $I(G)$ we have that $\operatorname{H}^i_{\mathfrak{a}}(I(G))$ is $\mathfrak{a}$-edge cominimax for all $i \geq 0$. Then, $\operatorname{Ext}^i_R(R/\mathfrak{a}, I(G))$ is $\mathfrak{a}$-edge minimax for all $i \geq 0$.*

Proof. The case $i = 0$ is clear, and so let $i > 0$ and do induction on i. We first we reduce to the case $\Gamma_{\mathfrak{a}}(I(G)) = 0$. To do this, let $I(\bar{G}) = I(G)/\Gamma_{\mathfrak{a}}(I(G))$. Then we have the long exact sequence

$$\cdots \to \operatorname{Ext}^i_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(I(G))) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, I(G)) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, I(\bar{G})) \to \cdots$$

and the isomorphism $\operatorname{H}^i_{\mathfrak{a}}(I(G)) \cong \operatorname{H}^i_{\mathfrak{a}}(I(\bar{G}))$ for $i > 0$. So in view of the Proposition 1, we may assume that $I(G)$ is $\mathfrak{a}$-torsion free. Let E be the injective envelope of $I(G)$ and we put $L = \operatorname{E}/I(G)$. Then,

$$\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{E}) = 0,$$

and we therefore get the following isomorphisms

$$\mathrm{H}^i_{\mathfrak{a}}(L) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(I(G)) \text{ for all } i \geq 0,$$

and also, we have that

$$\mathrm{Ext}^i_R(R/\mathfrak{a}, L) \cong \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, I(G)) \text{ for all } i \geq 0.$$

Now, the assertion it follows by induction. □

Now, we have the following proposition.

Proposition 4. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{a}$ be an ideal of R. Suppose that the R-module $I(G)$ is such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, I(G))$ is $\mathfrak{a}$-edge minimax for all $i \geq 0$. If t is a non-negative integer such that we have the R-module $\mathrm{H}^i_{\mathfrak{a}}(I(G))$ of usual local cohomology an $\mathfrak{a}$-edge cominimax module for all $i \neq t$, then $\mathrm{H}^t_{\mathfrak{a}}(I(G))$ is $\mathfrak{a}$-edge cominimax.*

Proof. We use induction on t. Let $I(\bar{G}) = I(G)/\Gamma_{\mathfrak{a}}(I(G))$. Then, we have that

$$\mathrm{H}^i_{\mathfrak{a}}(I(G)) \cong \mathrm{H}^i_{\mathfrak{a}}(I(\bar{G})),$$

for all $i > 0$. If $t = 0$, then $\mathrm{H}^i_{\mathfrak{a}}(I(\bar{G}))$ is $\mathfrak{a}$-edge cominimax for all i. Hence by Proposition 3, we have that

$$\mathrm{Ext}^i_R(R/\mathfrak{a}, I(\bar{G})) \text{ is } \mathfrak{a}\text{-edge minimax for all } i.$$

It follows that $\Gamma_{\mathfrak{a}}(I(G))$ is $\mathfrak{a}$-edge cominimax. So let $t > 0$ and suppose that the result has been proved for $t-1$. Since $\Gamma_{\mathfrak{a}}(I(G))$ is $\mathfrak{a}$-edge cominimax module, the exact sequence

$$\cdots \to \mathrm{Ext}^i_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(I(G))) \to \mathrm{Ext}^i_R(R/\mathfrak{a}, I(G)) \to \mathrm{Ext}^i_R(R/\mathfrak{a}, I(\bar{G})) \to \cdots$$

allows us to assume that $I(G)$ is $\mathfrak{a}$-torsion free. Let E be the injective envelope of $I(G)$ and we put $L = \mathrm{E}/I(G)$. Then, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{E}) = 0$ and $\Gamma_{\mathfrak{a}}(\mathrm{E}) = 0$, and we therefore get the isomorphisms

$$\mathrm{H}^i_{\mathfrak{a}}(L) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(I(G)) \text{ for all } i \geq 0,$$

and

$$\mathrm{Ext}^i_R(R/\mathfrak{a}, L) \cong \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, I(G)) \text{ for all } i \geq 0.$$

Now, the assertion it follows by induction. □

We finalize the article with two corollaries, where we will uses the previous results.

Corollary 4. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Let $\mathfrak{a}$ be an ideal of R, and suppose that the R-module $I(G)$ is an $\mathfrak{a}$-edge minimax module. If t is a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(I(G))$ is $\mathfrak{a}$-edge cominimax for all $i \neq t$, then $\mathrm{H}^t_{\mathfrak{a}}(I(G))$ is $\mathfrak{a}$-edge cominimax module.*

Proof. This it follows from Corollary 2 and Proposition 4. □

Corollary 5. *Let $R = K[v_1, \ldots, v_s]$ be the polynomial ring, $I(G)$ the edge ideal in R of a finite simple graph G, with no isolated vertices. Suppose that $\mathfrak{a}$ is a principal ideal of R and that $I(G)$ is an $\mathfrak{a}$-edge minimax R-module. Then, we have that $\mathrm{H}^i_{\mathfrak{a}}(I(G))$ is $\mathfrak{a}$-edge cominimax for all $i \geq 1$.*

Proof. Since $\mathrm{H}^0_{\mathfrak{a}}(I(G))$ is a submodule of $I(G)$, it turns out that $\mathrm{H}^0_{\mathfrak{a}}(I(G))$ is $\mathfrak{a}$-edge minimax module, by the Proposition 1. Also, we have that

$$\mathrm{H}^i_{\mathfrak{a}}(I(G)) = 0 \text{ for all } i \geq 1.$$

Therefore, the result follows from Corollary 4. □

Conclusion

In this article, we present a theory with applications within the theory of commutative algebra.

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