Journal of Algebra and Related Topics
Vol. , No. -, 2026, pp. -. Research Article

Hovey pairs in $\mathbb{C}_N(\mathcal{G})$

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Abstract. One approach to construct a model structure on $\mathbb{C}_N(\mathcal{A})$, the category of N-complexes over an abelian category $\mathcal{A}$, is to start with a complete hereditary cotorsion pair $(\mathcal{F},\mathcal{C})$ in $\mathcal{A}$ and then introduce Hovey pairs on $\mathbb{C}_N(\mathcal{A})$. There are three important pairs of cotorsion pairs in the literature, particularly in [2] and [33]. In this paper, we employ a different technique by considering $\mathcal{A}$ as a Grothendieck category to introduce these Hovey pairs. For these pairs of cotorsion pairs, we omit the hereditary conditions, the conditions of having enough $\mathcal{F}$-objects–essential conditions in [33]–as well as the condition of being closed under direct limits for the class $\mathcal{F}$. As a result, we can construct Hovey pairs on categories that do not necessarily have enough $\mathcal{F}$-objects or where the class of objects is not closed under direct limits such as the category of Cartesian modules over small categories and the category of quasi-coherent sheaves on a scheme $\mathbb{X}$.

Keywords: N-complexes, Complete cotorsion pairs, Model structure, Hovey pair.
AMS Subject Classification 2010: 18G35, 18E30, 1820F.

1 Introduction

A model category, often referred to as a Quillen model category, serves as a framework for conducting homotopy theory. Quillen introduced the concept to formalize the parallels between homotopy theory and homological algebra. His definition was primarily motivated by key examples such as the category of topological spaces, simplicial sets, and chain complexes. The fundamental issue that model categories address is as follows:

Given a category, one often has certain maps (weak equivalences) that are not isomorphisms, but one would like to consider them to be isomorphisms. One can always formally invert the weak equivalences, but in this case one loses control of the morphisms in the quotient category. If the weak equivalences are part of a model structure, however, then the morphisms in the quotient

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Received: 19 April 2025/ Revised: 09 September 2025/ Accepted: 13 September 2025
DOI: 10.22124/JART.2025.30423.1791

category from X to Y are simply homotopy classes of maps from a cofibrant replacement of X to a fibrant replacement of Y. Because this idea of inverting weak equivalences is so central in mathematics, model categories are extremely important.

In 2002, Hovey discovered a connection between the concept of cotorsion pairs and model structures on abelian categories. This connection was expressed through a correspondence between two cotorsion pairs, referred to as Hovey pairs, and the model structure on abelian categories (Hovey's Theorem [23]). The concept of cotorsion pairs (or cotorsion theory) was invented by Salce in [30] in the category of abelian groups and was rediscovered by Enochs and coauthors in the 1990's. In short, a cotorsion pair in an abelian category $\mathcal{A}$ is a pair $(\mathcal{F}, \mathcal{C})$ of classes of objects of $\mathcal{A}$ each of which is the orthogonal complement of the other with respect to the Ext functor. The study of cotorsion pairs is especially relevant to the study of covers and envelopes, particularly in the proof of the flat cover conjecture [6].

Given that the category of chain complexes plays a crucial role in homological algebra, introducing a model structure on this category is fundamental and significant. If $\mathcal{A}$ is an abelian category, the category $\mathbb{C}(\mathcal{A})$ (the category of complexes on $\mathcal{A}$) will also be an abelian category. Therefore, Hovey's theorem allows us to construct new model structures on this category by introducing new Hovey pairs. Many papers have been published in this area, including [3, 5, 8, 11, 12, 17–20, 31, 35].

As a generalization of the concept of complexes, the notion of N-complexes was introduced by Mayer [28] in the his study of simplicial complexes and its homological theory was studied by Kapranov and Dubois-Violette in [14, 27]. In recent years, many authors have generalized key concepts in homology, such as derived categories and homotopy categories, for N-complexes, see [2, 4, 21, 24–26, 33], and [34]. By an N-complex $\mathbf{X}$, we mean a sequence $\cdots \to X^{n-1} \to X^n \to X^{n+1} \to \cdots$ such that composition of any N consecutive maps gives the zero map in $\mathcal{A}$.

Gillespie introduced cotorsion pairs in [17–19], some of which formed Hovey pairs on chain complexes ($N = 2$). As a result, new model structures on this category were presented. His approach was to construct cotorsion pairs on $\mathbb{C}(\mathcal{A})$ from a hereditary cotorsion pair $(\mathcal{F}, \mathcal{C})$ in the abelian category $\mathcal{A}$ with enough $\mathcal{F}$-objects and enough $\mathcal{C}$-objects. Utilizing this idea, Yang and Cao introduced a series of these cotorsion pairs on the category of N-complexes in 2017. In the introduction of these cotorsion pairs, two conditions-completeness and hereditary property were essential in order to construct Hovey pairs in $\mathbb{C}_N(\mathcal{A})$. In 2020, Bahiraei considered the Grothendieck category $\mathcal{G}$, endowed with a faithful functor $U : \mathcal{G} \to \text{Set}$ instead of the Abelian category $\mathcal{A}$, which possesses enough $\mathcal{F}$-objects and $\mathcal{C}$-objects. He employed a different technique to construct cotorsion pairs in $\mathbb{C}_N(\mathcal{G})$ based on two cotorsion pairs cogenerated by a set in $\mathcal{G}$. This construction eliminated the hereditary condition from the paper by Yang and Cao, making their results a special case of Bahiraei's findings. Building on these results, he was able to regard $\mathcal{G}$ as the category of quasi-coherent sheaves on a scheme, which may not necessarily have enough $\mathcal{F}$-objects. However, he did not investigate which of these cotorsion pairs form Hovey pairs.

There are three important pairs of cotorsion pairs introduced in both [2] and [33]. In this paper, we aim to investigate which of these cotorsion pairs form Hovey pairs. These cotorsion pairs were previously shown to be Hovey pairs by Yang and Cao in [30, Corollary 3.14, Theorem 4.10] under the conditions of being complete and hereditary for the cotorsion pair $(\mathcal{F}, \mathcal{C})$. We employ a different technique to generalize their results. In fact, for these pairs of cotorsion pairs, we omit the hereditary conditions, the conditions of having enough $\mathcal{F}$-objects as well as the condition of being closed under direct limits for the class $\mathcal{F}$. More precisely:

Theorem 1. *Let $(\mathcal{F},\mathcal{C})$ be a cotorsion pair cogenerated by a set in a concrete Grothendieck category $\mathcal{G}$ endowed with a faithful functor $U : \mathcal{G} \to Set$ and such that $\mathcal{F}$ contains a generator G of $\mathcal{G}$. Then the pairs*

(1) $(\mathbb{C}_N(\mathcal{F}), \mathbb{C}_N(\mathcal{F})^{\perp})$ and $(ex\widetilde{\mathcal{F}}_N, (ex\widetilde{\mathcal{F}}_N)^{\perp})$ and

(1) $(^{\perp}\mathbb{C}_N(\mathcal{C}), \mathbb{C}_N(\mathcal{C}))$ and $(^{\perp}(ex\widetilde{\mathcal{C}}_N), ex\widetilde{\mathcal{C}}_N)$

are Hovey pairs. If furthermore $\mathcal{F}$ is closed under taking kernel of epimorphisms, then the pairs $(\widetilde{\mathcal{F}}_N, dg\widetilde{\mathcal{C}}_N)$ and $(dg\widetilde{\mathcal{F}}_N, \widetilde{\mathcal{C}}_N)$ are a Hovey pair.

We will also examine categories that do not satisfy the conditions of the theorems by Yang and Cao, but for which our theorems can be applied. The main focus of our paper is on categories of Cartesian modules over small categories and categories of quasi-coherent sheaves on a scheme $\mathbb{X}$. The paper is organized as follows: In Section 2, we recall some general concepts related to N-complexes and provide any background information needed throughout this paper. Our main result appears in Section 3 as Theorem 4. This result generalizes [33, Theorem 3.13] and [33, Propositions 4.8 and 4.9]. Finally, in Section 4, we will examine examples to show our framework and highlight the differences between our work and that of Yang and Cao.

2 Preliminaries

2.1 The category of N-complexes

Let $\mathcal{C}$ be an additive category. We fix a positive integer $N \geq 2$. An N-complex is a diagram

$$\cdots \xrightarrow{d_{\mathbf{X}}^{i-1}} X^i \xrightarrow{d_{\mathbf{X}}^{i}} X^{i+1} \xrightarrow{d_{\mathbf{X}}^{i+1}} \cdots$$

with $X^i \in \mathcal{C}$ and morphisms $d_{\mathbf{X}}^i \in \mathrm{Hom}_{\mathcal{C}}(X^i, X^{i+1})$ satisfy $d^N = 0$. That is, composing any N-consecutive maps gives 0. A morphism between N-complexes is a commutative diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d_{\mathbf{X}}^{i-1}} & X^i & \xrightarrow{d_{\mathbf{X}}^{i}} & X^{i+1} & \xrightarrow{d_{\mathbf{X}}^{i+1}} & \cdots \\ & & \downarrow f^i & & \downarrow f^{i+1} & & \\ \cdots & \xrightarrow{d_{\mathbf{Y}}^{i-1}} & Y^i & \xrightarrow{d_{\mathbf{Y}}^{i}} & Y^{i+1} & \xrightarrow{d_{\mathbf{Y}}^{i+1}} & \cdots \end{array}$$

We denote by $\mathbb{C}_N(\mathcal{C})$ the category of unbounded N-complexes on $\mathcal{C}$.

Remark 1. *It is easy to check that if $\mathcal{A}$ is an abelian category then $\mathbb{C}_N(\mathcal{A})$ is again abelian.*

For any object M of $\mathcal{C}$ and any j and $1 \leq i \leq N$, let

$$D_i^j(M) : \cdots \longrightarrow 0 \longrightarrow X^{j-i+1} \xrightarrow{d_{\mathbf{X}}^{j-i+1}} \cdots \xrightarrow{d_{\mathbf{X}}^{j-2}} X^{j-1} \xrightarrow{d_{\mathbf{X}}^{j-1}} X^j \longrightarrow 0 \longrightarrow \cdots$$

be an N-complex satisfying $X^n = M$ and $d_{\mathbf{X}}^n = 1_M$ for all $j - i + 1 \leq n \leq j$.

Notation 1. For $0 \leq r < N$ and $i \in \mathbb{Z}$, we define

$$d^i_{\mathbf{X},\{r\}} := d^{i+r-1}_{\mathbf{X}} \cdots d^i_{\mathbf{X}}$$

In this notation $d^i_{\{1\}} = d^i$ and $d^i_{\{0\}} = 1_{X_i}$.

Definition 1. *A morphism $f : \mathbf{X} \longrightarrow \mathbf{Y}$ of N-complexes is called null-homotopic if there exists $s^i \in \mathrm{Hom}_{\mathcal{C}}(X^i, Y^{i-N+1})$ such that*

$$f^i = \sum_{j=0}^{N-1} d^{i-(N-1-j)}_{\mathbf{Y},\{N-1-j\}} s^{i+j} d^i_{\mathbf{X},\{j\}}$$

We denote by $\mathbb{K}_N(\mathcal{C})$ the homotopy category of unbounded N-complexes on $\mathcal{C}$.

Remark 2. *Let $\mathcal{S}_N(\mathcal{C})$ be the collection of short exact sequences in $\mathbb{C}_N(\mathcal{C})$ with split short exact sequences in each degree. It is shown in [24] that $(\mathbb{C}_N(\mathcal{C}), \mathcal{S}_N(\mathcal{C}))$ is a Frobenius category and its stable category is the homotopy category $\mathbb{K}_N(\mathcal{C})$ of $\mathcal{C}$. So $\mathbb{K}_N(\mathcal{C})$ is a triangulated category, see [24, Theorem 2.3].*

Definition 2. *Let $\mathbf{X}$ be an N-complex of objects of $\mathcal{C}$ as $\cdots \xrightarrow{d^{i-1}_{\mathbf{X}}} X^i \xrightarrow{d^i_{\mathbf{X}}} X^{i+1} \xrightarrow{d^{i+1}_{\mathbf{X}}} \cdots$. We define*

$$\mathrm{Z}^i_r(\mathbf{X}) := \mathrm{Ker}(d^{i+r-1}_{\mathbf{X}} \cdots d^i_{\mathbf{X}}), \quad \mathrm{B}^i_r(\mathbf{X}) := \mathrm{Im}(d^{i-1}_{\mathbf{X}} \cdots d^{i-r}_{\mathbf{X}})$$
$$\mathrm{C}^i_r(\mathbf{X}) := \mathrm{Coker}(d^{i-1}_{\mathbf{X}} \cdots d^{i-r}_{\mathbf{X}}), \quad \mathrm{H}^i_r(\mathbf{X}) := \mathrm{Z}^i_r(\mathbf{X})/\mathrm{B}^i_{N-r}(\mathbf{X})$$

Therefore in each degree we have $N-1$ cycle and clearly $\mathrm{Z}^n_N(\mathbf{X}) = X^n$

Definition 3. *Let $\mathbf{X} \in \mathbb{C}_N(\mathcal{C})$. We say $\mathbf{X}$ is N-exact if $\mathrm{H}^i_r(\mathbf{X}) = 0$ for each $i \in \mathbb{Z}$ and all $r = 1, 2, \ldots, N-1$. We denote the full subcategory of $\mathbb{C}_N(\mathcal{C})$ consisting of all N-exact complexes by $\mathcal{E}_N$.*

Definition 4. *A morphism $f : \mathbf{X} \longrightarrow \mathbf{Y}$ in $\mathbb{K}_N(\mathcal{C})$ is called a quasi-isomorphism if the induced morphism $\mathrm{H}^i_r(f) : \mathrm{H}^i_r(\mathbf{X}) \longrightarrow \mathrm{H}^i_r(\mathbf{Y})$ is an isomorphism for any i and r. The derived category $\mathbb{D}_N(\mathcal{C})$ of N-complexes is defined by the quotient category $\mathbb{K}_N(\mathcal{C})/\mathcal{E}_N$.*

2.2 Cotorsion pair

Definition 5. *A pair of classes $(\mathcal{F}, \mathcal{C})$ in abelian category $\mathcal{A}$ is a cotorsion pair if the following conditions hold:*

1. $\mathrm{Ext}^1_{\mathcal{A}}(F, C) = 0$ *for all* $F \in \mathcal{F}$ *and* $C \in \mathcal{C}$.
2. *If* $\mathrm{Ext}^1_{\mathcal{A}}(F, X) = 0$ *for all* $F \in \mathcal{F}$, *then* $X \in \mathcal{C}$.
3. *If* $\mathrm{Ext}^1_{\mathcal{A}}(Y, C) = 0$ *for all* $C \in \mathcal{C}$, *then* $Y \in \mathcal{F}$.

We think of a cotorsion pair $(\mathcal{F}, \mathcal{C})$ as being "orthogonal with respect to $\mathrm{Ext}^1_{\mathcal{A}}$. This is often expressed with the notation $\mathcal{F}^{\perp} = \mathcal{C}$ and $\mathcal{F} = {}^{\perp}\mathcal{C}$.

Definition 6. *A cotorsion pair $(\mathcal{F}, \mathcal{C})$ in $\mathcal{A}$ is called complete if it has enough projectives and enough injectives. That is, for every $A \in \mathcal{A}$, there exist exact sequences*

$$0 \to Y \to W \to A \to 0 \quad \text{and} \quad 0 \to A \to Y' \to W' \to 0,$$

where $W, W' \in \mathcal{F}$ and $Y, Y' \in \mathcal{C}$.

Definition 7. *A cotorsion pair $(\mathcal{F}, \mathcal{C})$ in $\mathcal{A}$ is cogenerated by a set if there is a set $\mathcal{S} \subseteq \mathcal{F}$ such that $\mathcal{S}^{\perp} = \mathcal{C}$.*

We note that if $\mathcal{S}$ is any class of objects of $\mathcal{D}$ and if $\mathcal{S}^{\perp} = \mathcal{B}$ and $\mathcal{D} = {}^{\perp}\mathcal{B}$, then $(\mathcal{D}, \mathcal{B})$ is a cotorsion pair that is cogenerated by $\mathcal{S}$.

Definition 8. *A cotorsion pair $(\mathcal{F}, \mathcal{C})$ in $\mathcal{A}$ is called hereditary if for all $n \geq 1$ and all $F \in \mathcal{F}$ and $C \in \mathcal{C}$ we have $\operatorname{Ext}^n_{\mathcal{A}}(F, C) = 0$.*

Definition 9. *Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair.*

(1) If $\mathcal{F}$ is closed under kernels of epimorphisms, then $\mathcal{F}$ is called resolving.

(2) If $\mathcal{C}$ is closed under cokernels of monomorphisms, then $\mathcal{C}$ is called coresolving.

The following lemma is well known:

Lemma 1. *Let $\mathcal{A}$ be an abelian category and $(\mathcal{F}, \mathcal{C})$ a cotorsion pair.*

(1) Assume $\mathcal{A}$ has enough injectives and $\mathcal{C}$ is coresolving. Then $(\mathcal{F}, \mathcal{C})$ is hereditary.

(2) Assume $\mathcal{A}$ has enough projectives and $\mathcal{F}$ is resolving. Then $(\mathcal{F}, \mathcal{C})$ is hereditary.

Lemma 2. *Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair cogenerated by a set in a Grothendieck category $\mathcal{G}$ such that $\mathcal{F}$ contains a generator G of $\mathcal{G}$. Then the following are equivalent:*

(1) $\mathcal{F}$ is resolving and $\mathcal{C}$ is coresolving.

(2) $\operatorname{Ext}^n_{\mathcal{G}}(F, C) = 0$ for any $n > 0$ and any $F \in \mathcal{F}$ and $C \in \mathcal{C}$.

Proof. (1) $\Rightarrow$ (2): We will prove it by induction on n. Obviously $\operatorname{Ext}^1_{\mathcal{G}}(F, C) = 0$ for all $F \in \mathcal{F}$ and $C \in \mathcal{C}$. Suppose that $k > 0$ and $\operatorname{Ext}^k_{\mathcal{G}}(F, C) = 0$ for all $F \in \mathcal{F}$ and $C \in \mathcal{C}$. Let $F \in \mathcal{F}$ and $C \in \mathcal{C}$ be arbitrary but fix. Consider the short exact sequence $0 \to C \to I \to D \to 0$ where I is injective. Note that $I \in \mathcal{C}$, hence $D \in \mathcal{C}$. Now apply the functor $\operatorname{Hom}_{\mathcal{G}}(F, -)$ we have the exact sequence

$$\cdots \operatorname{Ext}^k_{\mathcal{G}}(F, D) \to \operatorname{Ext}^{k+1}_{\mathcal{G}}(F, C) \to \operatorname{Ext}^{k+1}_{\mathcal{G}}(F, I) \to \cdots$$

But by induction hypothesis and I is injective we have $\operatorname{Ext}^k_{\mathcal{G}}(F, D) = 0$ and $\operatorname{Ext}^{k+1}_{\mathcal{G}}(F, I) = 0$. So we are done.

(2) $\Rightarrow$ (1) is easy.

□

Definition 10. *A cotorsion pair $(\mathcal{F}, \mathcal{C})$ in $\mathcal{A}$ is said to be perfect if every object X of $\mathcal{A}$ has an $\mathcal{F}$-cover and a $\mathcal{C}$-envelope.*

It is clear by its definition that a perfect cotorsion pair (A, B) is complete. The converse holds when A is closed under direct limits [16, Corollary 2.3.7].

2.3 Hill lemma

Let $\mathcal{G}$ be a Grothendieck category endowed with a faithful functor $U : \mathcal{G} \to \mathbf{Set}$, where **Set** denotes the category of sets. By abuse of notation, we write $x \in G$ instead of $x \in U(G)$, for any object G in $\mathcal{G}$. Analogously, $|G|$ will denote the cardinality of $U(G)$. We will also assume that there exists an infinite regular cardinal λ such that for each $G \in \mathcal{G}$ and any set $S \subseteq U(G)$ (which we will write as $S \subseteq G$ for convenience) with $|S| < \lambda$, there is a subobject $X \subseteq G$ such that $S \subseteq X \subseteq G$ and $|X| < \lambda$. So in our setting if $\mathcal{G} = Qco(\mathbb{X})$, the category of quasi-coherent sheaves on a scheme $\mathbb{X}$, then $U(F) = \sqcup_{v \in \mathcal{V}} F(v)$, where $\mathcal{V}$ is a fixed open affine cover of X. Note that the existence of such cardinal in $Qco(\mathbb{X})$ is proved in [9, Proposition 3.3]. In the category of R-modules, we can consider a cardinal λ that is larger than both $\aleph_0$ and $|R|$.

Definition 11. *Let κ be an infinite regular cardinal.*

(1) A poset is called κ-directed provided that every subset of cardinality smaller than κ has an upper bound. A diagram whose scheme is a κ-directed poset is called a κ-directed diagram, and its colimit is called a κ-directed colimit.

(2) An object $X \in \mathcal{G}$ is called κ-presentable if the functor $\mathrm{Hom}_{\mathcal{G}}(X, -) : \mathcal{G} \to \boldsymbol{Ab}$ *preserves κ-directed colimits. An object $X \in \mathcal{G}$ is called κ-generated whenever* $\mathrm{Hom}_{\mathcal{G}}(X, -)$ *preserves κ-directed colimits of monomorphisms.*

By [1, Example 1.14 and Example 1.68] one can see that

$$|X| < \lambda \quad \Leftrightarrow \quad X \text{ is } \lambda\text{-presentable} \quad \Leftrightarrow \quad X \text{ is } \lambda\text{-generated}$$

Definition 12. *Let $\mathcal{S}$ be a class of objects of $\mathcal{G}$. An object $X \in \mathcal{G}$ is called $\mathcal{S}$-filtered if there exists a well-ordered direct system $(X_\alpha, i_{\alpha\beta} | \alpha < \beta \leq \sigma)$ indexed by an ordinal number σ such that*

(a) $X_0 = 0$ and $X_\sigma = X$,

(b) For each limit ordinal $\mu \leq \sigma$, the colimit of system $(X_\alpha, i_{\alpha\beta} | \alpha < \beta \leq \mu)$ is precisely X_μ, the colimit morphisms being $i_{\alpha\mu} : X_\alpha \to X_\mu$,

(c) $i_{\alpha\beta}$ is a monomorphism in $\mathcal{G}$ for each $\alpha < \beta \leq \sigma$,

(d) $\mathrm{Coker} i_{\alpha\alpha+1} \in \mathcal{S}$ for each $\alpha < \sigma$

The direct system $(X_\alpha, i_{\alpha\beta})$ is then called an $\mathcal{S}$-filtration of X. The class of all $\mathcal{S}$-filtered objects in $\mathcal{G}$ is denoted by Filt-$\mathcal{S}$.

The Hill Lemma is a way of creating a plentiful supply of a module with a given filtration, where these submodules have nice properties, see [22] and [15]. In the following we state the Hill Lemma for Grothendieck category which is known as the generalized Hill Lemma, see [31, Theorem 2.1].

Theorem 2. *Let $\mathcal{G}$ be as above and κ be a regular infinite cardinal such that $\kappa \geq \lambda$. Suppose that $\mathcal{S}$ is a set of κ-presentable objects and X is an object possessing an $\mathcal{S}$-filtration $(X_\alpha \mid \alpha \leq \sigma)$ for some ordinal σ. Then there is a complete sublattice $\mathcal{L}$ of $(\mathcal{P}(\sigma), \cup, \cap)$ and $\ell : \mathcal{L} \to Subobj(X)$ which assigns to each $S \in \mathcal{L}$ a subobject $\ell(S)$ of X, such that the following hold:*

(H1) For each $\alpha \leq \sigma$ *we have* $\alpha = \{\gamma \mid \gamma < \alpha\} \in \mathcal{L}$ *and* $\ell(\alpha) = X_\alpha$.

(H2) If $(S_i)_{i\in I}$ *is a family of elements of* $\mathcal{L}$*, then* $\ell(\cup S_i) = \sum \ell(S_i)$ *and* $\ell(\cap S_i) = \cap \ell(S_i)$.

(H3) If $S, T \in \mathcal{L}$ *are such that* $S \subseteq T$*, then the object* $N = \ell(T)/\ell(S) \in$ *Filt-*$\mathcal{S}$.

(H4) For each κ*-presentable subobject* $Y \subseteq X$*, there is* $S \in \mathcal{L}$ *of cardinal* $< \kappa$*(so* $\ell(S)$ *is* κ*-presentable by (H3)) such that* $Y \subseteq \ell(S) \subseteq X$.

Let $\mathcal{H} = \{\ell(S) \mid S \in \mathcal{L}\}$. We call $\mathcal{H}$ as the Hill Class of subobjects of X relative to κ.

Corollary 1. *If* $N \in \mathcal{H}$ *and* M *is a* κ*-presentable subobject of* X*, then there exists* $P \in \mathcal{H}$ *such that* $N + M \subseteq P$ *and* P/N *is* κ*-presentable.*

Proof. By using Theorem 2 (H4), we can find $S \in \mathcal{L}$ of cardinal $< \kappa$ such that $M \subseteq \ell(S)$. Denoting $W = \ell(S)$, $P = N+W$ and combining (H2) and (H3) of Theorem 2 with [31, Corollary A.5], we observe that $P \in \mathcal{H}$ and P/N is κ-presentable. □

Theorem 3. *Let* κ *be an uncountable regular cardinal such that* $\kappa > \lambda$*. Let* $(\mathcal{F}, \mathcal{C})$ *be a cotorsion pair in* $\mathcal{G}$ *such that* $\mathcal{F}$ *contains a family of* λ*-presentable generators of* $\mathcal{G}$*. Then the following conditions are equivalent:*

(1) The cotorsion pair $(\mathcal{F}, \mathcal{C})$ *is cogenerated by a class of* κ*-presentable objects in* $\mathcal{G}$.

(2) Every object in $\mathcal{F}$ *is* $\mathcal{F}^\kappa$*-filtered, where* $\mathcal{F}^\kappa$ *is the class of all* κ*-presentable objects in* $\mathcal{F}$.

Proof. We refer to [11, Theorem 2.1]. □

We need the following lemma. The idea of the proof is taken from [2, Lemma 2.14]. We provide here the argument for the reader's convenience.

Lemma 3. *Let* κ *be a regular infinite cardinal such that* $\kappa > \lambda$*. Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be* N*-complexes such that* $\mathbf{X} \subseteq \mathbf{Y}$*. For each* $i \in \mathbb{Z}$*, let* M^i *be a* κ*-presentable subobject of* Y^i*. Then there exists* N*-complex* $\mathbf{T}$ *such that* $\mathbf{X} \subseteq \mathbf{T} \subseteq \mathbf{Y}$ *and for each* $i \in \mathbb{Z}$*,* $M^i + X^i \subseteq T^i$ *and the object* T^i/X^i *is* κ*-presentable.*

Proof. We use the zig-zag technique to construct $\mathbf{T}$. First, consider the particular case $\mathbf{X} = 0$. We will construct $\mathbf{T}$ as the union of an increasing sequence of N-subcomplexes

$$\mathbf{C}_0 \subseteq \mathbf{C}_1 \subseteq \mathbf{C}_2 \subseteq \cdots$$

of $\mathbf{Y}$ where $M^i \subseteq C_0^i$, $|C_n^i| \leq \kappa$ for all $i, n \in \mathbb{Z}$. Then if $\mathbf{T} = \cup_{\mathbf{n}\in\mathbb{Z}}\mathbf{C_n}$, then $\mathbf{T} \subseteq \mathbf{Y}$ with $|T^i| < \kappa$ and $M^i \subseteq T^i$ for each $i \in \mathbb{Z}$. In this case clearly $M^i \subseteq T^i$ and $|\mathbf{T}| \leq \kappa$, since $|\mathbf{C}_n| \leq \kappa$. Let $\mathbf{C}_0 = (C_0^i)$ be such that $C_0^i = M^i + \sum_{k=1}^{N-1} d_{\{N-k\}}^{i-N+k}(M^{i-N+k})$. Then $\mathbf{C}_0$ is an N-subcomplex of $\mathbf{Y}$ and clearly $M^i \subseteq C_0^i$ and $|\mathbf{C}_0| \leq \kappa$. Having constructed $\mathbf{C}_n$ with $|\mathbf{C}_n| \leq \kappa$, we construct $\mathbf{C}_{n+1}$ with $\mathbf{C}_n \subseteq \mathbf{C}_{n+1}$ such that $|\mathbf{C}_{n+1}| \leq \kappa$.

By our assumption on κ, consider a subobject $S^{i-N+r} \subseteq Y^{i-N+r}$ for all $1 \leq r \leq N-1$. Now define $C_{n+1}^i = C_n^i + S^i + \sum_{k=1}^{N-1} d_{\{N-k\}}^{i-N+k}(S^{i-N+k})$ for all $i \in \mathbb{Z}$. Clearly $\mathbf{C}_n \subseteq \mathbf{C}_{n+1}$. Hence we have the desire $\mathbf{T} \subseteq \mathbf{Y}$.

In case $\mathbf{X} \neq 0$ let $\overline{\mathbf{Y}} = \mathbf{Y}/\mathbf{X}$ and $\overline{M^i} = \underline{(M^i + X^i)}/X^i$. According to the previous part, there is an N-complex $\overline{\mathbf{T}} \subseteq \overline{\mathbf{Y}}$, and for each $i \in \mathbb{Z}$, $\overline{M^i} \subseteq \overline{T^i}$, and the object $\overline{T^i}$ is κ-presentable. Then $\overline{\mathbf{T}} = \mathbf{T}/\mathbf{X}$ for an N-subcomplex $\mathbf{X} \subseteq \mathbf{T} \subseteq \mathbf{Y}$, and $\mathbf{T}$ clearly has the required properties. □

3 Hovey pairs in the category of N-complexes

In this section, we show that the cotorsion pairs introduced by Bahiraei in [2, Corollary 3.12] form Hovey pairs. Yang and Cao in [33, Corollary 3.14, Theorem 4.10] proved that these pairs of cotorsion pairs form hovey pairs when $(\mathcal{F}, \mathcal{C})$ is a complete hereditary cotorsion pair in the abelian category $\mathcal{A}$. We have omitted hereditary condition and the conditions of having enough $\mathcal{F}$-objects and $\mathcal{C}$-objects here. Moreover, the requirement that the class $\mathcal{F}$ is closed under direct limits is not necessary in our theorem. To begin, we will first recall the classes and cotorsion pairs from [2].

Definition 13. *Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair in $\mathcal{G}$. Let $\mathcal{E}_N$ be a class of N-exact complexes. We will consider the following subclasses of $\mathbb{C}_N(\mathcal{G})$:*

(1) The class of $\mathbb{C}_N(\mathcal{F})$ complexes (resp. $\mathbb{C}_N(\mathcal{C})$ complexes), consisting of all $\mathbf{X} \in \mathbb{C}_N(\mathcal{G})$ such that $X^i \in \mathcal{F}$ (resp. $X^i \in \mathcal{C}$) for each i.

(2) The class of $\mathcal{F}$-N-complex, that we denote by $\widetilde{\mathcal{F}}_N$, consisting of all $\mathbf{X} \in \mathcal{E}_N$ such that $\mathrm{Z}^i_r(\mathbf{X}) \in \mathcal{F}$ for all r, i.

(3) The class of $\mathcal{C}$-N-complex, that we denote by $\widetilde{\mathcal{C}}_N$, consisting of all $\mathbf{X} \in \mathcal{E}_N$ such that $\mathrm{Z}^i_r(\mathbf{X}) \in \mathcal{C}$ for all r, i.

(4) The class of dg-$\mathcal{F}$-N-complexes, that we denote by $\mathrm{dg}\widetilde{\mathcal{F}}_N$, consisting of all $\mathbf{X} \in \mathbb{C}_N(\mathcal{F})$ such that $\mathrm{Hom}_{\mathbb{K}_N(\mathcal{G})}(\mathbf{X}, \mathbf{C}) = 0$ whenever $\mathbf{C} \in \widetilde{\mathcal{C}}_N$.

(4) The class of dg-$\mathcal{C}$-N-complexes, that we denote by $\mathrm{dg}\widetilde{\mathcal{C}}_N$, consisting of all $\mathbf{X} \in \mathbb{C}_N(\mathcal{C})$ such that $\mathrm{Hom}_{\mathbb{K}_N(\mathcal{G})}(\mathbf{F}, \mathbf{X}) = 0$ whenever $\mathbf{F} \in \widetilde{\mathcal{F}}_N$.

(5) The class $ex_N(\mathcal{F}) = \mathbb{C}_N(\mathcal{F}) \cap \mathcal{E}_N$ (resp. $ex_N(\mathcal{C}) = \mathbb{C}_N(\mathcal{C}) \cap \mathcal{E}_N$).

In the following, we will introduce Hovey pairs in the category of N-complexes by using these classes. We will reiterate that these pairs were introduced by Yang and Cao in [33], but here we have omitted the hereditary condition. Additionally, the examination of how these pairs form complete cotorsion pairs in a more general context is shown in the article [2]. However, for the convenience of the reader, we will present it here in our case and then show that these pairs form Hovey pairs. The technique used in the article [2] to prove the completeness of cotorsion pairs is the use of Hill Lemma.

Theorem 4. *Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair cogenerated by a set in a concrete Grothendieck category $\mathcal{G}$ as above and such that $\mathcal{F}$ contains a generator G of $\mathcal{G}$. Then the induced pairs*

(1) $(\widetilde{\mathcal{F}}_N, dg\widetilde{\mathcal{C}}_N)$ and $(dg\widetilde{\mathcal{F}}_N, \widetilde{\mathcal{C}}_N)$

(2) $(\mathbb{C}_N(\mathcal{F}), \mathbb{C}_N(\mathcal{F})^{\perp})$ and $({}^{\perp}\mathbb{C}_N(\mathcal{C}), \mathbb{C}_N(\mathcal{C}))$

(3) $(ex\widetilde{\mathcal{F}}_N, (ex\widetilde{\mathcal{F}}_N)^{\perp})$ and $({}^{\perp}(ex\widetilde{\mathcal{C}}_N), ex\widetilde{\mathcal{C}}_N)$

are complete cotorsion pairs.

Proof. Let us see (1). By [2, Corollary 3.7] we have induced cotorsion pairs $(\widetilde{\mathcal{F}}_N, dg\widetilde{\mathcal{C}}_N)$ and $(dg\widetilde{\mathcal{F}}_N, \widetilde{\mathcal{C}}_N)$. By [2, Proposition 3.11] and [23, corollary 6.6] we have the completeness of the pair $(dg\widetilde{\mathcal{F}}_N, \widetilde{\mathcal{C}}_N)$. In order to show that $(\widetilde{\mathcal{F}}_N, dg\widetilde{\mathcal{C}}_N)$ is complete, we first prove that it is cogenerated by a set. To this point by Theorem 3 it is enough to show that each N-complex $\mathbf{F} \in \widetilde{\mathcal{F}}_N$ is $\widetilde{\mathcal{F}}_N^{\kappa}$-filtered(for some $\kappa \geq \lambda$ regular uncountable) i.e. we construct a filtration $(\mathbf{F}_\alpha \mid \alpha \leq \sigma)$ for $\mathbf{F}$ such that $\mathbf{F}_{\alpha+1}/\mathbf{F}_\alpha \in \widetilde{\mathcal{F}}_N^{\kappa}$. Let $\mathbf{F} = (F^i) \in \widetilde{\mathcal{F}}_N$. By definition $\mathbf{F}$ is an N-exact complex and $\mathrm{Z}_r^i(\mathbf{F}) \in \mathcal{F}$ for $i \in \mathbb{Z}$ and $1 \leq r \leq N-1$. Moreover, we have $\mathrm{Z}_N^i(\mathbf{F}) = F^i \in \mathcal{F}$. By assumption $\mathrm{Z}_r^i(\mathbf{F})$ has $\mathcal{F}^{\kappa}$- filtration $\mathcal{M}_{i,r} = (M_\alpha^{i,r} \mid \alpha \leq \sigma_{i,r})$ for each $i \in \mathbb{Z}$, $1 \leq r \leq N$. Using Hill Lemma, we obtain the corresponding families $\mathcal{H}^{i,r}$ for these filtrations.

Now, we recursively construct a filtration $(\mathbf{F}_\alpha \in \widetilde{\mathcal{F}}_N \mid \alpha \leq \sigma)$ for $\mathbf{F}$ with the property that, for each $\alpha < \sigma, i \in \mathbb{Z}$ and $1 \leq r \leq N$, the object $\mathrm{Z}_r^i(\mathbf{F}_\alpha)$ belongs to $\mathcal{H}^{i,r}$. First, put $\mathbf{F}_0 = 0$. If α is a limit ordinal and $\mathbf{F}_\beta$ is already defined for each $\beta < \alpha$, we simply put $\mathbf{F}_\alpha = \bigcup_{\beta<\alpha} \mathbf{F}_\beta$. This is again an N-exact complex and, by the properties of Hill families, we have $\mathrm{Z}_r^i(\mathbf{F}_\alpha) \in \mathcal{H}^{i,r}$ for all $i \in \mathbb{Z}$ and $1 \leq r \leq N$. We proceed to the crucial isolated step. Let $\mathbf{F}_\alpha$ be defined and assume that $\mathbf{F}_\alpha \neq \mathbf{F}$ (otherwise, we set $\sigma = \alpha$ and we are done). Put $\mathbf{G}_0 = \mathbf{F}_\alpha$.

For each $i \in \mathbb{Z}$, fix some $M_0^i \in \mathcal{H}^{i,N}$ such that $G_0^i \subseteq M_0^i$, M_0^i/G_0^i is κ-presentable and, if possible, $G_0^i \subsetneq M_0^i$. Assuming that M_n^i is defined for some nonnegative integer n and all $i \in \mathbb{Z}$, and M_n^i/G_0^i is κ-presentable, the objects $(M_n^i \cap \mathrm{Z}_r^i(\mathbf{F}))/\mathrm{Z}_r^i(\mathbf{F}_\alpha)$ are κ-presentable as well for all $1 \leq r \leq N-1$. Hence we can find $Z_n^{i,r} \in \mathcal{H}^{i,r}$, $1 \leq r < N$, such that $M_n^i \cap \mathrm{Z}_r^i(\mathbf{F}) \subseteq Z_n^{i,r}$ and $Z_n^{i,r}/\mathrm{Z}_r^i(\mathbf{F}_\alpha)$ is again κ-presentable. We define $M_{n+1}^i \in \mathcal{H}^{i,N}$ in such a way that $M_n^i \cup \bigcup_{r=1}^{N-1} Z_n^{i,r} \subseteq M_{n+1}^i$ and M_{n+1}^i/M_n^i is κ-presentable. This is possible by the properties of the Hill family $\mathcal{H}^{i,N}$. Consequently, M_{n+1}^i/G_0^i is κ-presentable. For each $i \in \mathbb{Z}$, put $M^i = \bigcup_{n=0}^{\infty} M_n^i$. Then M^i/G_0^i is κ-presentable. Moreover, $M^i \cap \mathrm{Z}_r^i(\mathbf{F}) = \bigcup_{n=0}^{\infty} Z_n^{i,r} \in \mathcal{H}^{i,r}$ for each $i \in \mathbb{Z}$ and $1 \leq r \leq N-1$ and $M^i = \bigcup_{n=0}^{\infty} M_n^i \in \mathcal{H}^{i,N}$.

Now, we use [2, Lemma 2.14] to obtain an N-exact complex $\mathbf{G}_1$ such that $\mathbf{G}_0 \subseteq \mathbf{G}_1 \subseteq \mathbf{F}$, the quotient Gb_1^i/Gb_0^i is κ-presentable and $M^i \subseteq Gb_1^i$ for each $i \in \mathbb{Z}$. We go back to the beginning of the previous paragraph and repeat the process with $\mathbf{G}_0$ replaced by $\mathbf{G}_1$. Using [2, Lemma 2.14], we obtain $\mathbf{G}_2$ and so on. Finally, we define $\mathbf{F}_{\alpha+1} = \bigcup_{n=0}^{\infty} \mathbf{G}_n$. This is an N-exact complex and, for all $i \in \mathbb{Z}$, $\mathrm{Z}_r^i(\mathbf{F}_{\alpha+1}) = F_{\alpha+1} \cap \mathrm{Z}_r^i(\mathbf{F})$ is the union of elements of the type $M^i \cap \mathrm{Z}_r^i(\mathbf{F}) \in \mathcal{H}^{i,r}$; thus $\mathrm{Z}_r^i(\mathbf{F}_{\alpha+1})$ is an element from $\mathcal{H}^{i,r}$ for all $i \in \mathbb{Z}$ and $1 \leq r \leq N$. Moreover, $F_{\alpha+1}^i/F_\alpha^i$ is κ-presentable.

This finishes the construction of the filtration $(\mathbf{F}_\alpha \mid \alpha \leq \sigma)$. Finally, we observe that, for each $\alpha < \sigma$, the quotient $\mathbf{F}_{\alpha+1}/\mathbf{F}_\alpha$ belongs to $\widetilde{\mathcal{F}}_N^{\kappa}$: here $\mathrm{Z}_r^i(\mathbf{F}_{\alpha+1})/\mathrm{Z}_r^i(\mathbf{F}_\alpha) \in \mathcal{F}$ since $\mathrm{Z}_r^i(\mathbf{F}_\alpha), \mathrm{Z}_r^i(\mathbf{F}_{\alpha+1}) \in \mathcal{H}^{i,r}$ for all $i \in \mathbb{Z}$ and $1 \leq r < N$.

Now we will prove (2). By [33, Proposition 4.1] $(\mathbb{C}_N(\mathcal{F}), \mathbb{C}_N(\mathcal{F})^{\perp})$ and $({}^{\perp}\mathbb{C}_N(\mathcal{C}), \mathbb{C}_N(\mathcal{C}))$ are cotorsion pairs. If $(\mathcal{F}, \mathcal{C})$ is cogenerated by $\{X_n\}_{n \in I_0}$ then $({}^{\perp}\mathbb{C}_N(\mathcal{C}), \mathbb{C}_N(\mathcal{C}))$ is cogenerated by the set $\{D_N^i(X_n) \mid i \in \mathbb{Z}, n \in I_0\}$. Therefore by [23, Corollary 6.6] it is complete. Finally we have to check that $(\mathbb{C}_N(\mathcal{F}), \mathbb{C}_N(\mathcal{F})^{\perp})$ is cogenerated by a set. This will be done by showing that every N-complex $\mathbf{X} \in \mathbb{C}_N(\mathcal{F})$ is $\mathbb{C}_N(\mathcal{F})^{\kappa}$-filtered. The argument to see this is similar to the case (1) to prove that every N-complex $\mathbf{X}$ in $\widetilde{\mathcal{F}}_N$ is $\widetilde{\mathcal{F}}_N^{\kappa}$-filtered. We have just to replace [2, Lemma 2.14] by Lemma 3 in the previous argument.

Let us see (3). By [2, Corollary 3.8] $(ex\widetilde{\mathcal{F}}_N, (ex\widetilde{\mathcal{F}}_N)^{\perp})$ and $({}^{\perp}(ex\widetilde{\mathcal{C}}_N), ex\widetilde{\mathcal{C}}_N)$ are cotorsion pairs. To see that the first one is complete it is enough to show that every complex in $ex\widetilde{\mathcal{F}}_N$ is

$ex\widetilde{\mathcal{F}}_N^\kappa$-filtered. Again this is achieved by following the same argument of the case (1) to see that every N-complex $\mathbf{X}$ in $\widetilde{\mathcal{F}}_N$ is $\widetilde{\mathcal{F}}_N^\kappa$-filtered. Finally, if $(\mathcal{F}, \mathcal{C})$ is cogenerated by $\{X_n\}_{n\in I_0}$ then $({}^\perp(ex\widetilde{\mathcal{C}}_N), ex\widetilde{\mathcal{C}}_N)$ is cogenerated by

$$\mathcal{S} = \{D_r^i(G)|\ i \in \mathbb{Z},\ 1 \leq r \leq N-1\} \cup \{D_r^i(X_n)|\ i \in \mathbb{Z},\ 1 \leq r \leq N-1,\ n \in I_0\}$$

See the proof of [2, Proposition 3.20] for more details. □

Corollary 2. *Let $(\mathcal{F}, \mathcal{C})$ be a complete cotorsion pair in the concrete Grothendieck category $\mathcal{G}$ as above and such that $\mathcal{F}$ contains a generator G of $\mathcal{G}$. If $\mathcal{F}$ is closed under direct limits then the induced pairs*

(1) $(\widetilde{\mathcal{F}}_N, dg\widetilde{\mathcal{C}}_N)$ and $(dg\widetilde{\mathcal{F}}_N, \widetilde{\mathcal{C}}_N)$

(2) $(\mathbb{C}_N(\mathcal{F}), \mathbb{C}_N(\mathcal{F})^\perp)$ and

(3) $(ex\widetilde{\mathcal{F}}_N, (ex\widetilde{\mathcal{F}}_N)^\perp)$

are perfect cotorsion pairs.

Proof. By [10, Theorem 2.6] we know that if $\mathcal{A}$ is a class of objects of a Grothendieck category $\mathcal{C}$ closed under direct sums, extensions and well ordered direct limits and such that the generator of $\mathcal{C}$ is in $\mathcal{A}$ and $(\mathcal{A}, \mathcal{A}^\perp)$ is cogenerated by a set, then every object M in $\mathcal{C}$ has an $\mathcal{A}$-cover and an $\mathcal{A}^\perp$-envelope. Here we can notice that since $\mathcal{F}$ is closed under direct limits, then all classes $\widetilde{\mathcal{F}}_N$, $dg\widetilde{\mathcal{F}}_N$, $\mathbb{C}_N(\mathcal{F})$, and $ex\widetilde{\mathcal{F}}_N$ are also closed under direct limits. So we are done. □

In the next step we will see that the induced cotorsion pairs in Theorem 4 give rise to an abelian model structure in $\mathbb{C}_N(\mathcal{G})$ which the homotopy category of this model structure is equal to $\mathbb{D}_N(\mathcal{G})$. We recall the definition of Hovey pair in $\mathbb{C}_N(\mathcal{G})$.

Definition 14. *Let $\mathcal{G}$ be a Grothendieck category. Suppose that $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{F}, \mathcal{C})$ are two complete cotorsion pairs in $\mathbb{C}_N(\mathcal{G})$. We say that they are compatible (we refer to these two cotorsion pairs as a **Hovey Pair**) if $\mathcal{B} = \mathcal{C} \cap \mathcal{E}_N$ and $\mathcal{F} = \mathcal{A} \cap \mathcal{E}_N$ where $\mathcal{E}_N$ is the class of all N-exact complexes in $\mathbb{C}_N(\mathcal{G})$.*

Now we introduce our main theorem:

Theorem 5. *Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair as in Theorem 4. Then the pairs*

(1) $(\mathbb{C}_N(\mathcal{F}), \mathbb{C}_N(\mathcal{F})^\perp)$ and $(ex\widetilde{\mathcal{F}}_N, (ex\widetilde{\mathcal{F}}_N)^\perp)$ and

(1) $({}^\perp\mathbb{C}_N(\mathcal{C}), \mathbb{C}_N(\mathcal{C}))$ and $({}^\perp(ex\widetilde{\mathcal{C}}_N), ex\widetilde{\mathcal{C}}_N)$

are Hovey pairs. If furthermore $\mathcal{F}$ is closed under taking kernel of epimorphisms, then the pairs $(\widetilde{\mathcal{F}}_N, dg\widetilde{\mathcal{C}}_N)$ and $(dg\widetilde{\mathcal{F}}_N, \widetilde{\mathcal{C}}_N)$ are a Hovey pair.

Proof. We only prove (1) and the proof of (2) is dual. By definition $ex\widetilde{\mathcal{F}_N} = \mathbb{C}_N(\mathcal{F}) \cap \mathcal{E}_N$. We will see that $\mathbb{C}_N(\mathcal{F})^{\perp} = (ex\widetilde{\mathcal{F}_N})^{\perp} \cap \mathcal{E}_N$. Since $dg\widetilde{\mathcal{F}_N} \subseteq \mathbb{C}_N(\mathcal{F})$ and $(dg\widetilde{\mathcal{F}_N}, \widetilde{\mathcal{C}_N})$ is a cotorsion pair, we see that $\mathbb{C}_N(\mathcal{F})^{\perp} \subseteq dg\widetilde{\mathcal{F}_N}^{\perp}$, hence $\mathbb{C}_N(\mathcal{F})^{\perp} \subseteq \widetilde{\mathcal{C}_N}$. So in particular, $\mathbb{C}_N(\mathcal{F})^{\perp} \subseteq \mathcal{E}_N$. Now since $ex\widetilde{\mathcal{F}_N} \subseteq \mathbb{C}_N(\mathcal{F})$ we get that $\mathbb{C}_N(\mathcal{F})^{\perp} \subseteq ex\widetilde{\mathcal{F}_N}^{\perp}$. Therefore, $\mathbb{C}_N(\mathcal{F})^{\perp} \subseteq ex\widetilde{\mathcal{F}_N}^{\perp} \cap \mathcal{E}_N$. Conversely, assume that $\mathbf{X} \in ex\widetilde{\mathcal{F}_N}^{\perp} \cap \mathcal{E}_N$. By Theorem 4 the pair $(\mathbb{C}_N(\mathcal{F}), \mathbb{C}_N(\mathcal{F})^{\perp})$ is complete, so there exists a short exact sequence in $\mathbb{C}_N(\mathcal{G})$

$$0 \to \mathbf{X} \to \mathbf{Y} \to \mathbf{F} \to 0$$

with $\mathbf{Y} \in \mathbb{C}_N(\mathcal{F})^{\perp}$ and $\mathbf{F} \in \mathbb{C}_N(\mathcal{F})$. But $\mathbf{X}$ and $\mathbf{Y}$ are N-exact and $\mathcal{E}_N$ is a thick subcategory of $\mathbb{C}_N(\mathcal{G})$ so $\mathbf{F} \in \mathbb{C}_N(\mathcal{F}) \cap \mathcal{E}_N = ex\widetilde{\mathcal{F}_N}$. So the short exact sequence splits and therefore $\mathbf{X}$ is a direct summand of $\mathbf{Y}$. Since $\mathbb{C}_N(\mathcal{F})^{\perp}$ is closed under direct summands we get that $\mathbf{X} \in \mathbb{C}_N(\mathcal{F})^{\perp}$.

For the last part, we first claim that $\mathrm{Ext}^n_{\mathcal{G}}(F, C) = 0$ for all $n > 0$, $F \in \mathcal{F}$ and $C \in \mathcal{C}$. Clearly $\mathrm{Ext}^1_{\mathcal{G}}(F, C) = 0$. Given any exact sequence

$$0 \to C \to X \to Y \to F \to 0$$

in $\mathrm{Ext}^2_{\mathcal{G}}(F, C)$. Since $\mathcal{F}$ contains a generator of $\mathcal{G}$ we have the exact sequence $0 \to K \to G \to F \to 0$ and by assumption $K \in \mathcal{F}$ (Since $\mathcal{F}$ is closed under kernels of epimorphisms). So $\mathrm{Ext}^1_{\mathcal{G}}(K, C) = 0$, hence $0 \to C \to M \to K \to 0$ splits. But this means that $0 \to C \to M \to G \to F \to 0$ (and therefore $0 \to C \to X \to Y \to F \to 0$) represents the zero element in $\mathrm{Ext}^2_{\mathcal{G}}(F, C)$. Proceeding inductively in this way we get our claim. So by Lemma 2 we can say that $\mathcal{F}$ is resolving and $\mathcal{C}$ is coresolving. Now by [33, Lemma 3.9] and Definition 13 one can see that $\widetilde{\mathcal{F}_N} \subseteq dg\widetilde{\mathcal{F}_N} \cap \mathcal{E}_N$ and $\widetilde{\mathcal{C}_N} \subseteq dg\widetilde{\mathcal{C}_N} \cap \mathcal{E}_N$. Now let $\mathbf{X} \in dg\widetilde{\mathcal{F}_N} \cap \mathcal{E}_N$ (resp. $\mathbf{Y} \in dg\widetilde{\mathcal{C}_N} \cap \mathcal{E}_N$). Then by [33, Lemma 3.4(i)] (resp. [33, Lemma 3.4(ii)]) we have $\mathbf{X} \in \widetilde{\mathcal{F}_N}$ (resp. $\mathbf{Y} \in \widetilde{\mathcal{C}_N}$). So The induced cotorsion pairs $(\widetilde{\mathcal{F}_N}, dg\widetilde{\mathcal{C}_N})$ and $(dg\widetilde{\mathcal{F}_N}, \widetilde{\mathcal{C}_N})$ are a Hovey pair.

□

Note that the previous results are improved versions of [33, Theorem 4.10] and [33, Theorem 3.13]. Essentially we do not assume that the cotorsion pair $(\mathcal{F}, \mathcal{C})$ is complete hereditary. Moreover we also do not assume closure under direct limits in our class $\mathcal{F}$. (See the proof of [33, Proposition 4.4] and [32, Corollary 2.7]). As we shall see in the next section, these improvement are crucial in finding new abelian model structures in $\mathbb{C}_N(\mathcal{G})$ with respect to a various classes that are not in general closed under arbitrary direct limits. By using Theorem 5 and [23, Theorem 2.2] we have the following corollaries:

Corollary 3. *Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair cogenerated by a set in a concrete Grothendieck category $\mathcal{G}$ and such that $\mathcal{F}$ contains a generator G of $\mathcal{G}$. The Hovey pair $(\mathbb{C}_N(\mathcal{F}), \mathbb{C}_N(\mathcal{F})^{\perp})$ and $(ex\widetilde{\mathcal{F}_N}, (ex\widetilde{\mathcal{F}_N})^{\perp})$ induce an abelian model structure in $\mathbb{C}_N(\mathcal{G})$. In this model structure the weak equivalences are the homology isomorphisms, the cofibrations (resp. trivial cofibrations) are the monomorphisms with cokernels in $\mathbb{C}_N(\mathcal{F})$ (resp, $ex\widetilde{\mathcal{F}_N}$), and the fibrations (resp. trivial fibrations) are the epimorphisms whose kernels are in $(ex\widetilde{\mathcal{F}_N})^{\perp}$) (resp. $\mathbb{C}_N(\mathcal{F})^{\perp}$).*

Corollary 4. *Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair cogenerated by a set in a concrete Grothendieck category $\mathcal{G}$ and such that $\mathcal{F}$ contains a generator G of $\mathcal{G}$ and $\mathcal{F}$ is closed under kernels of* *epimorphisms. The Hovey pair* $(\widetilde{\mathcal{F}}_N, dg\widetilde{\mathcal{C}}_N)$ *and* $(dg\widetilde{\mathcal{F}}_N, \widetilde{\mathcal{C}}_N)$ *induce an abelian model structure in* $\mathbb{C}_N(\mathcal{G})$*. In this model structure the weak equivalences are the homology isomorphisms, the cofibrations (resp. trivial cofibrations) are the monomorphisms with cokernels in* $dg\widetilde{\mathcal{F}}_N$ *(resp,* $\widetilde{\mathcal{F}}_N$*), and the fibrations (resp. trivial fibrations) are the epimorphisms whose kernels are in* $dg\widetilde{\mathcal{C}}_N$ *(resp.* $\widetilde{\mathcal{C}}_N$*).*

4 Examples

In this section, we will examine examples where, with the help of cotorsion pairs that do not satisfy the hereditary condition or where the class of objects is not closed under direct limits, a model structure can be constructed on the category of N-complexes.

4.1 Category of R-modules

Let R be a ring and $\mathcal{G}$ be the category of R-modules. Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair cogenerated by a set such that $\mathcal{F}$ is closed under kernels of epimorphisms. Let $M \in \mathcal{G} = \text{Mod-}R$. The projective dimension of M relative to $\mathcal{C}$, which is denoted by $\text{Pd}_{\mathcal{C}}(M)$ is defined as follow:

$$\text{Pd}_{\mathcal{C}}(M) = \min\{n \geq 0 \mid \text{Ext}^j_R(M, C) = 0 \text{ for any } j > n \text{ and } C \in \mathcal{C}\}$$

We denote by $\mathcal{P}^{\leq n}_{\mathcal{C}}$ the class of modules of finite projective dimension $\leq n$ relative to $\mathcal{C}$. By [7, Theorem 2.2, Corollary 2.7] $(\mathcal{P}^{\leq n}_{\mathcal{C}}, \mathcal{P}^{\leq n\perp}_{\mathcal{C}})$ is cogenerated by a set and $\mathcal{P}^{\leq n}_{\mathcal{C}}$ is closed under kernels of epimorphisms. So by Corollaries 3 and 4 we have two model structures on $\mathbb{C}_N(R)$ which the homotopy category of each model structure is $\mathbb{D}_N(R)$ the derived category of N-complexes of R-modules.

Note that in case we consider $(\mathcal{F}, \mathcal{C}) = (\text{Prj-}R, \text{Mod-}R)$ where Prj-R is the class of all projective R-modules, the model structure of Corollary 4 applies in the study of the so-called Finistic Dimension Conjectures.

4.2 Cartesian modules on small categories

A ringed category $(\mathbb{X}, \mathcal{O}_{\mathbb{X}})$ is a pair such that $\mathbb{X}$ is a small category and $\mathcal{O}_{\mathbb{X}}$ is a presheaf of commutative rings on $\mathbb{X}$. Then, the category of presheaves of $\mathcal{O}_{\mathbb{X}}$-modules on $\mathbb{X}$ can be defined. Note that a (not necessarily commutative) ring may be regarded as a special case of a small preadditive category so the category of modules over a preadditive category naturally arises.

Let $\mathcal{C}$ be a small category and consider a representation $R : \mathcal{C} \to \text{Add}$ to be a pseudofunctor from $\mathcal{C}$ to the category of small preadditive categories (see [13, Definition 3.1]. For each $(\alpha : c \to d) \in \text{Mor}(\mathcal{C})$ we denote by

$$\alpha_! : (R_c^{\text{op}}, \text{Ab}) \leftrightarrows (R_d^{\text{op}}, \text{Ab}) : \alpha^*$$

the change of base adjunction induced by R_α.

Definition 15. *A representation* R *of a small category* $\mathcal{C}$ *is right (resp., left) flat if, for any* $\alpha \in \mathcal{C}$*, the functor* R_α *is right (resp., left) flat.*

A right R-module $M = (M_c, M_\alpha)$ consist of the following data:

(1) for all $c \in \mathrm{Ob}\mathcal{C}$,a right R_c-module $M_c : R_c^{op} \to \mathrm{Ab}$;

(2) for any morphism $\alpha : c \to d$ in $\mathcal{C}$,a homomorphism $M_\alpha : M_c \to \alpha^* M_d$

Furthermore,we suppose that the following axioms hold:

(3) Given two morphism $\alpha : c \to d$, $\beta : d \to e$ in $\mathcal{C}$, then we have the following commutative diagram:

$$\begin{array}{ccc} M_c & \xrightarrow{M_\alpha} & \alpha^* M_d \\ \downarrow{\scriptstyle M_{\beta\alpha}} & & \downarrow{\scriptstyle \alpha^* M_\beta} \\ (\beta\alpha)^* M_e & \xrightarrow{\mu_{\alpha,\beta} * \mathrm{id}_{M_e}} & \alpha^*\beta^* M_e \end{array}$$

(4) $\delta_c * \mathrm{id}_{M_c} = \mathrm{id}_{M_c}$, for all $c \in \mathrm{Ob}\mathcal{C}$. where $\mu_{\alpha,\beta} * \mathrm{id}_{M_e}$ and $\delta_c * \mathrm{id}_{M_c}$ are horizontal pastings.

(5) Given two right R-modules M and N,a morphism $\varphi : M \to N$ consists of a family of morphisms $\{\varphi_c : M_c \to N_c\}_{c \in \mathrm{Ob}\mathcal{C}}$ in $(R(c)^{op}, \mathrm{Ab})$,such that the following square commutes for any morphism $\alpha : c \to d$ in $\mathcal{C}$:

$$\begin{array}{ccc} M_c & \xrightarrow{\varphi_c} & N_c \\ \downarrow{\scriptstyle M_\alpha} & & \downarrow{\scriptstyle N_\alpha} \\ \alpha^* M_d & \xrightarrow{\alpha^* \varphi_d} & \alpha^* N_d \end{array}$$

We denote by Mod-R the category of right R-modules. Notice that one can define analogously the category R-Mod of left R-modules. If $M \in$ Mod-R then we define $|M| = \sum_{c \in \mathrm{Ob}\mathcal{C}} |M_c|$.

Definition 16. *Given a representation $R : \mathcal{C} \to Add$ of a small category $\mathcal{C}$, the category of cartesian modules $Mod_{cart}(R)$ is the full subcategory of Mod-R whose objects are the modules M such that, for any given morphism $\alpha : c \to d$ in $\mathcal{C}$, the adjoint morphism $(M_\alpha)! : \alpha_!(M_c) \to M_d$ to the structural morphism $M_\alpha : M_c \to \alpha^*(M_d)$ is an isomorphism.*

Theorem 6. *Let $\mathcal{C}$ be a small category and $R : \mathcal{C} \to Add$ be a right flat representation. Then, the category $Mod_{cart}(R)$ is Grothendieck.*

Proof. We refer to [13, Theorem 3.23]. □

In the rest of the paper, we focus on the special case where R is a **flat presheaf of rings on the small category** $\mathcal{C}$. In this context, we can view R as a representation $R : \mathcal{C} \to \mathrm{Add}$. This perspective allows us to utilize the properties of flat presheaves while studying Cartesian modules. Our aim is to show that the category of Cartesian modules, as discussed in subsection 2.3, fits within our setting. Specifically, by Theorem 6, we established that $\mathrm{Mod}_{\mathrm{cart}}(R)$ is a Grothendieck category. Thus, it suffices to verify the existence of an infinite regular cardinal λ such that for each $M \in \mathrm{Mod}_{\mathrm{cart}}(R)$ and any set $S \subseteq M$ with $|S| < \lambda$, there is a subobject $M' \subseteq M$ such that $S \subseteq M' \subseteq M$ and $|M'| < \lambda$. We will see this fact in Proposition 1. To this end, we need to understand the concept of Cartesian modules on a quiver. A quiver $Q = (V, E)$ is a directed graph where V is the set of vertices and E is the set of edges. A quiver Q may be though as a category in which the objects of Q are the vertices of Q and the morphisms are the paths of Q. On the other hand, for any small category $\mathcal{C}$ there exists a quiver $Q_{\mathcal{C}}$ whose vertices are the objects of $\mathcal{C}$, and whose edges are the morphisms of $\mathcal{C}$. We refer to this quiver as the associated quiver to $\mathcal{C}$. The category of Cartesian modules on a quiver includes Cartesian R-modules as a full subcategory. As mentioned earlier, our objective is to prove Proposition 1. Since Enochs and Estrada established this proposition for Cartesian modules on quivers in [9], we first need to define this category before we can achieve our goal in this setting.

4.2.1 Cartesian modules on quivers:

Let $Q = (V, E)$ be a quiver and R be a presheaf from Q in the category of commutative rings, that is, for each vertex $v \in V$ we have a ring $R(v)$ and for an edge $a : v \to w$ we have a ring homomorphism $R(a^{op}) : R(w) \to R(v)$.

We shall say that we have an R-module M when we have $R(v)$-modules $M(v)$ and a morphism $M(a^{op}) : M(w) \to M(v)$ for each edge $a : v \to w$ that is $R(v)$-linear. The R-module M is said to be a Cartesian Q-module if for each edge $a : v \to w$ as above the morphism

$$R(v) \otimes_{R(w)} M(w) \to M(v)$$

given by $r_v \otimes m_w \mapsto r_v M(a^{op})(m_w)$ where $r_v \in R(v)$ and $m_w \in M(w)$ is an $R(v)$-isomorphism.

We denote by $Q\mathrm{Mod}_{\mathrm{Cart}}(R)$ the category of Cartesian Q-modules and this category is abelian when R is such that for an edge $v \to w$, $R(v)$ is a flat $R(w)$-module. It is also shown in [9] that $Q\mathrm{Mod}_{\mathrm{Cart}}(R)$ is a Grothendieck category. If $M \in Q\mathrm{Mod}_{\mathrm{Cart}}(R)$ then the cardinality of M is defined as $|M| = |\sqcup_{v\in V} M(v)|$.

Remark 3. *Let $\mathcal{C}$ be any small category with associated quiver $Q_{\mathcal{C}} = (V, E)$. It is clear that the category $Mod_{cart}(R)$ is a full subcategory of the category $Q_{\mathcal{C}}Mod_{Cart}(R)$. In case that $M \subseteq N$ in $Q_{\mathcal{C}}Mod_{Cart}(R)$ and N is a Cartesian R-module, then M will be automatically a Cartesian R-module as well. This easy observation is crutial in proving our main result in the next Proposition.*

Proposition 1. *Let $\mathcal{C}$ be any small category with associated quiver $Q_{\mathcal{C}} = (V, E)$ and M a Cartesian R-module. Let λ be an infinite cardinal such that $\lambda \geq |R(v)|$ for all v and such that $\lambda \geq max\{|E|, |V|\}$. Let $X_v \subseteq M(v)$ be subsets with $|X_v| \leq \lambda$ for all v. Then there is a Cartesian R-submodule $M' \subseteq M$ with $M'(v)$ pure for all v, with $X_v \subseteq M'(v)$ for all v and such that $|M'| \leq \lambda$.*

Proof. The proof of [9, Proposition 3.3] gives a Cartesian $Q_{\mathcal{C}}$-submodules M' of M satisfying the desired properties. Now by Remark 3, as M is Cartesian, M' will be also Cartesian R-module. □

With the fix λ as above and $\mathcal{G} = \mathrm{Mod}_{\mathrm{Cart}}(R)$ we can apply Theorem 4 and Corollary 4 to it. Indeed, we have the following corollary:

Corollary 5. *Let* $(\mathcal{A}, \mathcal{B})$ *be a cotorsion pair cogenerated by a set in* $Mod_{Cart}(R)$ *and such that* $\mathcal{A}$ *contains a generator of* $Mod_{Cart}(R)$ *and* $\mathcal{A}$ *is closed under kernels of epimorphisms. The Hovey pair* $(\widetilde{\mathcal{F}}_N, dg\widetilde{\mathcal{C}}_N)$ *and* $(dg\widetilde{\mathcal{F}}_N, \widetilde{\mathcal{C}}_N)$ *induces an abelian model structure in* $\mathbb{C}_N(Mod_{Cart}(R))$*. The corresponding homotopy category to this model structure is* $\mathbb{D}_N(Mod_{Cart}(R))$*, the derived category of* N*-complexes of* $Mod_{Cart}(R)$*.*

4.3 Category of quasi-coherent sheaves

Let $\mathbb{X}$ be an arbitrary scheme and $Qco(\mathbb{X})$ be the category of quasi-coherent sheaves on $\mathbb{X}$. It is well-known that $Qco(\mathbb{X})$ is a Grothendieck category. On the other hand, by [9, Proposition 3.3] there exists such a cardinal λ as we described before so $Qco(\mathbb{X})$ exactly fulfills the requirements of Theorems and note that $Qco(\mathbb{X})$ does necessarily not have enough projectives, but it does have enough flat objects. Let $\mathcal{F}$ be the class of all flat quasi-coherent sheaves. By [29] in case that $\mathbb{X}$ is quasi-compact and semi separated, we can say that $\mathcal{F}$ contains a generator of $Qco(\mathbb{X})$. In addition, by [9, Section 4] we see that $(\mathcal{F}, \mathcal{F}^{\perp})$ is cogenerated by a set. It is easy to check that $\mathcal{F}$ is closed under kernels of epimorphisms, so by Corollary 4 we have the following:

Corollary 6. *There is a model structure on* $\mathbb{C}_N(Qco(\mathbb{X}))$*, where weak equivalences are homology isomorphisms, the cofibrations (resp. trivial cofibrations) are the monomorphisms whose cokernels are dg-flat* N*-complexes (resp. flat* N*-complexes). The fibrations (resp. trivial fibrations) are the epimorphisms whose kernels are dg-cotorsion* N*-complexes (resp. Cotorsion* N*-complexes). The associated homotopy category is* $\mathbb{D}_N(Qco(\mathbb{X}))$*, the derived category of* N*-complexes of* $Qco(\mathbb{X})$

Remark 4. *We can also apply Corollary 3 to construct a model structure on* $\mathbb{C}_N(Qco(\mathbb{X}))$*, whose homotopy category is* $\mathbb{D}_N(Qco(\mathbb{X}))$*, the derived category of* N*-complexes of* $Qco(\mathbb{X})$*. We can also proceed in the same way and apply Corollaries 3 and 4 by considering* $\mathcal{F}$ *the class of (non-necessary finite dimensional) vector bundles and the class of 'restriced' Drinfeld vector bundles (see [8] for definition and terminology) on suitable schemes. These classes are not in general closed under direct limits and in all cases we get model structures compatible with the usal tensor product of* N*-complexes.*

Acknowledgments

The authors would like to thank the anonymous referee for their careful reading and valuable suggestions, which helped improve the quality of the paper.

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