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# Structure properties of posets involving skew derivations

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**Abstract.** The major goal of the present paper is to introduce and study skew derivations on partially ordered sets (posets), but with new restrictions. Some related properties of such derivations are introduced and investigated. Moreover, several examples are given to illustrate that various restrictions forced within the assumptions over these results cannot be ignored.

*Keywords*: Derivation, $id_T$, $\theta$-fixed point, Poset, Skew derivation.
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## 1 Introduction

Derivations were used in algebras and rings many years ago. In recent years, many papers (see, for example, [2, 5, 6, 12, 14], and their references) are generalized lattices notations for posets. Herstein [8] and Posner [11] obtained a number of important outcomes, especially for prime rings by 1957. Xin [13] studied the notion of Lattice specific fixed sets of derivations in 2012, and came up with a few exceptional results. Zhang and Li [15] created the idea of derivation on posets in 2017 and established a few essential attributes of ideals and procedures associated with derivations. For partially ordered sets, the idea of skew derivations is presented and discussed see [1] in 2021. The major goal of this paper is to introduce and study skew derivations on partially ordered sets (posets), but with new restrictions. Some related properties of such derivations are introduced and investigated.

The paper is organized as follows, in Section 2, some definitions and results are included, which are essential for the subsequent parts. In section 3, we study the notion of skew derivations on partially ordered sets but with the new restrictions and establish some important properties

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involving skew derivations. Furthermore, we discuss the notions of the $id_T, \theta$-fixed sets with skew derivations on posets using the new restrictions, in Section 4. Finally, in Section 5, we investigate the properties of ideals and the operations related with the skew derivation. Also, we prove that two skew derivations on posets always commute.

For the rest of the section, a few notations will be provided. $T$ always indicates a poset in this paper. $\downarrow s = \{t \in T : t \leq s\}$ and $\uparrow s = \{t \in T : s \leq t\}$ are written for an element $s \in T$. For a subset $U \subseteq T$, the lower cone of $U$ is represented by $l(U) = \{t \in T : t \leq w \text{ for all } w \in U\}$, and the upper cone of $U$ is represented by $u(U) = \{t \in T : w \leq t \text{ for all } w \in U\}$. It is immediately evident that both have monotone compositions $u(l(-))$ and $l(u(-))$ and are antitones. Additionally, using [4] we get $u(l(u(-))) = u(-)$ and $l(u(l(-))) = l(-)$. On the assumption that $W = \{w_1, w_2, \ldots, w_n\}$ is a finite subset of $T$, we write $l(W) = l(w_1, w_2, \ldots, w_n)$ and $u(W) = u(w_1, w_2, \ldots, w_n)$. Also, for $U \subseteq T$ and $W \subseteq T$, therefore $l(U, W)$ and $u(U, W)$ both will be indicated for $l(U \cup W)$ and $u(U \cup W)$ respectively. For a subset $W \subseteq T$, we write $\downarrow W = \{t \in T : t \leq w \text{ for some } w \in W\}$.

## 2 Some preliminaries

This section contains some definitions and results that are crucial for the following sections. Let's start out by talking about the next.

**Definition 1** ([3]). *Let $(T, \leq)$ be a poset and let $\psi : T \to T$ be a map. Then $\psi$ is said to be increasing if for any $r \leq s$ in $T$ one has $\psi(r) \leq \psi(s)$. Moreover, $\psi$ is an isomorphism in case $\psi$ is increasing, bijective and its inverse $\psi^{-1}$ is also increasing.*

**Remark 1.** *Be aware that if $(T, \leq)$ is a poset having the smallest element 0 and $\psi : T \to T$ is an isomorphism, then $\psi(0) = 0$.*

**Definition 2** ([7]). *Let $T$ be a poset and $C \subseteq T$. We say that $t \in T$ is an upper bound of $C$ if $c \leq t$ for every $c \in C$. If $U$ is a non empty subset of $T$ and for each finite subset of $U$ has an upper bound in $U$, then it is said to be directed. (It is enough to presume that each pair of items in $U$ has an upper bound in $U$ to infer non-emptiness). Additionally, when $U = \downarrow U$, we say that $U$ is a lower set. If $J$ is a directed lower subset of $T$, then it is referred to as an ideal.*

**Definition 3** ([10]). *Let $T$ be a poset and $\psi : T \to T$ be a map. Then $\psi$ is said to be a lower homomorphism if $u(\psi(l(s,t))) = u(l(\psi(s), \psi(t)))$ for every $s, t \in T$.*

**Definition 4** ([15], Definition 2.1). *Let $T$ be a poset and $\sigma : T \to T$ be a map. $\sigma$ is called a derivation on $T$, if it satisfies the following conditions:*

*(i) $\sigma(l(s,t)) = l(u(l(\sigma(s), t), l(s, \sigma(t))))$ for each $s, t \in T$;*

*(ii) $l(\sigma(u(s,t))) = l(u(\sigma(s), \sigma(t)))$ for each $s, t \in T$.*

The term skew derivations on posets is defined, and it is driven by the idea of skew derivations in rings [9], for more details see [1], as follows:

**Definition 5** ([1], Definition 2.2). *Let $T$ be a poset and $\sigma, \theta : T \to T$ be maps. We say that $\sigma$ is "a skew derivation" of $T$ associated with $\theta$ on $T$, if it satisfies the following assertions* *(i)* $\sigma(l(s,t)) = l(u(l(\sigma(s),\theta(t)), l(s,\sigma(t))))$ *for all* $s,t \in T$*;*

*(ii)* $l(\sigma(u(s,t))) = l(u(\sigma(s),\sigma(t)))$ *for all* $s,t \in T$*.*

**Corollary 1** ([15], Proposition 2.1)**.** *Let* $\sigma$ *be a derivation of* $T$*. Then the following statements hold:*

*(i)* $\sigma t \leq t$*, for each* $t \in T$*.*

*(ii)* $\sigma s \leq \sigma t$ *whenever* $s \leq t$*, for each* $s,t \in T$*.*

*(iii) If* $T$ *has an ideal* $J$*, then* $\sigma J \subseteq J$*.*

*(iv) If element 0 is the least in* $T$*, then* $\sigma 0 = 0$*.*

**Corollary 2** ([15], Theorem 2.2)**.** *Let* $T$ *be a poset and* $\sigma : T \to T$ *be a function. Then* $\sigma$ *is a derivation of* $T$ *if and only if*

*(i)* $\sigma(l(s,t)) = l(\sigma s,t) = l(s,\sigma t)$ *for every* $s,t \in T$;

*(ii)* $l(\sigma(u(s,t))) = l(u(\sigma s,\sigma t))$ *for every* $s,t \in T$*.*

Let $\sigma$ be a derivation of $T$. The fixed set of $\sigma$ in $T$ is defined by $\{s \in T : \sigma(s) = s\}$ and it is denoted by $Fix_\sigma(T)$.

**Corollary 3** ([15], Proposition 3.3)**.** *Let* $\sigma$ *be a derivation of* $T$*, where* 0 *is the smallest element. Then the subsequents hold:*

*(i)* $0 \in Fix_\sigma(T)$;

*(ii)* $s \in Fix_\sigma(T)$*, whenever* $t \in Fix_\sigma(T)$*, and* $s \leq t$;

*(iii) If* $T$ *is directed, then for any* $r,s \in Fix_\sigma(T)$*, there is* $t \in Fix_\sigma(T)$ *such that* $r \leq t, s \leq t$*.*

**Corollary 4** ([15], Corollary 3.3)**.** *Suppose that* $T$ *is a directed poset with the value* 0 *as the least element, thus* $Fix_\sigma(T)$ *is an ideal of* $T$*.*

**Corollary 5** ([15], Theorem 4.4)**.** *Let* $\sigma_1$ *and* $\sigma_2$ *be two derivations of* $T$*. Then they commute.*

**Corollary 6** ([15], Theorem 4.5)**.** *Let* $\sigma_1$ *and* $\sigma_2$ *be two derivations on* $T$*. Then* $\sigma_1 \leq \sigma_2$ *if and only if* $\sigma_2\sigma_1 = \sigma_1$*.*

## 3 Skew derivations on posets

In this section, $T$ is a poset and $\theta : T \to T$ is an isomorphism such that $\theta \leq id_T$, on $T$, where $id_T$ is the identity map on $T$.

**Lemma 1.** *Let* $\theta : T \to T$ *be an isomorphism associated with a skew derivation* $\sigma$ *of* $T$ *such that* $\theta \leq id_T$ *on* $T$*. Then, the following hold:*

*(i)* $\sigma(t) \leq t$ *for each* $t \in T$;

*(ii)* *If* $s \leq t$*, then* $\sigma(s) \leq \sigma(t)$*;*

*(iii)* *If* $T$ *has an ideal* $J$*, then* $\sigma(J) \subseteq J$*;*

*(iv)* *If the least element* $0$ *is in* $T$*, then* $\sigma(0) = 0$*.*

*Proof.* (i) Since $\sigma$ is a skew derivation associated with $\theta$ of $T$ and $\theta \leq id_T$ on $T$ then,

$$\begin{aligned}\sigma(l(t,t)) &= l(u(l(\sigma(t),\theta(t)), l(t,\sigma(t)))),\\ &= l(u(l(\sigma(t),t))), \text{ where } \theta \leq id_T \text{ on } T.\\ &= l(\sigma(t),t) \text{ for all } t \in T.\end{aligned}$$

But, from the definition of $l(-)$ we have $l(t) = l(t,t)$ for all $t \in T$. So,

$$\begin{aligned}\sigma(l(t)) &= \sigma(l(t,t))\\ &= l(\sigma(t),t) \text{ for all } t \in T.\end{aligned}$$

Also $\sigma(t) \in \sigma(l(t))$, and hence $\sigma(t) \in l(\sigma(t),t)$. Additionally, we conclude that $\sigma(t) \leq t$ for all $t \in T$.
(ii) Suppose that $s \leq t$. Then we obtain

$$\begin{aligned}l(\sigma(u(s,t))) &= l(\sigma(u(t)))\\ &= l(u(\sigma(s),\sigma(t))).\end{aligned}$$

But $\sigma(s) \in l(u(\sigma(s),\sigma(t)))$, so $\sigma(s) \in l(\sigma(u(t)))$. Hence, $\sigma(s) \leq \sigma(t)$ for all $s,t \in T$.
(iii) If $J$ is an ideal of $T$ and $t \in \sigma(J)$, there exists $j \in J$ such a way that $\sigma(j) = t$. Using $(i)$ to obtain $t(j) \leq j$ so $t \leq j$, but $J$ is an ideal of $T$. Hence $t \in J$. This has shown $\sigma(J) \subseteq J$.
(iv) If the least element 0 is in $T$, then by applying $(i)$, we get $0 \leq \sigma(0) \leq 0$. Hence $\sigma(0) = 0$. □

Note that by putting $\theta = id_T$, in Lemma 1, we obtain Corollary 1.
Now, we verify the first principal outcome of this part.

**Theorem 1.** *Let* $T$ *be a poset and* $\sigma, \theta : T \to T$ *be maps such that* $\theta$ *is an isomorphism of* $T$*,* $\theta \leq \sigma$ *and* $\theta \leq id_T$ *on* $T$*. Consequently,* $\sigma$ *is a skew derivation of* $T$ *associated with* $\theta$ *on* $T$ *if and only if*

*(i)* $\sigma(l(s,t)) = l(s,\sigma(t))$ *for all* $s,t \in T$*;*

*(ii)* $l(\sigma(u(s,t))) = l(u(\sigma(s),\sigma(t)))$ *for all* $s,t \in T$*.*

*Proof.* It is sufficient to demonstrate the condition (i) exists in the assumptions and the condition (i) in definition 5 are the same.
First assuming that the circumstance in the hypothesis is satisfied. Then

$$\begin{aligned}\sigma(l(s,t)) &= l(s,\sigma(t))\\ &= l(u(l(s,\sigma(t))))\end{aligned}$$

$$= \quad l(u(l(\sigma(s), \theta(t)), l(s, \sigma(t))), \text{ by Lemma 1 } (i)$$

Conversely, assume that $\sigma$ is a skew derivation of $T$ associated with $\theta$ on $T$. Then, we have

$$\begin{aligned} l(s, \sigma(t)) &= l(u(l(s, \sigma(t)))) \\ &\subseteq l(u(l(\sigma(s), \theta(t)), l(s, \sigma(t))) \\ &= \sigma(l(s,t)) \text{ for every } s, t \in T. \end{aligned}$$

This implies that

$$l(s, \sigma(t)) \subseteq \sigma(l(s,t)) \text{ for every } s, t \in T. \quad (1)$$

On the other hand, assume that $s, t \in T$ and $w \in \sigma(l(s,t))$, therefore, there exists $v \in l(s,t)$ satisfying $\sigma(v) = w$. But by Using Lemma 1 $(i)$, $(ii)$, we get $\sigma(v) \leq \sigma(s) \leq s$ and $\sigma(v) \leq \sigma(t)$. This leads to $w = \sigma(v) \in l(s, \sigma(t))$. Thus,

$$\sigma(l(s,t)) \subseteq l(s, \sigma(t)) \text{ for every } s, t \in T. \quad (2)$$

By the Equations (3.1) and (3.2), we get $\sigma(l(s,t)) = l(s, \sigma(t))$ for every $s, t \in T$. Hence the proof is complete. □

We have Corollary 2 as a direct result of Theorem 1.
The example below demonstrates that the terms $\theta$ is an isomorphism and $\theta \leq id_T$ on $T$ are vital in theorem 1.

**Example 1.** Assume that $(T, \ \leq) = (\{0, 2, 4, 6\}, \ \leq)$ is a poset as in Figure 1.

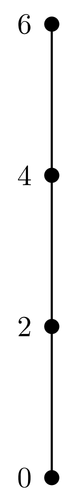


**Figure 1**

Define the functions $\sigma, \ \theta : T \to T$ by $\sigma(0) = \theta(0) = 0$, $\sigma(2) = 0, \theta(2) = 4$, $\sigma(4) = 6, =$ $\theta(4) = 2$ and $\sigma(6) = 6 = \theta(6)$. Consequently, it is directed to verify that $\sigma$ is a skew derivation associated with $\theta$ on $T$, and $id_T \leq \sigma$ of $T$. But $\sigma(l(2,6)) = \sigma(l(2)) = \{0\} \neq \{0,2\} = l(2) = l(2,6) = l(2, \sigma(6))$. As a result, the restriction implied by Theorem 1's assumption cannot be ignored.

**Example 2.** A poset is defined as $(T, \leq) = ([1, \infty], \leq)$. Describe the maps $\sigma,\ \theta : T \to T$ by $\sigma(t) = \theta(t) = 2t$ for all $t \in T$. It is also simple to verify that $\sigma$ is a skew derivation associated with $\theta$ on $T$ such that $\theta \leq \sigma$ on $T$. But $\sigma(l(4,6)) = \sigma(l(4)) = [2,8] \neq [1,4] = l(4) = l(4,12) = l(4, \sigma(6))$. Therefore, the suggested constraints in Theorem 1's hypothesis cannot be removed.

Within the next example we show that the condition $\theta \leq \sigma$ on $T$ is required by theorem 1.

**Example 3.** Consider that $(T, \ \leq) = ([0, \infty), \ \leq)$ is a poset. Define the maps $\sigma,\ \theta : T \to T$ by $\sigma(t) = \frac{1}{4}t$ and $\theta(t) = \frac{1}{2}t$ for all $t \in T$. After that, it is easy to determine whether $\sigma$ is a skew derivation of $T$ associated with $\theta$ on $T$. Also $\theta \leq id_T$ on $T$. But $\sigma(l(8,16)) = \sigma(l(8)) = [0,2] \neq [0,4] = l(4) = l(8,4) = l(8, \sigma(16))$. Therefore it's not possible to take out the conditions that are presumed in Theorem 1's hypothesis.

**Theorem 2.** *Let $\sigma$ be a skew derivation of $T$ associated with an isomorphism $\theta : T \to T$ such that $\theta \leq \sigma$ and $\theta \leq id_T$ on $T$. Then $\sigma(l(s,t)) = l(\sigma(s), \sigma(t))$ for every $s, t \in T$.*

*Proof.* First suppose that $s, t \in T$ and $w \in l(\sigma(s), \sigma(t))$, so $w \leq \sigma(s)$, $w \leq \sigma(t)$. By Lemma 1 (i), $w \leq t$. Then we have $w \in l(\sigma(s), t)$. By Theorem 1(i), $l(\sigma(s), \sigma(t)) \subseteq \sigma(l(s,t))$. Secondly, we suppose that $w \in \sigma(l(s,t))$. Consequently, $v \in l(s,t)$ exists such that $\sigma(v) = w$. Then by Lemma 1(ii), we obtain $\sigma(v) \leq \sigma(s), \sigma(v) \leq \sigma(t)$. Thus, $w \in l(\sigma(s), \sigma(t))$. Hence $\sigma(l(s,t)) \subseteq l(\sigma(s), \sigma(t))$ for every $s, t \in T$. □

**Corollary 7.** *Assuming $\sigma$ is a skew derivation of $T$ associated with an isomorphism $\theta$ on $T$, such that $\theta \leq \sigma$ and $\theta \leq id_T$ on $T$. Then, $\sigma(l(t)) = l(\sigma(t))$ for all $t \in T$.*

The following example makes it obvious that the requirements $\theta \leq id_T$ and $\theta \leq \sigma$ on $T$ are essential in Theorem 2 that follows.

**Example 4.** Consider that $(T, \ \leq) = ([0, \infty), \ \leq)$ is a poset. Define the functions $\sigma, \theta : T \to T$ by $\sigma(t) = 2$ and $\theta(t) = 2t$ for all $t \in T$. Therefore, it is simple to show that $\sigma$ is a skew derivation associated with an isomorphism $\theta$ of $T$. But $\sigma(l(0,2)) = \sigma(l(0)) = \sigma(\{0\}) = \{0\} \neq [0,2] = l(2) = l(2,2) = l(\sigma(0), \sigma(2))$. Thus, the expected assumptions in the hypothesis of Theorem 2 cannot be neglected.

## 4 The $\mathrm{id}_T$, $\theta$-fixed points of a skew derivation

Consider that $T$ is a poset in this section, $\theta : T \to T$ is a map such that $\theta \leq id_T$ on $T$, and a skew derivation of $T$ is $\sigma$ associated with $\theta$ of $T$ such that $\theta \leq \sigma$ on $T$. An element $s \in T$ is known as a $id_T, \theta$-fixed point of $\sigma$ in $T$ if $\sigma(s) = \theta(s)$ or $\sigma(s) = s$. We will denote $Fix_{id_T, \theta, \sigma}(T)$ for the set $\{s \in T : \sigma(s) = \theta(s)$ or $\sigma(s) = s\}$ of all $id_T, \theta$-fixed points of $\sigma$ in $T$, and $\sigma(T) = \{\sigma(s) : s \in T\}$. Obviously, $Fix_\sigma(T) = \{s \in T : \sigma(s) = s\} = Fix_{id_T, id_T, \sigma}(T)$.

**Theorem 3.** *Let $T$ be a poset and $\theta : T \to T$ be a bijective map. Also, let $\sigma$ be a skew derivation on $T$ associated with $\theta$ such that $\theta \leq \sigma$, $\theta \leq id_T$ and $\theta\sigma = \sigma\theta$ on $T$. Then the following hold:*

*(i)* $\sigma(t) \in Fix_{id_T, \theta, \sigma}(T)$ *for each* $t \in T$;

*(ii)* $Fix_\sigma(T) = \sigma(T)$.

*Proof.* (i) We will prove that $\sigma(\sigma(t)) = \theta(\sigma((t)))$ or $\sigma(\sigma(t)) = \sigma(t)$ for all $t \in T$. Let $t \in T$, then

$$\begin{aligned} \sigma(l(\sigma(t))) &= \sigma(l(\sigma(t), t)) \\ &= l(\sigma(t), \sigma(t)) \\ &= l(\sigma(t)) \text{ for all } t \in T. \end{aligned}$$

So by using Corollary 7 we get $l(\sigma(\sigma(t))) = l(\sigma(t))$ for all $t \in T$. Then $\sigma(\sigma(t)) = \sigma(t)$ for all $t \in T$. Hence $\sigma(t) \in Fix_{id_T,\theta,\sigma}(T)$ for every $t \in T$.
(ii) By (i), $\sigma(t) \in Fix_{id_T,\theta,\sigma}(T)$ for all $t \in T$, then $\sigma(T) \subseteq Fix_{id_T,\theta,\sigma}(T)$. On the other hand, imagine $t \in Fix_{id_T,\theta,\sigma}(T)$, so $t \in Fix_{id_T,\theta,\sigma}(T)$ i.e. $\sigma(t) = \theta(t)$ or $\sigma(t) = t$. If $\sigma(t) = \theta(t)$, then by commuting $\theta, \sigma$ and $\theta$ is bijective map, so $\theta^{-1}, \sigma$ commute and $\sigma(t) = \theta^{-1}(\sigma(t)) = \sigma(\theta^{-1}(t)) \in \sigma(T)$. Thus $Fix_{id_T,\theta,\sigma}(T) \subseteq \sigma(T)$. Hence $\sigma(T) = Fix_{id_T,\theta,\sigma}(T)$. □

The next example demonstrates that the conditions $\theta \leq \sigma$ and $\theta \leq id_T$ are essential for Theorem 3.

**Example 5.** Let $(T, \leq) = ((0, \infty), \leq)$ be a poset. Describe the maps $\sigma, \theta : T \to T$ by $\sigma(t) = 2t$ and $\theta(t) = 4t$ for all $t \in T$. After that, the fact that $\sigma$ is a skew derivation associated with $\theta$ on $T$ can be easily confirmed. Also $\theta$ is a bijective map and $\theta\sigma = \sigma\theta$. But $\sigma(\sigma(t)) = 4t \neq 8t = \theta(\sigma(t))$ and $\sigma(\sigma(t)) = 4t \neq 2t = \sigma(t)$ i.e., $\sigma(t) \notin Fix_{id_T,\theta,\sigma}(T)$. Therefore, the presumptive limitations in the Theorem 3 hypothesis cannot be omitted.

It is to remark that Proposition 3.2 of [15] is an immediate result of the next proposition:

**Proposition 1.** *Let $\sigma_1, \sigma_2$ be two skew derivations on $T$ associated with an isomorphism $\theta$ of $T$ such that $\theta$ commutes with $\sigma_1, \sigma_2$ and $\theta \leq \sigma_1$, $\theta \leq \sigma_2$, $\theta \leq id_T$, on $T$. Then $\sigma_1 = \sigma_2$ if and only if $Fix_{id_T,\theta,\sigma_1}(T) = Fix_{id_T,\theta,\sigma_2}(T)$.*

*Proof.* It is evident that if $\sigma_1 = \sigma_2$, then $Fix_{id_T,\theta,\sigma_1}(T) = Fix_{id_T,\theta,\sigma_2}(T)$. Conversely, let $Fix_{id_T,\theta,\sigma_1}(T) = Fix_{id_T,\theta,\sigma_2}(T)$ and $t \in T$. Applications of Theorem 3(i) implies that $\sigma_1(t) \in Fix_{id_T,\theta,\sigma_1}(T) = Fix_{id_T,\theta,\sigma_2}(T)$. So $\sigma_2(\sigma_1(t)) = \theta(\sigma_1(t))$ or $\sigma_2(\sigma_1(t)) = \sigma_1(t)$. So, by using $\theta, \sigma_1$ commute then we obtain $\sigma_2(\sigma_1(t)) = \sigma_1(\theta(t))$ or $\sigma_2(\sigma_1(t)) = \sigma_1(t)$. Similarly, we have $\sigma_1(\sigma_2(t)) = \sigma_2(\theta(t))$ or $\sigma_1(\sigma_2(t)) = \sigma_2(t)$. So there are four cases.
Case(1): If $\sigma_2(\sigma_1(t)) = \theta(\sigma_1(t)) = \sigma_1(\theta(t))$, and $\sigma_1(\sigma_2(t)) = \sigma_2(\theta(t)) = \theta(\sigma_2(t))$ then utilizing Lemma 1 (ii), $\theta \leq \sigma_1$ and $\theta \leq \sigma_2$ on $T$ we obtain $\sigma_1(\theta(t)) \leq \sigma_2(\theta(t))$ and $\sigma_2(\theta(t)) \leq \sigma_1(\theta(t))$. Thus $\sigma_1(\theta(t)) = \sigma_2(\theta(t))$ i.e. $\theta(\sigma_1(t)) = \theta(\sigma_2(t))$. By $\theta$ is a bijective map we get $\sigma_1(t) = \sigma_2(t)$, i.e. $\sigma_1 = \sigma_2$.
Case (2): If $\sigma_2(\sigma_1(t)) = \sigma_1(t)$ and $\sigma_1(\sigma_2(t)) = \sigma_2(t))$, then Lemma 1 (i) gives $\sigma_2(\sigma_1(t)) \leq \sigma_1(\sigma_2(t))$ and $\sigma_1(\sigma_2(t)) \leq \sigma_2(\sigma_1(t))$. Hence $\sigma_1(t) = \sigma_2(t)$.
Case (3): If $\sigma_2(\sigma_1(t)) = \sigma_1(\theta(t)) = \theta(\sigma_1(t))$ and $\sigma_1(\sigma_2(t)) = \sigma_2(t)$ then by Lemma 1 (i),(ii) we have $\sigma_1(\sigma_2(t)) = \sigma_2(t) \leq \sigma_1(t)$, i.e. $\sigma_2 \leq \sigma_1$ on $T$. Vice versa, by applying Lemma 1(i), and Theorem 2, we obtain $l(\sigma_2(\sigma_1(t), \sigma_1(\sigma_2(t))) = l(\sigma_2(\sigma_1(t)), \sigma_2(t)) = \sigma_2(l(\sigma_1(t), t)) = \sigma_2(l(\sigma_1(t)))$. But by Corollary 7, we have $\sigma_2(l(\sigma_1(t))) = l(\sigma_2(\sigma_1(t)))$. This leads to $\sigma_2(\sigma_1(t) \leq \sigma_1(\sigma_2(t)))$.

Similarly we can prove $\sigma_1(\sigma_2(t)) \leq \sigma_2(\sigma_1(t))$. Thus, $\sigma_2(\sigma_1(t)) = \sigma_1(\sigma_2(t))$, i.e. $\theta(\sigma_1(t)) = \sigma_2(t)$. Then,

$$\begin{aligned} l(\theta(\sigma_1(t))) &= l(\theta(\sigma_1(t)), \sigma_2(t)), \\ &= l(\sigma_1(\theta(t)), \sigma_2(t)), \text{ since } \theta \text{ commutes with } \sigma_1, \\ &= \sigma_2(l(\sigma_1(\theta(t)), t), \text{ by Theorem 1(i)}, \\ &= \sigma_2(l(\sigma_1(\theta(t))), \text{ by Lemma 1(i) and } \theta \leq id_T \text{ on } T, \\ &= l(\sigma_2(\sigma_1(\theta(t))), \text{ Corollary 7}. \end{aligned}$$

This implies that $l(\theta(\sigma_1(t)))) = l(\sigma_2(\sigma_1(\theta(\sigma_2(t)))$, i.e. $\theta(\sigma_1(t)) = \sigma_2(\sigma_1(\theta(t)))$. But $\theta$ commutes with $\sigma_1, \sigma_2$ and it is an isomorphism on $T$, so we have $\sigma_1(t) = \sigma_2(\sigma_1(t))$. Then using Lemma 1(i),(ii) we obtain $\sigma_1(t) = \sigma_2(\sigma_1(t)) \leq \sigma_2(t)$. That implies that $\sigma_1 \leq \sigma_2$. Hence, $\sigma_1 = \sigma_2$.
Case (4): Similarly as case (3).

The following example demonstrates how the circumstances $\theta \leq \sigma_1$, $\theta \leq \sigma_2$ and $\theta \leq id_T$ are essential for Theorem 1. □

**Example 6.** Let $(T, \leq) = ([0, \infty), \leq)$ be a poset. Define the maps $\sigma_1, \sigma_2, \theta : T \to T$ by $\sigma_1(t) = 3t$, $\sigma_2(t) = 2t$ and $\theta(t) = 4t$ for all $t \in T$. Then, it is simple to verify that the skew derivations of $T$ associated with $\theta$ on $T$ are $\sigma_1$ and $\sigma_2$. Also $\theta$ is an isomorphism, $\theta\sigma_1 = \sigma_1\theta$ and $\theta\sigma_2 = \sigma_2\theta$. But $\sigma_1 \neq \sigma_2$ although $Fix_{id_T,\theta,\sigma_1}(T) = Fix_{id_T,\theta,\sigma_2}(T) = \{0\}$.

**Proposition 2.** *Let $\theta$ be an isomorphism such that $\sigma$ is a skew derivation on $T$ with 0 as the smallest element and $\sigma\theta = \theta\sigma$, $\theta \leq \sigma$ and $\theta \leq id_T$ of $T$. Then the next assertions hold:*

*(i)* $0 \in Fix_{id_T,\theta,\sigma}(T)$;

*(ii) If $t \in Fix_{id_T\theta,\sigma}(T)$, and $s \leq t$, then $s \in Fix_{id_T,\theta,\sigma}(T)$;*

*(iii) If $T$ is directed, then, for any $r, s \in Fix_{id_T,\theta,\sigma}(T)$, there is a $t \in Fix_{id_T,\theta,\sigma}(T)$ such that $r \leq t, s \leq t$.*

*Proof.* It is clear to see that (i) is obvious, since $\sigma(0) = \theta(0) = 0$. (ii) Assume that $t \in Fix_{id_T,\theta,\sigma}(T)$, and $s \leq t$ consequently, $\sigma(t) = \theta(t)$ or $\sigma(t) = t$. Then by Theorem 1(i), Lemma 1(ii) and $\theta \leq id_T$ on $T$ we get $\sigma(l(\theta(s))) = \sigma(l(\theta(s), t)) = l(\theta(s), \sigma(t)) = l(\theta(s), \theta(t)) = l(\theta(s))$. But using Corollary 7 we obtain $l(\sigma(\theta(s))) = l(\theta(s))$. This implies that $\sigma(\theta(s)) = \theta(s)$. Applications of $\theta, \sigma$ commute and $\theta$ is an isomorphism on $T$ imply that $\sigma(s) = s$. Hence $s \in Fix_{id_T,\theta,\sigma}(T)$. As opposed to that, if $\sigma(t) = t$, then we have $\sigma(l(s)) = \sigma(l(s, t)) = l(s, \sigma(t)) = l(s, t) = l((s))$. Thus we obtain $t(s) = s$ and hence $s \in Fix_{id_T,\theta,\sigma}(T)$.
(iii) Suppose that $T$ is directed, in which case any $r, s \in T$, $v \in T$ exists such that $r \leq v$ and $s \leq v$. Since $r, s \in Fix_{id_T,\theta,\sigma}(T)$, then $(\sigma(r) = \theta(r)$ or $\sigma(r) = r)$ and $(\sigma(s) = \theta(s)$ or $\sigma(s) = s)$. Then we have four cases.
Case(1): If $\sigma(r) = \theta(r)$ and $\sigma(s) = \theta(s)$, then by using Lemma 1 (ii) we obtain $\theta(r) = \sigma(r) \leq \sigma(v)$ and $\theta(s) = \sigma(s) \leq \sigma(v)$. But $\theta$ is an automorphism and commutes with $\sigma$, then $r \leq \sigma(\theta^{-1}(v))$ and $s \leq \sigma(\theta^{-1}(v))$. Then put $t = \sigma(\theta^{-1}(v))$. Consequently, by applying Theorem 3(ii), we obtain $t \in Fix_{id_T,\theta,\sigma}(T)$. By utilizing $\theta, \sigma$ commute and $\theta$ is an isomorphism we have $r \leq \theta^{-1}(\sigma(v)) = \sigma(\theta^{-1}(v)) = t$, i.e. $r \leq t$. Also for $s$.

Case (2): If $\sigma(r) = r$ and $\sigma(s) = s$. So based on Lemma 1 we also have $r = \sigma(r) \leq \sigma(v)$ and $s = \sigma(s) \leq \sigma(v)$. Now put $t = \sigma(v)$, then again by Theorem 3(ii) we get $t \in Fix_{id_T,\theta,\sigma}(T)$. Hence we have $r \leq t$ and $s \leq t$.
Case(3): If $\sigma(r) = \theta(r)$ and $\sigma(s) = s$, then for Lemma 1 (ii) we obtain $\theta(r) = \sigma(r) \leq \sigma(v)$ and $s = \sigma(s) \leq \sigma(v)$. By $\theta$ is an isomorphism and commutes with $\sigma$, then $r \leq \sigma(\theta^{-1}(v))$ and $s \leq \sigma(v)$. Then put $t = \sigma(\theta^{-1}(v))$. Theorem 3(ii) yields the following result, $t \in Fix_{id_T,\theta,\sigma}(T)$. But $\theta, \sigma$ commute and $\theta$ is an isomorphism then we have $r \leq \theta^{-1}(\sigma(v)) = \sigma(\theta^{-1}(v)) = t$, i.e. $r \leq t$ and also for $s$.
Case (4): Similarly as case(3). Hence the proof is completed. □

Corollaries 3 and 4 are the immediate consequences of Theorem 2.

## 5 The ideals and operations related with skew derivations

The least element in the poset $T$ in this section is 0, and $\theta : T \to T$ is an isomorphism such that $\theta \leq id_T$ on $T$.
We will begin with the following result which is an extension of [15, Proposition 4.3].

**Proposition 3.** *Let $\sigma$ be a skew derivation of $T$ associated with an isomorphism $\theta$ on $T$ such that $\sigma$ commutes with $\theta$. Also, let $I_1, I_2$ be two ideals of $T$ such that $\theta^{-1}(I_2) \subseteq I_2$, $\theta \leq \sigma$ and $\theta \leq id_T$ on $T$. Then*

*(i)* $\sigma(I_1) \subseteq \sigma(I_2)$, *whenever* $I_1 \subseteq I_2$,

*(ii)* $\sigma(I_1 \cap I_2) = \sigma(I_1) \cap \sigma(I_2)$.

*Proof.* (i) It is clear to prove $(i)$ by using $I_1, I_2$ are ideals of $T$.
(ii) It is evident that $\sigma(I_1 \cap I_2) \subseteq \sigma(I_1) \cap \sigma(I_2)$. However, suppose that $t \in \sigma(I_1) \cap \sigma(I_2)$, then there exist $r \in I_1, s \in I_2$ such that $\sigma(r) = t$ and $\sigma(s) = t$. Thus by Theorem1 (i) we obtain $\sigma(l(\sigma(r), s)) = l(\sigma(r), \sigma(s)) = l(t, t) = l(t)$. But $t \in l(t)$, so $t \in \sigma(l(\sigma(r), s)))$. As a result, there is $z \in l(\sigma(r), s)$ such that $\sigma(z) = t$. But $z \leq \sigma(r)$ and $z \leq s$ so by utilizing $\sigma \leq id_T$ we have $z \leq \sigma(r) \leq r$ and $z \leq s$, i.e., $z \in I_1 \cap I_2$. Hence $t \in \sigma(I_1 \cap I_2)$, and $\sigma(I_1) \cap \sigma(I_2) \subseteq \sigma(I_1 \cap I_2)$. □

The subsequent Theorem is an extension of Theorem 4.4 in [15].

**Theorem 4.** *Let $\sigma_1$ and $\sigma_2$ represent two skew derivations of $T$ that are associated with an isomorphism $\theta$ on $T$. Also, let $\theta\sigma_1 = \sigma_1\theta$, $\theta\sigma_2 = \sigma_2\theta$, $\theta \leq \sigma_1$, $\theta \leq \sigma_2$ and $\theta \leq id_T$ on $T$. Then $\sigma_1$ and $\sigma_2$ commute.*

*Proof.* Suppose that $\sigma_1$ and $\sigma_2$ are two skew derivations on $T$ associated with an isomorphism $\theta$ of $T$ such that $\theta \leq \sigma_1$ and $\theta \leq \sigma_2$ on $T$. Thus, for each $t \in T$, there are

$$\begin{aligned} \sigma_1(l(\sigma_2(t))) &= \sigma_1(l(\sigma_2(t), t)), \text{ from Lemma 1}(i), \\ &= l(\sigma_2(t), \sigma_1(t))) \quad \text{for all} \quad t \in T, \end{aligned}$$

and

$$\sigma_2(l(\sigma_1(t))) = \sigma_2(l(\sigma_1(t), t)), \text{ from Lemma 1}(i),$$

$$= \quad l(\sigma_1(t)), \sigma_2(t))) \text{ for all } t \in T.$$

This implies that $\sigma_1(l(\sigma_2(t))) = \sigma_2(l(\sigma_1(t)))$. But $\sigma_1\sigma_2(t) = \sigma_1(\sigma_2(t)) \in \sigma_2(l(\sigma_1(t)))$, thus $\sigma_1\sigma_2(t) \in \sigma_2(l(\sigma_1(t)))$. Then there exists $s \in l(\sigma_1(t))$ such that $\sigma_1\sigma_2(t) = \sigma_2(s)$. From Lemma 1 $(ii)$, $\sigma_2(s) \leq \sigma_2(\sigma_1(t))$, and therefore $\sigma_1(\sigma_2(t)) \leq \sigma_2(\sigma_1(t))$. Similarly the case $\sigma_2(\sigma_1(t)) \leq \sigma_1(\sigma_2(t))$. Hence, $\sigma_1(\sigma_2(t)) = \sigma_2(\sigma_1(t))$ for all $t \in T$. □

The following example demonstrates how the circumstances of $\theta \leq \sigma_1$, $\theta \leq \sigma_2$ are essential in Theorem 4 and can not be omitted.

**Example 7.** Let $T = \{0, 1, 2, 3\}$ then $(T, \leq)$ is a poset. Define the functions $\sigma_1,\ \sigma_2,\ \ \theta : T \to T$ by $\sigma_1(0) = \sigma_1(1) = \sigma_1(3) = 0, \sigma_1(2) = 1$, $\sigma_2(0) = \sigma_2(1) = \sigma_2(2) = 0, \sigma_2(3) = 2$ and $\theta(t) = t$ for all $t \in T$. Then, it is straightforward to confirm that $\sigma_1$ and $\sigma_2$ are skew derivations of $T$ associated with $\theta$ on $T$, $\theta \leq id_T$ on $T$ and $\theta$ commutes with $\sigma_1, \sigma_2$. But $\sigma_1\sigma_2 \neq \sigma_2\sigma_1$ i.e., $\sigma_1, \sigma_2$ does not commute. Hence, the conditions $\theta \leq \sigma_1$ and $\theta \leq \sigma_2$ in Theorem 4 are essential and can not be omitted.

The key conclusion of this study is the theorem that follows which generalizes Theorem 4.5, proved in [15].

**Theorem 5.** *Let $\sigma_1$ and $\sigma_2$ be two skew derivations of $T$ associated with an isomorphism $\theta$ on $T$. Also, let $\theta\sigma_1 = \sigma_1\theta$, $\theta\sigma_2 = \sigma_2\theta$, $\theta \leq \sigma_1$, $\theta \leq \sigma_2$ and $\theta \leq id_T$. Then $\sigma_1 \leq \sigma_2$ if and only if $\sigma_2\sigma_1 = \theta\sigma_1$ or $\sigma_2\sigma_1 = \sigma_1$*

*Proof.* Let $\sigma_1$ and $\sigma_2$ be two skew derivations on $T$ associated with $\theta$ of $T$ such that $\sigma_1 \leq \sigma_2$ on $T$. So, for every $t \in T$, there is $\sigma_1(t) \in Fix_{id_T,\theta,\sigma_1}(T)$. Then $\sigma_1(t) = \sigma_1(\sigma_1(t))$ or $\theta\sigma_1(t) = \sigma_1(\sigma_1(t))$ We have

$$\begin{aligned} l(\sigma_2(\sigma_1(t))) &= l(\sigma_1(t), \sigma_2(\sigma_1(t))), \\ &= l(\sigma_1(t), \sigma_1(\sigma_2(t))), \text{ Theorem 4}, \\ &= \sigma_1(l(\sigma_1(t), \sigma_2(t))), \text{ Theorem1}, \\ &= \sigma_1(l(\sigma_1(t))), \text{ since } \sigma_1 \leq \sigma_2 \text{ on } T, \\ &= l(\sigma_1(\sigma_1(t))), \text{ Corollary7}. \end{aligned}$$

Thus we obtain $l(\sigma_2(\sigma_1(t))) = l(\sigma_1(\sigma_1(t)))$ for all $t \in T$. This implies that $\sigma_2(\sigma_1(t)) = \sigma_1(\sigma_1(t))$ for all $t \in T$. Hence we have $\sigma_2\sigma_1 = \theta\sigma_1$ or $\sigma_2\sigma_1 = \sigma_1$. Conversely if $\sigma_2\sigma_1 = \theta\sigma_1$ then for any $t \in T$, we obtain from Lemma 1(i),(ii) $\theta(\sigma_1(t)) = \sigma_2(\sigma_1(t)) \leq \sigma_2(t)$. So

$$\begin{aligned} l(\theta(\sigma_1(t))) &= l(\theta(\sigma_1(t)), \sigma_2(t)), \text{ since } \theta\sigma_1 \leq \sigma_2 \text{ on } T, \\ &= l(\sigma_1(\theta(t)), \sigma_2(t)), \text{ from } \theta, \sigma_1 \text{ commute on } T, \\ &= \sigma_2(l(\sigma_1(\theta(t)), t)), \text{ Theorem1}, \\ &= \sigma_2(l(\sigma_1(\theta(t))), \text{ by using Lemma 1 (i) and } \theta \leq id_T, \\ &= l(\sigma_2(\sigma_1(\theta(t)))), \text{ by Corollary7}. \end{aligned}$$

Then we get $l(\theta(\sigma_1(t))) = l(\sigma_2(\sigma_1(\theta(t))))$, for all $t \in T$. This leads to $\theta(\sigma_1(t)) = \sigma_2(\sigma_1(\theta(t)))$ for all $t \in T$. But $\theta$ commutes with $\sigma_1, \sigma_2$ so we have $\theta(\sigma_1(t)) = \theta(\sigma_2(\sigma_1(t)))$ for all $t \in T$. Which means that $\theta\sigma_1 = \theta\sigma_2\sigma_1$ on $T$. But $\theta$ is an isomorphism so $\sigma_1 = \sigma_2\sigma_1$ of $T$. Thus by using Theorem 4 and Lemma 1(i), we get $\sigma_1(t) = \sigma_2(\sigma_1(t)) = \sigma_1(\sigma_2(t)) \leq \sigma_2(t)$ for all $t \in T$. This implies that $\sigma_1 \leq \sigma_2$ on $T$. On the other hand if $\sigma_1(t) = \sigma_2(\sigma_1(t))$ then $\sigma_1(t) = \sigma_2(\sigma_1(t)) = \sigma_1(\sigma_2(t)) \leq \sigma_2(t)$. This implies that $\sigma_1 \leq \sigma_2$. Thus $\sigma_1 \leq \sigma_2$. □

The subsequent conclusion is a straight effect of Theorem 5.

**Corollary 8.** *Allow $\sigma$ to be a skew derivation of $T$ associated with an isomorphism $\theta$ on $T$ such that $\sigma \leq \theta$, $\theta \leq id_T$ and $\theta\sigma = \sigma\theta$ on $T$. Then $\sigma^2 = \sigma\sigma = \theta\sigma$ or $\sigma^2 = \sigma$.*

The most recent illustration demonstrates that the condition $\theta \leq id_T$ is essential in Theorem 5.

**Example 8.** Let $(T, \leq) = (\mathbb{R}, \leq)$ be a poset. Define the maps $\sigma_1,\ \sigma_2,\ \theta : T \to T$ by $\sigma_1(t) = t + 1$, $\sigma_2(t) = t + 2$ and $\theta(t) = t + 3$ for all $t \in T$. So, it is direct to check that $\sigma_1$ and $\sigma_2$ are skew derivations of $T$ associated with $\theta$ on $T$ such that $\sigma_2 \leq \theta$ and $\sigma_1 \leq \theta$ on $T$. Also we have $\sigma_1 \leq \sigma_2$, but $\sigma_2(\sigma_1(t)) = t + 3 \neq t + 4 = \theta(\sigma_1(t))$ and $\sigma_2(\sigma_1(t)) = t + 3 \neq t + 1 = \sigma_1(t)$ for all $t \in T$. Hence, the mentioned restriction can not be ignore .

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