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On the extendibility of $D(4)$-pair of Pell numbers

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Abstract. Let ℓ be a non-zero integer, a set of m distinct positive integers $\{a_1, a_2, \ldots, a_m\}$ is called a $D(\ell)$-Diophantine m-tuple, if $a_i a_j + \ell$ is a perfect square for any distinct $i, j \in \{1, 2, \ldots, m\}$. Let P_n denote the n^{th} Pell number, defined by the recurrence relation $P_{n+1} = 2P_n + P_{n-1}$, with initial conditions $P_0 = 0$ and $P_1 = 1$. This paper investigates the extendibility of the pair P_{2n+4}, $4P_{2n+2}$ to a D(4)-Diophantine triple by another Pell number.

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1 Introduction

Let l be a non-zero integer, a set of m integer elements $\{a_1, a_2, \ldots, a_m\}$ is called an integer Diophantine m-tuple for non-zero integer ℓ with property $D(\ell)$, or alternatively a $D(\ell)$ m-tuple, provided the product of any two different elements increased by ℓ, $a_i a_j + \ell$ where $i \neq j$, is an integer square.

The Pell numbers are defined by the recurrence relation:

$$P_n = 2P_{n-1} + P_{n-2} \qquad \text{with initial conditions} \qquad P_0 = 0, \quad P_1 = 1.$$

The n^{th} term of this sequence is given by:

$$P_n = \frac{1}{2\sqrt{2}}((1 + \sqrt{2})^n - (1 - \sqrt{2})^n).$$

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Herein we will denote $\alpha = 1 + \sqrt{2}$, which gives the following representation:

$$P_k = \frac{\alpha^k - \overline{\alpha}^k}{\alpha - \overline{\alpha}} \qquad \text{and} \qquad P_k = \frac{\alpha^k - (-1)^k \alpha^{-k}}{2\sqrt{2}}, \quad \text{where} \quad \overline{\alpha} = 1 - \sqrt{2}. \tag{1}$$

The Pell numbers satisfy an analogue of Catalan's identity:

Proposition 1. *(Catalan's identity) For positive integers n and r with $n \geq r$, the Pell numbers satisfy*

$$P_n^2 - P_{n-r}P_{n+r} = (-1)^{n-r} P_r^2.$$

Proof. Using (1),

$$\begin{aligned} P_n^2 - P_{n-r}P_{n+r} &= \left(\frac{\alpha^n - \overline{\alpha}^n}{\alpha - \overline{\alpha}}\right)^2 - \frac{\alpha^{n-r} - \overline{\alpha}^{n-r}}{\alpha - \overline{\alpha}} \frac{\alpha^{n+r} - \overline{\alpha}^{n+r}}{\alpha - \overline{\alpha}} \\ &= \frac{(\alpha^n - \overline{\alpha}^n)^2 - (\alpha^{n-r} - \overline{\alpha}^{n-r})(\alpha^{n+r} - \overline{\alpha}^{n+r})}{(\alpha - \overline{\alpha})^2} \\ &= (\alpha\overline{\alpha})^{n-r} \left(\frac{\alpha^r - \overline{\alpha}^r}{\alpha - \overline{\alpha}}\right)^2 \\ &= (-1)^{n-r} P_r^2. \end{aligned}$$

□

The Pell numbers also satisfy the following bounds in terms of α:

Lemma 1. $\alpha^{n-2} + 1 \leq P_n < \alpha^{n-1}$ *for $n \geq 2$. The first inequality is strict for $n > 2$.*

Proof. For $P_n < \alpha^{n-1}$, start with $P_n = \dfrac{\alpha^n - \overline{\alpha}^n}{\alpha - \overline{\alpha}}$, multiply top and bottom by α^n and use $(\alpha\overline{\alpha})^n = (-1)^n$,

$$P_n = \frac{\alpha^n - \overline{\alpha}^n}{\alpha - \overline{\alpha}} = \frac{\alpha^{2n} - (-1)^n}{\alpha^{n+1} + \alpha^{n-1}} = \alpha^{n-1} \frac{\alpha^{2n} - (-1)^n}{\alpha^{2n} + \alpha^{2n-2}} < \alpha^{n-1} \frac{\alpha^{2n} + 1}{\alpha^{2n} + 1} = \alpha^{n-1}.$$

For $\alpha^{n-2} + 1 \leq P_n$, equality when $n = 2$. For $n > 2$, use induction. The inequality is strict when $n = 3$ and $n = 4$. For $k > 4$, assume that $\alpha^{k-2} + 1 < P_k$. Then

$$\begin{aligned} P_{k+1} - 1 &= 2P_k + P_{k-1} - 1 \\ &> 2(P_k - 1) + (P_{k-1} - 1) \\ &> 2\alpha^{k-2} + \alpha^{k-3} \\ &= (2\alpha^{-1} + \alpha^{-2})\alpha^{k-1} \\ &= \alpha^{k-1}. \end{aligned}$$

□

The following is the main result of this paper:

Theorem 1. *Let n be a non-zero integer, the set $\{P_{2n+4}, 4P_{2n+2}, P_k\}$ is a $D(4)$-triple exactly when $k = 2n$.*

To prove this theorem, we need to establish that $k = 2n$ is the only solution which will make $\{P_{2n+4}, 4P_{2n+2}, P_k\}$ a $D(4)$-triple.

2 Preliminaries

We state several results that will be required in order to prove Theorem (1). The following result gives a general solution to a certain class of Pellian equations. It is a generalization of a result from [7], and is used and proven in [4].

Lemma 2 (Lemma on Pellian equations)**.** *Let ℓ be a non-zero integer and $\{a, b, c\}$ be a $D(\ell^2)$-triple, that is there exist positive integers r, s, t such that*

$$ab + \ell^2 = r^2, \qquad ac + \ell^2 = s^2, \qquad bc + \ell^2 = t^2.$$

Suppose that $a < b < a\left(4 + \frac{4}{\ell^2}\right)$. If one of the following conditions holds:

- $\ell = 2$
- *ℓ is an odd prime and $\ell \mid ab$, or*
- *$\ell^2 \mid a$ or $\ell^2 \mid b$*

then all the solutions of the equation

$$at^2 - bs^2 = \ell^2(a - b)$$

have the form

$$t\sqrt{a} + s\sqrt{b} = (\pm\ell\sqrt{a} + \ell\sqrt{b})\left(\frac{r + \sqrt{ab}}{\ell}\right)^{\upsilon}, \qquad \upsilon \in \mathbb{Z}^+.$$

Two results for lower bounds are stated below. The first is due to Matveev [6] and the second is due to Laurent [5]. Both of these theorems require the definition of absolute logarithmic height which is defined as follow,

Definition 1 (Definition of absolute logarithmic height)**.** *Let γ be an algebraic number with minimal polynomial over $\mathbb{Z}$ is $a\prod_{j=1}^{d}(x - \gamma^j)$, where $\gamma^1, \gamma^2, \ldots, \gamma^d$ are the conjugates of γ, including γ. Define*

$$h(\gamma) = \frac{1}{d}\left(\log|a| + \sum_{j=1}^{d} \log\left(\max\left\{1, |\gamma^j|\right\}\right)\right).$$

The following theorem is due to Matveev from [6].

Theorem 2. *Let N be a non-zero natural number and Λ be a linear form in logarithms of multiplicatively independent, totally real algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_N$ with non-zero rational integer coefficients $b_1, b_2, \ldots, b_N$. Let $h(\alpha_j)$ denote the absolute logarithmic height of α_j. Let D be the degree of $\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_N)$ as a field extension over $\mathbb{Q}$. Define the numbers A_j and E so that*

$$A_j \geq \max\{Dh(\alpha_j), \log(|\alpha_j|)\}, 1 \leq j \leq N \quad \textit{and} \quad E = \max\left\{1, \max\{|b_j|, \frac{A_j}{A_N}; 1 \leq j \leq N\}\right\}$$

Then,

$$\log(\Lambda) > -C(N)C_0W_0D^2\Omega,$$

where,

$$C(N) = \frac{8}{(N-1)!}(N+2)(2N+3)(4e(N+1))^{N+1} \qquad C_0 = \log\left(e^{4.4N+7}N^{5.5}D^2\log(eD)\right)$$

$$W_0 = \log(1.5eED\log(eD)) \qquad W_0 = \log(1.5eED\log(eD))$$

The following theorem on linear forms in two logarithms is due to Laurent [5].

Theorem 3. *Let $\gamma_1 > 1$ and $\gamma_2 > 1$ be two real multiplicatively independent algebraic numbers $b_1, b_2 \in \mathbb{Z}$ not both 0, and $\Lambda = b_2\log(\gamma_2) - b_1\log(\gamma_1)$. Let $D = [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]$. Let*

$$h_i \geq \max\left\{h(\gamma_i), \frac{|\log(\gamma_i)|}{D}, \frac{1}{D}\right\} \quad \textit{for} \quad i = 1, 2, \quad b' \geq \frac{|b_1|}{D} + \frac{|b_2|}{D}.$$

then

$$\log(|\Lambda|) \geq -17.9.D^4\left(\max\left\{\log(b') + 0.38, \frac{30}{D}, 1\right\}\right)^2 h_1h_2.$$

The following result due to Dujella and Pethő [3] is a variation of a method firstly used by Baker and Davenport in [1] to reduce the bounds to a more manageable size.

Dujella's original lemma in [3] had the requirement that $q > 6M$ (M is a positive integer). His lemma had two parts, and this requirement is only necessary for part (b) of his lemma, which is not needed in this paper.

Lemma 3. *Assume κ and μ are real numbers and M is a positive integer. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 1$, and let*

$$\varepsilon = \|\mu q\| - M.\|\kappa q\|$$

where $\|.\|$ denotes the distance to the nearest integer. If $\varepsilon > 0$, then there is no solution to the inequality

$$0 < j\kappa - k + \mu < AB^{-j}$$

in integers j and k with

$$\frac{\log(Aq/\varepsilon)}{\log B} \leq j \leq M.$$

3 The $D(4)$-triple

Using Catalan's identity, it can be shown that $\{P_{2n+4}, 4P_{2n+2}, P_k\}$ is a $D(4)$-triple, so it remains to show that $k = 2n$ is the only solution which makes $\{P_{2n+4}, 4P_{2n+2}, P_k\}$ a $D(4)$-triple.

If $\{P_{2n+4}, 4P_{2n+2}, P_k\}$ is a $D(4)$-triple, then from some integers X and Y we have

$$P_{2n+4}P_k + 4 = X^2 \qquad \textit{and} \qquad 4P_{2n+2}P_k + 4 = Y^2.$$

We can eliminate P_k to obtain the Pellian equation:

$$4P_{2n+2}X^2 - P_{2n+4}Y^2 = 4(4P_{2n+2} - P_{2n+4})$$

Now apply lemma (2) with the quantities $a = 4P_{2n+2}$, $b = P_{2n+4}$ and $\ell = 2$. Since $4P_{2n+2} < P_{2n+4} < 20P_{2n+2}$, this equation has general solution:

$$Y\sqrt{P_{2n+4}} + 2X\sqrt{P_{2n+2}} = \left(\pm 2\sqrt{P_{2n+4}} + 4\sqrt{P_{2n+2}}\right)\left(P_{2n+3} + \sqrt{P_{2n+2}P_{2n+4}}\right)^{\upsilon}, \upsilon \in \mathbb{Z}^+. \quad (2)$$

Define the sequences V_j, U_j by:

$$V_j + U_j\sqrt{P_{2n+2}P_{2n+4}} = \left(P_{2n+3} + \sqrt{P_{2n+2}P_{2n+4}}\right)^j. \quad (3)$$

This result in:

$$\begin{aligned} Y\sqrt{P_{2n+4}} + 2X\sqrt{P_{2n+2}} &= \left(\pm 2\sqrt{P_{2n+4}} + 4\sqrt{P_{2n+2}}\right)\left(V_j + U_j\sqrt{P_{2n+2}P_{2n+4}}\right) \\ &= (\pm 2V_j + 4P_{2n+2}U_j)\sqrt{P_{2n+4}} + (\pm 2U_jP_{2n+4} + 4V_j)\sqrt{P_{2n+2}} \end{aligned}$$

which gives the expressions for X_j and Y_j:

$$X = X_j = 2V_j \pm P_{2n+4}U_j \qquad \textit{and} \qquad Y = Y_j = 4P_{2n+2}U_j \pm 2V_j \quad (4)$$

Thus

(A) $\quad P_{2n+4}P_k + 4 = X^2 = (2V_j \pm P_{2n+4}U_j)^2,$

(B) $\quad 4P_{2n+2}P_k + 4 = Y^2 = (4P_{2n+2}U_j \pm 2V_j)^2.$

Subtracting (B) from (A) gives

$$P_k = (P_{2n+4} + 4P_{2n+2})\, U_j^2 \pm 4U_jV_j \tag{5}$$

We call the resulting expression $C_j^{\pm}$:

$$C_j^{\pm} = \pm 4U_jV_j + (P_{2n+4} + 4P_{2n+2})\, U_j^2$$

The goal is to find the values of j such that $C_j^{\pm}$ results in a Pell number, that is to find a pair of integers (j, k) such that $P_k = C_j^{\pm}$.
Note that when $j = 1$ we have:

$$\begin{aligned} C_1^- &= -4P_{2n+3} + P_{2n+4} + 4P_{2n+2} \\ &= P_{2n+2} - 2P_{2n+1} - 4P_{2n+3} + P_{2n+4} + 3P_{2n+2} + 2P_{2n+1} \\ &= P_{2n+2} - 2P_{2n+1} - 2P_{2n+3} + 4P_{2n+2} + 2P_{2n+1} \\ &= P_{2n+2} - 2P_{2n+1} - 2P_{2n+3} + 2P_{2n+3} \\ &= P_{2n}, \end{aligned}$$

and also $P_{2n+6} < C_1^+ < P_{2n+7}$, which follows from:

$$\begin{aligned} C_1^+ &= 4P_{2n+3} + P_{2n+4} + 4P_{2n+2} \\ &= P_{2n+6} + 4P_{2n+3} - (P_{2n+6} - P_{2n+4}) + 4P_{2n+2} \\ &= P_{2n+6} + 4P_{2n+3} - 2P_{2n+5} + 4P_{2n+2} \\ &= P_{2n+6} + 2P_{2n+3} + 4(P_{2n+2} - P_{2n+4}) \\ &= P_{2n+6} - 6P_{2n+3} \\ &\quad < P_{2n+6} + 10P_{2n+3} \\ &\quad < P_{2n+6} + 3P_{2n+3} + 7P_{2n+4} \\ &= P_{2n+6} + 3P_{2n+5} + P_{2n+4} \\ &= 2P_{2n+6} + P_{2n+5} \\ &= P_{2n+7}. \end{aligned}$$

So C_1^- is the already-known solution of $k = 2n$ and C_1^+ cannot be a Pell number.
From equation (2) we can obtain recursive forms for X_j and Y_j:

$$\begin{aligned} Y_{j+1}\sqrt{P_{2n+4}} + 2X_{j+1}\sqrt{P_{2n+2}} &= (Y_j\sqrt{P_{2n+4}} + 2X_j\sqrt{P_{2n+2}})(P_{2n+3} + \sqrt{P_{2n+2}P_{2n+4}}) \\ &= (Y_jP_{2n+3} + 2X_jP_{2n+2})\sqrt{P_{2n+4}} + (Y_jP_{2n+4} + 2X_jP_{2n+3})\sqrt{P_{2n+2}}. \end{aligned}$$

Which gives $2X_{j+1} = 2X_jP_{2n+3} + Y_jP_{2n+4}$. Therefore,

$$\begin{aligned} X_{j+1} &= X_jP_{2n+3} + Y_j\frac{1}{2}P_{2n+4} \\ &= (2V_j \pm P_{2n+4}U_j)P_{2n+3} + (2P_{2n+2}U_j \pm V_j)P_{2n+4} \end{aligned}$$

$$\begin{aligned}
&= 2V_jP_{2n+3} \pm P_{2n+4}P_{2n+3}U_j + 2P_{2n+2}P_{2n+4}U_j \pm V_jP_{2n+4}\\
&= (2P_{2n+3} \pm P_{2n+4})V_j + (2P_{2n+2} \pm P_{2n+3})P_{2n+4}U_j\\
&\quad > 2V_j \pm P_{2n+4}U_j\\
&= X_j^{\pm}.
\end{aligned}$$

So we have that $X_{j+1}^{\pm} > X_j^{\pm}$ and $X_1^{+} > X_1^{-} > 0$. We are interested in solutions (j,k) that satisfy

$$P_{2n+4}P_k = (X_j^{\pm})^2.$$

When $j = 1$ we have $P_{2n+4}P_{2n} = (X_1^{-})^2$. For any solution (j,k) with $j > 1$,

$$P_{2n+4}P_k + 4 = (X_j^{\pm})^2 > (X_1^{-})^2 = P_{2n+4}P_{2n} + 4$$

which implies that $P_k > P_{2n}$. Based on this, we conclude that any solution different from $(j,k) = (1,2n)$, would have $j \geq 2$ and $k > 2n$.

Define $\beta_n = P_{2n+3} + \sqrt{P_{2n+3}^2 - 1}$, then $\beta_n^{-1} = P_{2n+3} - \sqrt{P_{2n+3}^2 - 1}$, so from (3),

$$\begin{aligned}
V_j + U_j\sqrt{P_{2n+2}P_{2n+4}} &= \beta_n^j\\
V_j - U_j\sqrt{P_{2n+2}P_{2n+4}} &= \beta_n^{-j}
\end{aligned} \quad \Rightarrow \quad V_j = \frac{\beta_n^j + \beta_n^{-j}}{2} \quad \textit{and} \quad U_j = \frac{\beta_n^j - \beta_n^{-j}}{2\sqrt{P_{2n+2}P_{2n+4}}}$$

So $C_j^{\pm}$ can be rewritten as

$$\begin{aligned}
C_j^{\pm} &= \pm\frac{\beta_n^{2j} - \beta_n^{-2j}}{\sqrt{P_{2n+2}P_{2n+4}}} + (P_{2n+4} + 4P_{2n+2})\frac{\beta_n^{2j} + \beta_n^{-2j} - 2}{4P_{2n+2}P_{2n+4}}\\
&= \pm\frac{\beta_n^{2j}}{\sqrt{P_{2n+2}P_{2n+4}}} + \frac{(4P_{2n+2} + P_{2n+4})\beta_n^{2j}}{4P_{2n+2}P_{2n+4}} - \frac{\pm\beta_n^{-2j}}{\sqrt{P_{2n+2}P_{2n+4}}} + \frac{(P_{2n+4} + 4P_{2n+2})\beta_n^{-2j}}{4P_{2n+2}P_{2n+4}}\\
&\quad - \frac{P_{2n+4} + 4P_{2n+2}}{2P_{2n+2}P_{2n+4}}\\
&= \left(\frac{\pm 1}{\sqrt{P_{2n+2}P_{2n+4}}} + \frac{P_{2n+4} + 4P_{2n+2}}{4P_{2n+2}P_{2n+4}}\right)\beta_n^{2j} - \frac{P_{2n+4} + 4P_{2n+2}}{2P_{2n+2}P_{2n+4}} + \frac{\pm 1}{\sqrt{P_{2n+2}P_{2n+4}}}\\
&\quad + \frac{P_{2n+4} + 4P_{2n+2}}{4P_{2n+2}P_{2n+4}}.
\end{aligned}$$

Define $\gamma_n^{\pm} = \frac{\pm 1}{\sqrt{P_{2n+2}P_{2n+4}}} + \frac{P_{2n+4}+4P_{2n+2}}{4P_{2n+2}P_{2n+4}}$, so that the problem may be expressed as finding $j \geq 2$ and $k > 2n$ which satisfy the equation:

$$\gamma_n^{\pm}\beta_n^{2j} - \frac{P_{2n+4} + 4P_{2n+2}}{2P_{2n+2}P_{2n+4}} + \gamma_n^{\pm}\beta_n^{-2j} = \frac{\alpha^k - \overline{\alpha}^k}{2\sqrt{2}} \qquad \text{where,} \qquad \alpha = 1 + \sqrt{2}. \tag{6}$$

4 A linear form in three logarithms

From (6) we find that

$$1 - \frac{1}{\gamma_n^{\pm}\beta_n^{2j}}\frac{\alpha^k}{2\sqrt{2}} = \frac{1}{\gamma_n^{\pm}\beta_n^{2j}}\left(\frac{P_{2n+4} + 4P_{2n+2}}{2P_{2n+2}P_{2n+4}} - \frac{\overline{\alpha}^k}{2\sqrt{2}}\right) - \frac{\gamma_n^{\pm}\beta_n^{-2j}}{\gamma_n^{\pm}\beta_n^{2j}}. \tag{7}$$

The left-hand side of equation (7) can be written as follows:

$$\begin{aligned} 1-\frac{1}{\gamma_n^{\pm}\beta_n^{2j}}\frac{\alpha^k}{2\sqrt{2}} &= 1-\frac{\beta_n^{-2j}\times\alpha^k}{2\sqrt{2}\times\gamma_n^{\pm}} \\ &= 1-e^{\log\left(\frac{\beta_n^{-2j}\times\alpha^k}{2\sqrt{2}\times\gamma_n^{\pm}}\right)} \\ &= 1-e^{\log(\beta_n^{-2j}\alpha^k)-\log(2\sqrt{2}\gamma_n^{\pm})} \\ &= 1-e^{-2j\log(\beta_n)+k\log(\alpha)-\log(2\sqrt{2}\gamma_n^{\pm})} \\ &= 1-e^{-\left(2j\log(\beta_n)-k\log(\alpha)+\log(2\sqrt{2}\gamma_n^{\pm})\right)} \\ &= 1-e^{-\Lambda}, \end{aligned}$$

where Λ is the following linear form in three logarithms:

$$\Lambda = 2j\log(\beta_n)-k\log(\alpha)+\log(2\sqrt{2}\gamma_n^{\pm}).$$

It will be established in lemma (6) that Λ is positive. We will be able to get an upper bound for the left-hand side of equation (7), which will give us an upper bound for Λ using the following lemma.

Lemma 4. *For $0 < x < \dfrac{4}{3}$, the inequality $2(1-e^{-x}) > x$ is true.*

Proof. Using the Taylor series of the logarithm, we have:

$$-\ln\left(1-\frac{x}{2}\right) = \frac{x}{2}+\sum_{k=2}^{\infty}\frac{x^k}{2^k k} < \sum_{k=2}^{\infty}\frac{x^k}{2^{k+2}} = \frac{x}{2}+\frac{x^2}{4(2-x)} < x.$$

which is equivalent to $2(1-e^{-x}) > x$. □

Using theorem (2), we will be able to get a lower bound for Λ. Once we have an upper and lower bound for Λ, we will be able to compare these two bounds to obtain the following bound for n and j :

Proposition 2. *If equation (5) has a positive integer solution (j,k) with $j>1$, then*

$$j < 1.92441\times 10^{12}(4n+7)\log(39j(4n+7)).$$

To get an upper bound for $1-e^{-\Lambda}$, we will start by finding the bounds on $\gamma_n^{\pm}$. Using the identity
$\pm\frac{1}{xy}+\frac{y^2+4x^2}{4x^2y^2} = (\frac{1}{y}\pm\frac{1}{2x})^2$ we have

$$\sqrt{\gamma_n^{\pm}} = \pm\frac{1}{\sqrt{P_{2n+4}}}+\frac{1}{2\sqrt{P_{2n+2}}}.$$

Lemma 5. *$\gamma_n^{\pm}$ satisfy the following:*

$$0.02081\alpha^{-2n-2} < \gamma_n^- < 0.02093\alpha^{-2n-2} \quad \textit{and} \quad 2.36395\alpha^{-2n-2} < \gamma_n^+ < 2.3651\alpha^{-2n-2}.$$

Proof. We have

$$\begin{aligned}\sqrt{\gamma_n^{\pm}} &= \pm\frac{1}{\sqrt{P_{2n+4}}} + \frac{1}{2\sqrt{P_{2n+2}}}\\ &= \pm\frac{1}{\sqrt{(\alpha^{2n+4}-\alpha^{-2n-4})/(2\sqrt{2})}} + \frac{1}{2\sqrt{(\alpha^{2n+2}-\alpha^{-2n-2})/(2\sqrt{2})}}\\ &= 2^{3/4}\alpha^{-n-1}\left(\pm\frac{1}{\alpha\sqrt{1-\alpha^{-4n-8}}} + \frac{1}{2\sqrt{1-\alpha^{-4n-4}}}\right)\end{aligned}$$

From the Taylor series of $(1-x)^{-1/2})$, for $0 < x < 1$ we have

$$1 + \frac{1}{2}x < \frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots < 1 + \frac{x}{2}\left(\frac{1}{1-x}\right)$$

so

$$\frac{1}{\alpha}\left(1 + \frac{1}{2}\alpha^{-4n-8}\right) < \frac{1}{\alpha\sqrt{1-\alpha^{-4n-8}}} < \frac{1}{\alpha}\left(1 + \frac{\alpha^{-4n-8}}{2(1-\alpha^{-4n-8})}\right).$$

Hence,

$$0.41421356 < \frac{1}{\alpha} < \frac{1}{\alpha\sqrt{1-\alpha^{-4n-8}}} < \frac{1}{\alpha}\left(1 + \frac{1}{2}\frac{1}{\alpha^{4n+8}-1}\right) < 0.414218846$$

Similarly,

$$0.5 < \frac{1}{2}\left(1 + \frac{1}{2}\alpha^{-4n-4}\right) < \frac{1}{2}\sqrt{1-\alpha^{-4n-4}} < \frac{1}{2}\left(1 + \frac{\alpha^{-4n-4}}{2(1-\alpha^{-4n-4})}\right) < 0.50021665$$

We then obtain bounds for $\sqrt{\gamma^{\pm}}$

$$0.085781154 < 2^{-3/4}\alpha^{n+1}\sqrt{\gamma_n^-} = -\frac{1}{\alpha\sqrt{1-\alpha^{-4n-8}}} + \frac{1}{2\sqrt{1-\alpha^{-4n-4}}} < 0.08600309$$

$$0.91421356 < 2^{-3/4}\alpha^{n+1}\sqrt{\gamma_n^+} = \frac{1}{\alpha\sqrt{1-\alpha^{-4n-8}}} + \frac{1}{2\sqrt{1-\alpha^{-4n-4}}} < 0.941435496$$

So we get the following bounds:

$$0.02081\alpha^{-2n-2} < \gamma_n^- < 0.02093\alpha^{-2n-2} \qquad \textit{and} \qquad 2.363995\alpha^{-2n-2} < \gamma_n^+ < 2.36514\alpha^{-2n-2}$$

□

With this we can show that Λ is positive and obtain an upper bound for it.

Lemma 6. $0 < \Lambda < 4046\beta_n^{-2j}$ *for* $j \geq 2$.

Proof. Firstly, we show that $\Lambda > 0$. $\Lambda = \log(2\sqrt{2}\gamma_n^{\pm}\beta_n^{2j}\alpha^{-k}) > 0$ if and only if $2\sqrt{2}\gamma_n^{\pm}\beta_n^{2j}\alpha^{-k} > 1$. For an argument by contradiction, suppose this is not the case. This implies that

$$\gamma_n^{\pm}\beta_n^{2j} \leq \frac{\alpha^k}{2\sqrt{2}} \qquad \text{and} \qquad \frac{2\sqrt{2}}{\alpha^k} \leq \frac{\beta_n^{-2j}}{\gamma_n^{\pm}} \leq \frac{\beta_n^{-2j}}{\gamma_n^{-}} \tag{8}$$

this gives

$$\begin{aligned}
\frac{1}{P_{2n+2}} &< \frac{14}{11}\left(\frac{1}{2P_{2n+2}} + \frac{2}{P_{2n+4}}\right) && \bullet \text{ because } P_{2n+4} < 7P_{2n+2} \\
&= \frac{14}{11}\left(\frac{P_{2n+4} + 4P_{2n+2}}{2P_{2n+2}P_{2n+4}}\right) \\
&= \frac{14}{11}\left(\frac{\overline{\alpha}^k}{2\sqrt{2}} + \gamma_n^{\pm}\beta_n^{-2j} + \left(\gamma_n^{\pm}\beta_n^{2j} - \frac{\alpha^k}{2\sqrt{2}}\right)\right) \\
&\leq \frac{14}{11}\left(\gamma_n^{\pm}\beta_n^{-2j} + \frac{\overline{\alpha}^k}{2\sqrt{2}}\right) \\
&\leq \frac{14}{11}\left(\gamma_n^{\pm}\beta_n^{-2j} + \frac{2\sqrt{2}}{8\alpha^k}\right) && \bullet \text{ since } \overline{\alpha}^k \leq \alpha^{-k} \\
&\leq \frac{14}{11}\beta_n^{-2j}\left(\gamma_n^{\pm} + \frac{1}{8\gamma_n^{-}}\right)
\end{aligned}$$

which we apply below, along with the bounds on $\gamma_n^{\pm}$ from lemma (5)

$$\begin{aligned}
P_{2n+2}^{j}P_{2n+4}^{j} &= (P_{2n+3}^2 - 1)^j \\
&< \beta_n^{2j} \\
&< \frac{14}{11}P_{2n+2}\left(\gamma_n^{\pm} + \frac{1}{8\gamma_n^{-}}\right) \\
&< P_{2n+2}(3.01\alpha^{-2n-2} + 7.64\alpha^{2n+2}) && \bullet \text{ by lemma(5)} \\
&< P_{2n+2} \cdot 21.26\left(\frac{\alpha^{2n+3} + \alpha^{-2n-3}}{2\sqrt{2}}\right) \\
&= 21.26P_{2n+2}P_{2n+3} && \bullet \text{ by equation (1)} \\
&< 21.26P_{2n+2}P_{2n+4}.
\end{aligned}$$

which implies that $21.26 > P_{2n+2}^{j-1}P_{2n+4}^{j-1} \geq P_4P_6 = 12 \cdot 70$, which is a contradiction. We now establish the upper bound for Λ, firstly by finding an upper bound for $1 - e^{-\Lambda} = 1 - \frac{\alpha^k}{2\sqrt{2}}\frac{1}{\gamma_n^{\pm}\beta_n^{2j}}$.

$$\begin{aligned}
0 &\leq 1 - \frac{\alpha^k}{2\sqrt{2}}\frac{1}{\gamma_n^{\pm}\beta_n^{2j}} && \bullet \text{ since } \Lambda > 0 \\
&= \frac{1}{\gamma_n^{\pm}\beta_n^{2j}}\left(\frac{P_{2n+4} + 4P_{2n+2}}{2P_{2n+2}P_{2n+4}}\frac{\overline{\alpha}^k}{2\sqrt{2}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\gamma_n^{\pm}\beta_n^{2j}}\left(\frac{P_{2n+4}+4P_{2n+2}}{2P_{2n+2}P_{2n+4}}\frac{1}{2\sqrt{2}\alpha^k}\right)\\
&< \frac{1}{\gamma_n^{\pm}\beta_n^{2j}}\left(\frac{5}{2P_{2n+2}}+\frac{1}{2\sqrt{2}\alpha^{2n+1}}\right)\\
&= \frac{1}{\beta_n^{2j}\times 0.12081\alpha^{2n+1}}\left(\frac{5}{2P_{2n+2}}+\frac{1}{2\sqrt{2}\alpha^{2n+1}}\right) \qquad \bullet \text{ by lemma}(2)\\
&< \frac{2023}{\beta_n^{2j}}\\
&< \frac{1}{2}.
\end{aligned}
$$

Note that $1-e^{-x}<\frac{1}{2}$ implies that $x<\ln 2$, so by lemma (2) we must have $\Lambda < 2(1-e^{-\Lambda}) < 4046\beta_n^{-2j}$. □

We will now work towards using Matveev's theorem (2) to get a lower bound for Λ. To apply this theorem, we will take $(\alpha_1,\alpha_2,\alpha_3)=(\beta_n,\alpha,2\sqrt{2}\gamma^{\pm})$. We will need to find the degree D of $\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3)$ as a field extension over $\mathbb{Q}$, and to establish that $(\alpha_1,\alpha_2,\alpha_3)$ are multiplicatively independent, that is, for $p,q,r\in\mathbb{Z}$, that $\alpha_1^p\alpha_2^q\alpha_3^r=1$ if and only if $p=q=r=1$. In order to do this, we will need to establish the following result:

Lemma 7. *$P_{2n+2}P_{2n+4}$ is neither a square nor 2 times a square.*

Proof. The fact that $P_{2n+2}P_{2n+4}$ is not square follows simply from Catalan's identity:

$$P_{2n+2}P_{2n+4}=P_{2n+3}^2-1$$

as we know that consecutive integers will not both be squares. For the sake of contradiction, suppose that $P_{2n+2}P_{2n+4}=2Y^2$ for some integer Y. Catalan's identity gives:

$$X^2-2Y^2=1 \qquad \text{where,} \quad X=P_{2n+3} \quad \text{for some } n\in\mathbb{Z}. \tag{9}$$

We find the fundamental solution $(X,Y)=(3,2)$, and with this we obtain the general solution $X_j+Y_j\sqrt{2}=(3+2\sqrt{2})^j, \quad j\in\mathbb{Z}^+$. With this, we can obtain the general solution for X_j:

$$\begin{aligned} X+Y\sqrt{2}&=(3+2\sqrt{2})^j\\ X-Y\sqrt{2}&=(3-2\sqrt{2})^j \end{aligned} \quad \xrightarrow{\textit{eliminate } Y} \quad X=\frac{(3+2\sqrt{2})^j+(3-2\sqrt{2})^j}{2}.$$

Noting that $3+2\sqrt{2}=\alpha^2$, we have $X_j=\frac{\alpha^{2j}+\alpha^{-2j}}{2}$. Thus we have

$$X_j=P_{2n+3} \quad \Rightarrow \frac{\alpha^{2j}+\alpha^{-2j}}{2}=\frac{\alpha^{2n+3}+\alpha^{-2n-3}}{2\sqrt{2}}.$$

Following from this result,

$$\frac{\alpha^{2n+2}+\alpha^{-2n-2}}{2}=\frac{\alpha^{-1}\alpha^{2n+3}+\alpha^{-2n-3}\alpha}{2}$$

$$
\begin{aligned}
&\leq \frac{\frac{1}{\sqrt{2}}\alpha^{2n+3} + \frac{1}{\sqrt{2}}\alpha^{-2n-3} + (\alpha^{-1} - \frac{1}{\sqrt{2}})\alpha^{2n+3} + (\alpha - \frac{1}{\sqrt{2}})\alpha^{-2n-3}}{2} \\
&< \frac{\frac{1}{\sqrt{2}}\alpha^{2n+3} + \frac{1}{\sqrt{2}}\alpha^{-2n-3} - 0.29\alpha^{2n+3} + 1.71\alpha^{-2n-3}}{2} \\
&< \frac{\alpha^{2n+3} + \alpha^{-2n-3}}{2\sqrt{2}} \\
&= \frac{\alpha^{2j} + \alpha^{-2j}}{2} \quad \Rightarrow \quad 2n+2 < 2j. \\
\frac{\alpha^{2n+3} + \alpha^{-2n-3}}{2} &> \frac{\alpha^{2n+3} + \alpha^{-2n-3}}{2\sqrt{2}} \\
&= \frac{\alpha^{2j} + \alpha^{-2j}}{2} \quad \Rightarrow \quad 2j < 2n+3.
\end{aligned}
$$

Therefore, $2n+2 < 2j < 2n+3$, which means that j cannot be an integer, so equation (9) does not have a solution. With this, we conclude that $P_{2n+2}P_{2n+4}$ is neither a square nor 2 times a square. □

With this fact established, we can take $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{Q}(\sqrt{d}, \sqrt{2})$ where d is the square-free part of $P_{2n+2}P_{2n+4}$. This field extension has the basis $\{1, \sqrt{2}, \sqrt{d}, \sqrt{2d}\}$ as a vector space over $\mathbb{Q}$, so its degree D is 4.

Proposition 3. *$\alpha_1, \alpha_2, \alpha_3$ are multiplicatively independent.*

Proof. Suppose, to the contrary, that there exist $p, q, r \in \mathbb{Z}$, not all zero, such that $\alpha_1^p\alpha_2^q\alpha_3^r = 1$. By lemma (7) it follow that $\mathbb{Q}(\sqrt{d})$ is a quadratic field different from $\mathbb{Q}(\sqrt{2})$. Since $\gamma^{\pm}, \beta_n \in \mathbb{Q}(\sqrt{d})$ for all n, by closure under multiplication we have:

$$\beta_n^p(\gamma^{\pm})^r = (2\sqrt{2})^{-r} \quad \alpha_1^p\alpha_3^r \in \mathbb{Q}(\sqrt{d}).$$

However, by rearranging $\alpha_1^p\alpha_2^q\alpha_3^r = 1$ we find that $(2\sqrt{2})^{-r}\alpha_1^p\alpha_3^r = (2\sqrt{2})^{-r}\alpha_2^{-q}$, and the right-hand side of this equation is always irrational in $\mathbb{Q}(\sqrt{2})$ unless $q = 0$ and r is even (in this case r is rational). Thus, letting $q = 0$ and $r = 2k$ we have that $\alpha_1^{-p}\alpha_3^{2k} = 1$, or moreover that $\alpha_1^{-p} = \alpha_3^{2k}$.
Note that α_1 and α_1^{-1} are both algebraic integers, so α_1^{-p} is an algebraic integer, and thus α_3^{2k} must also be. However, the minimal polynomial of α_3^{2k} has constant term:

$$(8\gamma^+\gamma^-)^{2k} = \left(\frac{P_{2n+4} - 4P_{2n+2}}{\sqrt{2}P_{2n+2}P_{2n+4}}\right)^{4k} < 1 \text{ for all } n.$$

The constant term is never an integer, so α_1^{-p} is not an algebraic integer a contradiction.
To see that the constant term is always less than 1, the inequality $0 < \left(\frac{P_{2n+4} - 4P_{2n+2}}{\sqrt{2}P_{2n+2}P_{2n+4}}\right)^4 < 1$, can be verified by noting $P_{2n+4} - 4P_{2n+2} = 2P_{2n+1} + P_{2n+2} > 1$, and

$$
\begin{aligned}
(P_{2n+4} - 4P_{2n+2})^2 &= 16P_{2n+2}^2 - 8P_{2n+2}P_{2n+4} + P_{2n+4}^2 \\
&< 16P_{2n+2}P_{2n+4} - 8P_{2n+2}P_{2n+4} + P_{2n+4}^2
\end{aligned}
$$

$$\begin{aligned} &= 8P_{2n+2}P_{2n+4} + P_{2n+4}^2 \\ &< (P_{2n+2}P_{2n+4})^2 + P_{2n+4}^2 \\ &< (2P_{2n+2}P_{2n+4})^2 \\ &= \left(\sqrt{2}P_{2n+2}P_{2n+4}\right)^2. \end{aligned}$$

Rearrange this and raise to the power $4k$ to obtain $\left(\frac{P_{2n+4}-4P_{2n+2}}{\sqrt{2}P_{2n+2}P_{2n+4}}\right)^{4k} < 1$. □

From lemma (1) we now that $\alpha^{\lambda-2} + 1 < P_\lambda < \alpha^{\lambda-1}$. With this fact we find that

$$\begin{aligned} \beta_n &= P_{2n+3} + \sqrt{P_{2n+3}^2 - 1} < 2P_{2n+3} < 2\alpha^{2n+2} < \alpha^{2n+3}, \quad \textit{and} \\ \beta_n &= P_{2n+3} + \sqrt{P_{2n+3}^2 - 1} > 2P_{2n+3} - 1 > 2\alpha^{2n+1}. \end{aligned} \tag{10}$$

With this we can prove the following fact which will be useful when applying theorem (2).

Lemma 8. *$k \leq j(4n+7)$.*

Proof. First note that $\frac{1}{9} > \frac{1}{16} + \left(\frac{1}{12}\right)^2 \geq \frac{1}{16} + \left(\frac{1}{P_{2n+2}}\right)^2$. Multipliying by $(P_{2n+2}P_{2n+4})^2$ shows that

$$\frac{1}{9}(P_{2n+2}P_{2n+4})^2 > \frac{1}{16}(P_{2n+2}P_{2n+4})^2 + P_{2n+4}^2.$$

Also note that $24{\cdot}16 < 12{\cdot}70 = P_4P_6 \leq P_{2n+2}P_{2n+4}$ implies that $24P_{2n+2}P_{2n+4} < \left(\frac{1}{4}P_{2n+2}P_{2n+4}\right)^2$. Therefore,

$$\begin{aligned} (P_{2n+2} + 4P_{2n+4})^2 &= 16P_{2n+2}^2 + 8P_{2n+2}P_{2n+4} + P_{2n+4}^2 \\ &< 16P_{2n+2}P_{2n+4} + 8P_{2n+2}P_{2n+4} + P_{2n+4}^2 \\ &= 24P_{2n+2}P_{2n+4} + P_{2n+4}^2 \\ &< \left(\frac{1}{4}P_{2n+2}P_{2n+4}\right)^2 + P_{2n+4}^2 \\ &< \frac{1}{9}\left(\frac{1}{4}P_{2n+2}P_{2n+4}\right)^2. \end{aligned}$$

Which means that $P_{2n+4} + 4P_{2n+2} > \frac{1}{3}P_{2n+2}P_{2n+4}$, using this we establish the result:

$$\begin{aligned} \alpha^{k-1} &< 3\alpha^{k-2} \\ &< 3P_k && \bullet \text{ by lemma}(1) \\ &= \pm 12U_jV_j + 3U_j^2(P_{2n+4} + 4P_{2n+2}) \\ &\quad < 12U_jV_j + U_j^2P_{2n+4}P_{2n+2} \\ &\quad < (V_j + U_j\sqrt{P_{2n+4}P_{2n+2}})^2 - V_j^2 && \bullet \text{ because } 12 < 2\sqrt{P_4P_6} \leq 2\sqrt{P_{2n+2}P_{2n+4}} \end{aligned}$$

$$\begin{aligned} &< (V_j + U_j\sqrt{P_{2n+4}P_{2n+2}})^2 \\ &= \beta_n^{2j} \\ &< \alpha^{2j(2n+3)} \end{aligned} .$$

Therefore, $k \leq j(4n+7)$. □

We have everything needed to apply theorem (2), to get an upper bound for $-\log|\Lambda|$. From lemma (2) we have the lower bound $2\log\beta_n - \log 4046 < -\log|\Lambda|$. Combining these two bounds will allow us to prove proposition (2).

Proof. (Proof of Proposition (2))
To prove this, we will apply theorem Matveev's (2) to the linear form in logarithms

$$\Lambda = 2j\log\beta_n - k\log\alpha + \log(2\sqrt{2}\gamma_n^{\pm}),$$

with the following quantities as specified by theorem (2):

$$N = 3 \qquad D = 4 \qquad b_1 = 2j \qquad b_2 = -k \qquad b_3 = 1$$

$$\alpha_1 = \beta_n \qquad \alpha_2 = \alpha \qquad \alpha_3 = 2\sqrt{2}\gamma_n^{\pm}.$$

We have already established that $D = 4$ and $\alpha_1, \alpha_2, \alpha_3$ are multiplicatively independent.
Since α_1 and α_2 are both algebraic integers with degree 2 and their conjugates are less than 1, their absolute logarithmic heights are:

$$h(\alpha_1) = \frac{1}{2}\log\beta_n \qquad \textit{and} \qquad h(\alpha_2) = \frac{1}{2}\log\alpha.$$

For α_3, note that γ_n^+ and γ_n^- are roots of the polynomial

$$(x-\gamma_n^+)(x-\gamma_n^-) = x^2 + 2\left(\frac{P_{2n+4}+4P_{2n+2}}{4P_{2n+2}P_{2n+4}}\right)x + \left(\frac{P_{2n+4}-4P_{2n+2}}{4P_{2n+2}P_{2n+4}}\right)^2.$$

Clearing the denominators, we find that the minimal polynomial of $\gamma_n^{\pm}$ has constant term $16P_{2n+2}^2P_{2n+4}^2$.
Since $|\gamma_n^{\pm}| < 1$ for all n, and $P_\lambda < \alpha^\lambda/2^{3/2}$ for positive even λ, we have

$$h(\gamma_n^{\pm}) = \frac{1}{2}(16P_{2n+2}^2P_{2n+4}^2) = \log(4P_{2n+2}P_{2n+4}) < (4n+6)\log(\alpha) + \log(\frac{1}{2}).$$

Thus we can take

$$\begin{aligned} h(\alpha_3) &= h(2\sqrt{2}\gamma_n^{\pm}) \\ &< h(2\sqrt{2}) + h(\gamma_n^{\pm}) \\ &< \frac{3}{2}\log 2 + \log(4n+6)\log(\alpha) + \log(\frac{1}{2}) \end{aligned}$$

$$
\begin{aligned}
&< \frac{3}{2}\log 2 + \log(\frac{1}{2}) + \log(4n+6)\log(\alpha) \\
&< \log(\alpha) + (4n+6)\log(\alpha) \\
&= (4n+7)\log(\alpha).
\end{aligned}
$$

Since we need $A_i \geq \max\{Dh(\alpha_i), |\alpha_i|\}$, where $D = 4$, we take

$$A_1 = 2\log\beta_n \qquad A_2 = 2\log\alpha \qquad A_3 = 4(4n+7)\log\alpha.$$

Note that the requirement that $A_3 > |\log\alpha_3|$ is met:

$$
\begin{aligned}
\left|\log\left(2\sqrt{2}\gamma_n^{\pm}\right)\right| &\leq \left|\log\left(2\sqrt{2}\times 0.02081\alpha^{-2n-2}\right)\right| \\
&< \left|\log\left(2\sqrt{2}\times 0.02081\alpha^{-4(4n+4)}\right)\right| \\
&< \left|\log\left(\alpha^{-4(4n+7)}\right)\right| \\
&= 4(4n+7)\log\alpha.
\end{aligned}
$$

For E (see theorem(2)), we have

$$
\begin{aligned}
E = \max\left\{1, \max\left\{|b_j|\frac{A_j}{A_3} : 1 \leq j \leq 3\right\}\right\} &= \max\left\{1, \max\left\{\frac{|b_1|A_1}{A_3}, \frac{|b_2|A_2}{A_3}, \frac{|b_3|A_3}{A_3}\right\}\right\} \\
&= \max\left\{\frac{j\log\beta_n}{(4n+7)\log\alpha}, \frac{k}{2(4n+7)}, 1\right\} \\
&< \max\left\{\frac{j}{2}, \frac{j}{2}, 1\right\} \qquad \bullet \text{ by equation (5) and lemma(6)} \\
&< j(4n+7).
\end{aligned}
$$

To apply theorem (2), consider the quantities

$$C(3) = \frac{8}{2}\cdot 5\cdot 9\cdot(16\cdot\exp)^4 < 6.45\times 10^8 \qquad C_0 = \log\left(e^{20.2}3^{5.5}(16)\log(4e)\right) < 30$$
$$W_0 = \log(1.5eE\cdot 4\log(4e)) < \log(39j(4n+7)) \qquad \Omega = (2\log\beta_n)\log(2\log\alpha)(4(4n+7)\log\alpha).$$

Therefore,

$$
\begin{aligned}
2j\log\beta_n - \log 4046 &< -\log|\Lambda| \qquad \bullet \text{ by lemma(6)} \\
&< C(3)C_0W_0D^2\Omega \\
&< 3.8481\times 10^{12}(4n+7)(\log\beta_n)\log(39j(4n+7)),
\end{aligned}
$$

which implies

$$j < 1.9241\times 10^{12}(4n+7)\log(39j(4n+7)).$$

□

5 Linear form in two logarithms

Firstly, define a new linear form in three logarithms Λ_0, by substituting $(j,k) = (1, 2n)$ into Λ:

$$\Lambda_0 = 2\log\beta_n - 2n\log\alpha + \log(2\sqrt{2}\gamma_n^{\pm}).$$

By the easily verifiable identity $x + \sqrt{x^2-1} = 2x\left(1 - \frac{1}{2x(x+\sqrt{x^2-1})}\right)$, and that $P_{2n+3} = \frac{\alpha^{2n+3}}{2\sqrt{2}}\left(1 + \frac{1}{\alpha^{4n+6}}\right)$, we find that

$$\begin{aligned}\beta_n &= P_{2n+3} + \sqrt{P_{2n+3}^2 - 1}\\ &= 2P_{2n+3}\left(1 - \frac{1}{2P_{2n+3}(P_{2n+3} + \sqrt{P_{2n+3}^2-1})}\right)\\ &= \frac{1}{\sqrt{2}}\alpha^{2n+3}\left(1 + \frac{1}{\alpha^{4n+6}}\right)\left(1 - \frac{1}{2P_{2n+3}\beta_n}\right).\end{aligned}$$

Let's define $\delta_n = \left(1 + \frac{1}{\alpha^{4n+6}}\right)\left(1 - \frac{1}{2P_{2n+3}\beta_n}\right)$, we then obtain

$$\begin{aligned}\Lambda - \Lambda_0 &= \left(2j\log\beta_n - k\log\alpha + \log(2\sqrt{2}\gamma_n^{\pm})\right) - \left(2\log\beta_n - 2n\log\alpha + \log(2\sqrt{2}\gamma_n^{\pm})\right)\\ &= (2j-2)\log\beta_n - (k-2n)\log\alpha\\ &= (2j-2)\left(\log\left(\frac{1}{\sqrt{2}}\right) + (2n+3)\log\alpha + \log\delta_n\right) - (k-2n)\log\alpha\\ &= (2j-2)\log\left(\frac{1}{\sqrt{2}}\right) + ((2j-2)(2n+3) - (k-2n))\log\alpha + (2j-2)\log\delta_n.\end{aligned}$$

If we define the linear form in two logarithms:

$$\Lambda_1 = K\log\alpha - (j-1)\log(2) \quad \text{where,} \quad K = (2j-1)(2n+3) - k - 3.$$

This means that

$$\Lambda_1 = \Lambda - \Lambda_0 - (2j-2)\log\delta_n.$$

Which, by the triangle inequality, implies that $|\Lambda_1| \leq |\Lambda| + |\Lambda_0| + (2j-2)|\log\delta_n|$.
In this section the task is to find an upper bound for $|\Lambda_1|$, and then Laurent's theorem (3) gives a lower bound. Combining these bounds with the result from proposition (2) will allow us to get the following bounds for n and j:

Proposition 4. *If equation (5) has a positive integer solution (j,k) with $j > 1$, then*

$$j < 9.19 \times 10^{18} \qquad \textit{and} \qquad n < 20358.$$

We already have an upper bound for Λ from lemma (6). In order to get an upper bound for $|\Lambda_1|$, it remains to find an upper bound for $|\Lambda_0|$ and for $|\log\delta_n|$. We will begin with $|\Lambda_0|$.

Lemma 9. *$|\Lambda_0| < 15911\beta_n^{-2}$.*

Proof. For now assume $n \geq 2$. Substituting $(j,k) = (1,2n)$ into (7) we obtain

$$1 - \frac{1}{\gamma_n^{\pm}\beta_n^2}\frac{\alpha^{2n}}{2\sqrt{2}} = \frac{1}{\gamma_n^{\pm}\beta_n^2}\frac{P_{2n+4}+4P_{2n+2}}{2P_{2n+2}P_{2n+4}} - \frac{1}{\gamma_n^{\pm}\beta_n^2}\frac{\overline{\alpha}^{2n}}{2\sqrt{2}} - \frac{1}{\gamma_n^{\pm}\beta_n^2}\gamma_n^{\pm}\beta_n^{-2}. \tag{11}$$

Observe that $1 - e^{-\Lambda_0} = 1 - e^{-\log(\beta_n^2\alpha^{-2n}2\sqrt{2}\gamma_n^{pm})} = 1 - \frac{1}{\gamma_n^{\pm}\beta_n^2}\frac{\alpha^{2n}}{2\sqrt{2}}$, the left-hand side of the above equation. This part of the proof is split into two cases:

$$1 - e^{-\Lambda_0} \leq 0 \qquad \textit{and} \qquad 1 - e^{-\Lambda_0} > 0$$

- **Case 1:** If $1 - e^{-\Lambda_0} \leq 0$, then

$$\begin{aligned}
0 &\leq \frac{1}{\gamma_n^{\pm}\beta_n^2}\frac{\alpha^{2n}}{2\sqrt{2}} - 1 \\
&= \frac{\frac{\overline{\alpha}^{2n}}{2\sqrt{2}} + \gamma_n^{\pm}\beta_n^2}{\gamma_n^{\pm}\beta_n^2} - \frac{1}{\gamma_n^{\pm}\beta_n^2}\frac{P_{2n+4}+4P_{2n+2}}{2P_{2n+2}P_{2n+4}} \\
&< \frac{\frac{\beta_n^2\alpha^{-2n}2\sqrt{2}}{+}\gamma_n^{\pm}}{\gamma_n^{\pm}\beta_n^4} \\
&< \frac{\frac{1}{8\gamma_n^{\pm}} + \gamma_n^{\pm}}{\gamma_n^{\pm}\beta_n^4} \\
&< \frac{\frac{1}{8\times 0.02081\alpha^{-2n-2}} + 2.365614\alpha^{-2n-2}}{\beta_n^4 0.02081\alpha^{-2n-2}} \\
&< \frac{113.654 + 288.646\alpha^{4n+4}}{\beta_n^4} \\
&< \frac{113.654 + 288.646\alpha^{4n+4}}{4\beta_n^4\alpha^{4n}} \\
&< 2480\beta_n^{-2}.
\end{aligned}$$

  Note that $1 - e^{-x} \leq 0$ implies $x \leq 0$. It follows in this case that

$$|\Lambda_0| = \Lambda_0 < e^{-\Lambda} - 1 < 2480\beta_n^{-2}.$$

- **Case 2:** If $1 - e^{-\Lambda_0} > 0$, then

$$\begin{aligned}
0 &< 1 - \frac{1}{\gamma_n^{\pm}\beta_n^2}\frac{\alpha^{2n}}{2\sqrt{2}} \\
&= 1 - \frac{1}{\gamma_n^{\pm}\beta_n^2}\frac{P_{2n+4}+4P_{2n+2}}{2P_{2n+2}P_{2n+4}} - \frac{\frac{\overline{\alpha}^{2n}}{2\sqrt{2}} + \gamma_n^{\pm}\beta_n^{-2}}{\gamma_n^{\pm}\beta_n^2} \\
&< \frac{1}{\gamma_n^{\pm}\beta_n^2}\left(\frac{1}{2P_{2n+2}} + \frac{2}{P_{2n+4}}\right)
\end{aligned}$$

$$< \frac{1}{\gamma_n^{\pm}\beta_n^2}\left(\frac{5}{2P_{2n+2}}\right)$$
$$< 701\beta_n^{-2}$$
$$< \frac{1}{2}.$$

Note that $0 < 1 - e^{-x} < \frac{1}{2}$ implies that $0 < x < \ln 2$, so by lemma (4) we must have

$$|\Lambda_0| = \Lambda_0 < 2(1 - e^{-\Lambda_0}) < 1402\beta_n^{-2j}.$$

If $n = 1$, then

$$|\Lambda_0| = 2\log(29+\sqrt{29^2-1}) - 2\log(1+\sqrt{2}) + \log\left(2\sqrt{2}\left(\pm\frac{1}{\sqrt{70}} + \frac{1}{2\sqrt{12}}\right)\right) < 4.7326 < 15911\beta_n^{-2}.$$

in any case, we have $|\Lambda_0| < 15911\beta_n^{-2}$.

□

To find an upper bound for $|\log\delta_n|$, note that for $0 < x < \frac{1}{2}$ and $0 < y$ we have the following:

$$-\log(1-x) < 2x \qquad \textit{and} \qquad \log(1+y) < y.$$

Using this and the bounds from lemma (1) and equation (10), we obtain the bound for $|\log\delta_n|$:

$$|\log\delta_n| \leq \left|\log\left(1 - \frac{1}{2P_{2n+3}\beta_n}\right)\right| + \left|\log\left(1 + \frac{1}{\alpha^{4n+6}}\right)\right| < \frac{1}{P_{2n+3}\beta_n} + \frac{1}{\alpha^{2n+6}} < \frac{1}{2\alpha 4n+2} + \frac{1}{\alpha^{4n+6}} < \frac{18}{\alpha^{4n+6}}.$$

Using the bounds for $|\Lambda_0|$, $|\log\delta_n|$, and $|\Lambda|$, we prove the following result:

Lemma 10. $|\Lambda_1| < \frac{7j+29074}{\alpha^{4n+4}}$.

Proof. Bringing together $|\Lambda_1| < 4046\beta_n^{+2j}$, $|\Lambda_0| < 15911\beta_1^{-2}$ and $|\delta_n| < \frac{18}{\alpha^{4n+6}}$, we get:

$$\begin{aligned}|\Lambda_1| &\leq |\Lambda| + |\Lambda_0| + (2j-2)|\log\delta_n| \\ &< 4046\beta_n^{-2j} - 15911\beta_n^{-2} + (2j-2)\frac{18}{\alpha^{4n+6}} \\ &< \frac{4046}{(2\alpha^{2n+1})^2 j} + \frac{15911}{(2\alpha^{2n+1})^2} + (2j-2)\frac{18}{\alpha^{4n+6}} \\ &< \frac{7j+29074}{\alpha^{4n+6}}.\end{aligned}$$

□

Now, using the theorem (3) due to Laurent on linear forms in two logarithms, we can prove proposition (4).

Proof. (Proof of proposition (4)) We will apply theorem (3) to

$$\Lambda_1 = K \log \alpha - (j-1)\log(2).$$

We have,

$$D = 2 \qquad \gamma_1 = 2 \qquad \gamma_2 = \alpha \qquad b_1 = j-1 \qquad b_2 = K.$$

Also we take h_1 and h_2 as shown below,

$$h_1 = \log(2) \geq \max\left\{h(\gamma_1), \frac{\log(\gamma_1)}{D}, \frac{1}{D}\right\} = \max\left\{\log(2), \frac{\log(2)}{4}, \frac{1}{4}\right\} = \log(2),$$

$$h_2 = \frac{1}{2} \geq \max\left\{h(\gamma_2), \frac{\log(\gamma_2)}{D}, \frac{1}{D}\right\} = \max\left\{\frac{1}{2}\log(\alpha), \frac{1}{4}\log(\alpha), \frac{1}{4}\right\}.$$

By lemma (8),

$$K < \frac{(j-1)\log(2) + (7j+29074)\alpha^{-4n-4}}{\log(\alpha)} < 0.794j + 27.799.$$

And because

$$\frac{|b_1|}{Dh_2} + \frac{|b_2|}{Dh_1} = (j-1) + \frac{|k|}{2\log 2} < 1.573j + 19.0523 = b'.$$

Applying theorem (3) we obtain the bound,

$$\log|\Lambda_1| > -17.9 \cdot 8 \log 2 \cdot \left(\max\left\{\log(1.573j + 19.0523) + 0.38, 15\right\}\right)^2.$$

And from lemma (10) we have the bound,

$$\log|\Lambda_1| < \log(7j + 29074) - (4n+4)\log\alpha.$$

Combining these two bounds yields

$$n < 10.279\left(\max\left\{\log(1.573j + 19.0523) + 0.38, 15\right\}\right)^2 + 0.104\log(7j + 27074).$$

If $\log(1.573j + 19.0523) + 0.38 < 15$, then $j < 2.81554 \times 10^6$. Otherwise,

$$n < 10.279(\log(1.573j + 19.0523) + 0.38)^2 + 0.104\log(7j + 29074).$$

In proposition (4) we found the bound $j < 1.9241 \times 10^{12}(4n+7)\log(39j(4n+7))$. Bringing these two results together, we have

$$\begin{aligned} j < {} & 1.9241 \times 10^{12}\left(4\left(10.279(\log(1.573j + 19.0523) + 0.38)^2 + 0.1041\log(7j + 29074)\right) + 7\right) \\ & \times \log\left(39j\left(4\left(10.279(\log(1.573j + 19.0523) + 0.38)^2 + 01041\log(7j + 29074)\right) + 7\right)\right), \end{aligned}$$

which implies $j < 9.19 \times 10^{18}$ and therefore $n < 20358$. □

6 Refining the bounds

In this section, the bounds on n and j are improved before Baker-Davenport reduction wich will be applied in the next section.

Lemma (10) gives

$$|K\log\alpha-(j-1)\log 2|<\frac{7j+29074}{\alpha^{4n+4}}\quad\xrightarrow{\text{divide by } j-1}\quad\left|\frac{\log 2}{\log\alpha}-\frac{K}{j-1}\right|<\frac{7j+29074}{(j-1)\alpha^{4n+4}\log\alpha}.$$

Assume that

$$\frac{7j+29074}{(j-1)\alpha^{4n+4}\log\alpha}<\frac{1}{2(j-1)^2}.$$

Then by inequality above,

$$\left|\frac{\log 2}{\log\alpha}-\frac{K}{j-1}\right|<\frac{1}{2(j-1)^2}.$$

By Legendre's criterion, $\frac{K}{j-1}$ is a convergent of the continued fraction of $\frac{\log 2}{\log\alpha}$. The 38^{th} convergent of continued fraction of $\frac{\log 2}{\log\alpha}$ is

$$\frac{7486685157270191075}{9519719241472897252}.$$

Its denominator is larger than the upper bound of 9.19×10^{18} established for j, so $\frac{K}{j-1}$ cannot be equal to the 38^{th} convergent, nor any convergent that follows it. Therefore $\frac{K}{j-1}$ is a convergent that occurs among the first 37 convergents of $\frac{\log 2}{\log\alpha}$. By theorem on the continued fraction [2,8], we can use the denominator of the 37^{th} convergent

$$\frac{506355234091671513}{9519719241472897252},$$

to obtain the lower bound

$$\left|\frac{\log 2}{\log\alpha}-\frac{K}{j-1}\right|\geq\left|\frac{\log 2}{\log\alpha}-\frac{506355234091671513}{9519719241472897252}\right|>1.00\times10^{-38}.$$

Combining these bounds we obtain

$$10^{-38}<\frac{7j+29074}{(j-1)\alpha 4n+4\log\alpha}<\frac{29200}{\alpha^{4n+4}\log\alpha},$$

which implies that $n<27$.
Since we know that $\frac{p_r}{q_r}$ is the r^{th} convergent of $\frac{\log 2}{\log\alpha}$,

$$\left|\frac{\log 2}{\log\alpha}-\frac{p_r}{q_r}\right|\geq\frac{1}{(a_{r+1}+2)q_r^2}$$

and a_{r+1} is the $(r+1)^{st}$, partial quotient of $\frac{\log 2}{\log\alpha}$. Therefore, since $\frac{K}{j-1}$ is among the first 37 convergents of $\frac{\log 2}{\log\alpha}$, we have for $2\leq r\leq 37$ that

$$\min_{2\leq r\leq 37}\left\{\frac{1}{(a_{r+1}+2)(j-1)^2}\right\}<\left|\frac{\log 2}{\log\alpha}-\frac{K}{j-1}\right|<\frac{7j+29074}{(j-1)\alpha^{4n+4}\log\alpha}.$$

Since $\max\{a_{r+1} : 2 \leq r \leq 37\} = a_{27} = 100$,

$$\alpha^{4n+4} \leq 2(j-1)(7j+29074)(\log\alpha)^{-1}.$$

In either case, $\alpha^{4n+4} < 9 \times 10^5 j^2$. This leads to the following result.

Proposition 5. *If equation (5) has a positive integer solution (j,k) with $j > 1$, then*

$$n < 0.568 \log j + 3.889.$$

Combining this result with the bound for j found in proposition (2), we get

$$j < 1.9241 \times 10^{12}(4(0.5681 \log j + 3.889) + 7) \log(39j(4(0.5681 \log j + 3.889) + 7))$$

which implies $j < 9.21 \times 10^{15}$ and $n < 25$.

7 Baker-Davenport reduction

To prove theorem (1), it just remains to apply Baker-Davenport reducation described in lemma (3). We know from lemma (6) that

$$0 < 2j\log\beta_n - k\log\alpha + \log(2\sqrt{2}\gamma_n^{\pm}) < 4046\beta_n^{-2j}.$$

So we can apply lemma (3), with the quantities

$$\kappa = \frac{2\log\beta_n}{\log\alpha}, \quad \mu = \frac{\log(2\sqrt{2}\gamma_n^{\pm})}{\log\alpha} \quad A = \frac{4046}{\log\alpha}, \quad B = \beta_n^2,$$

$$M = 9.21 \times 10^{15}, \quad 1 \leq n \leq 24.$$

We again use procedures written in Maple™ to carry out the reduction for each $1 \leq n \leq 24$ in the case of γ_n^+ and the case of γ_n^-.

In each case we find that $j \leq 6$, which implies $n \leq 4$.

Proposition 6. *If equation (5) has solutions (j,k) with $j > 1$, then $j \leq 6$ and $n \leq 4$.*

Applying this result to the equation $P_k = C_j^{\pm}$, we can prove theorem (1).

Proof. (Proof of theorem (1)) By testing each case one-by-one, we find that no combination of n and j with $1 < j \leq 6$ and $1 \leq n \leq 4$ result in $C_j^{\pm}$ being a Pell number. When $j = 1$, we have already seen that $C_1^- = P_{2n}$ and $P_{2n+6} < C_1^+ < P_{2n+7}$. □

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