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Complemented and completely regular $\Gamma-$ semirings

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Abstract. This paper investigates the additive and multiplicative properties of complemented and completely regular $\Gamma-$ semirings. We prove many results on different structures of $\Gamma-$ semirings like anti-inverse, quasi-separative, distributive, and partial order. Boolean $\Gamma-$ semiring is demonstrated by applying the concept of a completely regular $\Gamma-$ semiring. Finally, by using the idea of simple and completely regular $\Gamma-$ semiring, we define a relation $\leq$ on R such that $x \leq y$ if and only if $x + y + 1 = x\alpha y$ for all $x, y \in R, \alpha \in \Gamma$ and prove that R is a partially ordered $\Gamma-$ semiring.

Keywords: Regular band, $\Gamma-$semigroup, Quasi-separative, Partially ordered $\Gamma-$ semiring.
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1 Introduction

Nobusawa [6] first proposed the concept of $\Gamma-$ring in 1964 as a generalization of the ring. Sen [9, 10] first developed the idea of $\Gamma-$semigroup. Rao [7] introduced the notion of $\Gamma-$semiring as a generalization of $\Gamma-$semiring, ternary semiring, and semirings. Because of the significance of results on $\Gamma-$semirings, which were proved by Dutta and Sardar [2–4] and many others, we were inspired to study $\Gamma-$semirings. Later, Sharma and Gupta [13] examined the implications of a commutative, simple, additive idempotent and multiplicative $\Gamma-$idempotent on a $\Gamma-$semiring R and studied the consequences of these implications on R. Further, more restrictions were imposed on $\Gamma-$semirings by Sharma and Ranote [17], to generalize the work of Sharma and Gupta [13]. Furthermore, Sharma and Gupta [16] introduced the concept of complementation of $\Gamma-$ semirings. Because the complemented elements play an important role in lattices, another major source of inspiration for the theory of $\Gamma-$ semirings is lattice theory. Thus, Sharma [12] introduced the concept of lattices in $\Gamma-$semirings.

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This paper is divided into two sections. In the first part of this paper, we extend some of the results regarding complemented $\Gamma-$ semirings. We determine the additive and multiplicative structure of complemented $\Gamma-$ semiring from [5] by assuming different properties on the additive and multiplicative structures. In the second section, we generalize some of the results of [1] to completely regular $\Gamma-$ semirings. We proved many criteria on different structures of $\Gamma-$ semirings like, anti-inverse, quasi-separative, distributive, and Boolean $\Gamma-$ semirings.

2 Preliminaries and examples

We recall from [1, 14–17] some basic notations, definitions, and examples of Γ-semirings needed for this paper.

Let R and Γ be two additive commutative semigroups. Then R is called a $\Gamma-$ semiring if there exists a mapping $R \times \Gamma \times R \to R$ denoted by $x\alpha y$ for all $x, y \in R$ and $\alpha \in \Gamma$ satisfying (i) $(x + y)\alpha z = x\alpha z + y\alpha z$ (ii) $x(\alpha + \beta)z = x\alpha z + x\beta z$ (iii) $x\alpha(y + z) = x\alpha y + x\alpha z$ (iv)$(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.

Every semiring R is a $\Gamma-$ semiring. Let R be a semiring and Γ be a commutative semigroup. Define a mapping $R \times \Gamma \times R \to R$ denoted by $x\gamma y = xy$ for all $x, y, z \in R$ and $\gamma \in \Gamma$. Then R is a $\Gamma-$ semiring. Let A and B be semirings and let $R = Hom(A, B)$ and $\Gamma = Hom(B, A)$ denote the sets of homomorphisms from A to B and B to A, respectively. Then R is a $\Gamma-$ semiring with operations of pointwise addition and composition of mappings. Let M be a $\Gamma-$ ring and R the set of all ideals of M. Define addition in the natural way and if $A, B \in R$, $\gamma \in \Gamma$, let $A\gamma B$ denote the ideal generated by $\{x\gamma y | x, y \in M\}$. Then R is a $\Gamma-$ semiring.

R is said to be commutative if $x\gamma y = y\gamma x$ and $x + y = y + x$ for all $x, y \in R$ and $\gamma \in \Gamma$. A $\Gamma-$ semiring R is said to have a zero element if $0\gamma x = 0 = x\gamma 0$ and $x + 0 = x = 0 + x$ for all $x \in R$ and $\gamma \in \Gamma$. A $\Gamma-$ semiring R is said to have an identity element if there exists $\gamma \in \Gamma$ such that $1\gamma x = x = x\gamma 1$ for all $x \in R$. A $\Gamma-$ semiring R is said to have a strong identity element if for all $x \in R$, $1\alpha x = x = x\alpha 1$ for all $\alpha \in \Gamma$. A $\Gamma-$ semiring R is said to $\Gamma-$ multiplicatively cancellative if for all $x, y, z \in R, x \neq 0$ and $\alpha \in \Gamma$ we have, $x\alpha y = x\alpha z$ and $y\alpha x = z\alpha x$ implies $y = z$. An element x of a $\Gamma-$semiring R is said to be additive idempotent if $x + x = x$. If every element of R is additive idempotent, then R is called additive idempotent(band). An element x of a $\Gamma-$ semiring R is said to be multiplicative $\Gamma-$ idempotent if there exists $\gamma \in \Gamma$ such that $x = x\gamma x$. If every element of R is multiplicative $\Gamma-$ idempotent, then R is called multiplicative $\Gamma-$ idempotent($\Gamma-$band). An element x of R is said to be strongly multiplicative $\Gamma-$ idempotent if $x = x\gamma x$ for all $\gamma \in \Gamma$.

An element x of a $\Gamma-$ semiring R is multiplicatively sub $\Gamma-$ idempotent if $x + x\alpha x = x$ for all $\alpha \in \Gamma$. R is multiplicatively sub $\Gamma-$idempotent if each of its element is multiplicatively sub $\Gamma-$idempotent. A semigroup S is said to be a rectangular band if $x + y + x = x$, for all $x, y \in S$. S is called a rectangular $\Gamma-$band if for every pair $x, y \in S$ there exists $\alpha, \beta \in \Gamma$ such that $x\alpha y\beta x = x$. R is a left(right) singular if $x + y = x(x + y = y)$ for all $x, y \in R$. R is singular if R is both left and right singular. R is $\Gamma-$singular if $x\alpha y = x(x\alpha y = y)$ for all $x, y \in R, \alpha \in \Gamma$.

Remark 1. *Throughout this paper, R will denote a $\Gamma-$ semiring with zero elements "0" and identity element "1" unless otherwise stated.*

3 Complemented $\Gamma-$ semirings

In this section, we discuss properties of complemented $\Gamma-$ semirings. We determine the additive and multiplicative structures of complemented $\Gamma-$ semirings by assuming different properties on the additive and multiplicative structures.

Recall the following definitions from [16].

Definition 1 ([16])**.** *Let x, y be elements of a $\Gamma-$ semiring R. Then x is $\Gamma-$ interior y denoted by $x\nabla y$ if there exists an element $z \in R, \alpha \in \Gamma$ such that $x\alpha z = z\alpha x = 0$, and $z + y = 1$.*

Definition 2 ([16])**.** *An element x is complemented if $x\nabla x$. That is, there exists an element $y \in R, \alpha \in \Gamma$ such that $x\alpha y = y\alpha x = 0$ and $x + y = 1$. This element $y \in R$ is the complement of x in R. We will denote the complement of x by $x^{\perp}$.*

The following theorem is easy to prove, so we omit the proof.

Theorem 1. *Let R be commutative and complemented $\Gamma-$ semiring(semiring). Then $x + 1 = 1 + x = x$ if and only if R is a rectangular band.*

If R is complemented and a $\Gamma-$band, then by Theorem 1, R is a band.

Theorem 2. *Let R be a complemented $\Gamma-$semiring with strong identity. If R contains an additive identity zero, then R is a $\Gamma-$band.*

Proof. Let R be a complemented $\Gamma-$ semiring. Then for all $x \in R$, there exists an element $y \in R, \alpha \in \Gamma$ such that $x\alpha y = y\alpha x = 0$ and $x + y = 1$. Since $x + y = 1$ for all $x, y \in R$. So by Theorem 1, we have $x = x + 1$. Therefore, $x + 1 + y = x + y$ gives that $x + 1 + y = 1$. Now, for all $\alpha \in \Gamma$ we have $x\alpha(x + 1 + y) = x\alpha 1$ implies that $x = x\alpha x + x\alpha 1 + x\alpha y = x\alpha x + x + x\alpha y = x\alpha x + x\alpha 1 = x\alpha(x + 1) = x\alpha x$. Hence, R is a $\Gamma-$band. □

Example 1. Consider the set $R = \{0, 1, x, y\}$ and $\alpha \in \Gamma$. We define additive and multiplicative structure on R as

$+$	0	1	x	y
0	0	1	x	y
1	1	1	x	y
x	x	x	x	1
y	y	y	1	y

α	0	1	x	y
0	0	0	0	0
1	0	1	x	y
x	0	x	x	0
y	0	y	0	y

Then R is a $\Gamma-$ band.

Theorem 3. *Let R be a complemented $\Gamma-$ band with strong identity. Then R is a singular, commutative, and rectangular band.*

Proof. Let R be a complemented $\Gamma-$ band. Then for all $x \in R$, $\alpha \in \Gamma$, $x\alpha x = x$ and there exists an element $y \in R, \alpha \in \Gamma$ such that $x\alpha y = y\alpha x = 0$ and $x + y = 1$.

First, we show that R is singular. By Theorem 1, we have $x+1 = x = 1+x$, for all $x \in R$. This implies that $x+1+y = x+y$ gives that $x+1+y = 1$. Now, for $\alpha \in \Gamma$ we have $x\alpha(x+1+y) = x\alpha 1$

implies that $x = x\alpha x + x\alpha 1 + x\alpha y = x\alpha x + x + x\alpha y = x\alpha x + x\alpha(1+y) = x\alpha x + x\alpha y$. Therefore, $x + y = x\alpha x + x\alpha y + y = x\alpha x + (x+1)\alpha y = x\alpha x + x\alpha y = x\alpha x = x$. Thus, R is left singular. Similarly, we can prove R is a right singular matrix. Hence, R is singular.

Now, we prove that R is commutative. Consider $x\alpha y + 1\alpha y + y\alpha y + x\alpha x + 1\alpha x + x\alpha y = (x+1+y)\alpha y + x\alpha(x+1+y)$. This implies that $0 + y + y\alpha y + x\alpha x + x = (x+1+y)\alpha(y+x)$. Therefore, $y\alpha(1+y) + x\alpha(x+1) = x\alpha(y+x) + 1\alpha(y+x) + y\alpha(y+x)$. Thus, $y\alpha y + x\alpha x = x\alpha y + x\alpha x + y + x + y\alpha y + y\alpha x$. Now, since R is a $\Gamma-$band so, $y + x = x\alpha y + x\alpha x + y + x + y\alpha y + y\alpha x = 0 + x + y + x + y + y\alpha x = x + y + x + y\alpha(1+x) = x + y + x + y\alpha x = x + y + (1+y)\alpha x = x + y + y\alpha x = x + y + 0 = x + y$. Hence, R is commutative.

Finally, we show that R is a rectangular band. Consider $x + 1 + y = 1$ for all $x, y \in R$. Now, $y = y\alpha 1 = y\alpha(x+1+y) = y\alpha x + y + y\alpha y = y\alpha x + y + y\alpha y = y\alpha(x+1) + y\alpha y = y\alpha x + y\alpha y$. Again, $x + y = x + y\alpha x + y\alpha y = (1+y)\alpha x + y\alpha y = y\alpha x + y\alpha y = y\alpha(x+1) = y\alpha x$. This implies that $x + y + x = y\alpha x + x = 0 + x = x$. Hence, R is a rectangular band.

□

Theorem 4. *Let R be a complemented $\Gamma-$semiring with a strong identity. Then R is multiplicative sub $\Gamma-$idempotent.*

Proof. Let R be a complemented $\Gamma-$semiring. Then for all $x \in R, \alpha \in \Gamma$ there exists an element $y \in R$ such that $x\alpha y = y\alpha x = 0$ and $x + y = 1$ for all $x \in R$. So, by Theorem 1, $x + 1 = x$. This implies that $x + 1 + y = x + y = 1$ for all $x, y \in R$. Now, for all $\alpha \in \Gamma$, $x = x\alpha 1 = x\alpha(x+1+y) = x\alpha x + x\alpha 1 + x\alpha y = x\alpha x + x + x\alpha y = x\alpha x + x = x + x\alpha x$. Hence, R is a multiplicative sub-Γ-idempotent. □

Theorem 5. *Let R be a rectangular band. If R is complemented $\Gamma-$ semiring with strong identity, then $(x\alpha)^{n-1}x + (y\alpha)^{n-1}y = [(x+1+y)\alpha]^{n-1}(x+1+y) = 1$ for all n.*

Proof. Let R be a complemented $\Gamma-$semiring. Then for all $x \in R, \alpha \in \Gamma$ there exists an element $y \in R$ such that $x\alpha y = y\alpha x = 0$ and $x+y = 1$ for all $x \in R$. Now, by Theorem 1, $x+1 = x = 1+x$ for all $x \in R$. This implies that $x+1+y = x+y = 1$ for all $x, y \in R$. Now, we will prove the result by the induction process. Let $n = 1$. Then clearly, $x+y = x+1+y = 1$. Assume that the result is true for $n-1$. That is, $(x\alpha)^{n-2}x + (y\alpha)^{n-2}y = [(x+1+y)\alpha]^{n-2}(x+1+y) = 1$ for all n. Now, we will prove that the result is true for n. That is, $(x\alpha)^{n-1}x + (y\alpha)^{n-1}y = [(x+1+y)\alpha]^{n-1}(x+1+y) = 1$ for all n. Now,

$$
\begin{aligned}
(x\alpha)^{n-1}x + (y\alpha)^{n-1}y &= (x\alpha)^{n-1}x + (x\alpha)^{n-3}x.0 + (y\alpha)^{n-3}y.0 + (y\alpha)^{n-1}y \\
&= (x\alpha)^{n-1}x + (x\alpha)^{n-3}x\alpha(x\alpha y) + (y\alpha)^{n-3}y\alpha(y\alpha x) + (y\alpha)^{n-1}y \\
&= (x\alpha)^{n-1}x + (x\alpha)^{n-2}x\alpha y + (y\alpha)^{n-2}y\alpha x + y\alpha)^{n-1}y \\
&= (x\alpha)^{n-1}x + (x\alpha)^{n-2}x\alpha(1+y) + (y\alpha)^{n-2}y\alpha(x+1) + (y\alpha)^{n-1}y \\
&= (x\alpha)^{n-1}x + (x\alpha)^{n-2}x\alpha 1 + (x\alpha)^{n-2}x\alpha y + (y\alpha)^{n-2}y\alpha x + (y\alpha)^{n-2}y\alpha 1 \\
&\quad + (y\alpha)^{n-1}y \\
&= [(x\alpha)^{n-2}x + (y\alpha)^{n-2}y]\alpha[x+1+y] \\
&= [(x+1+y)\alpha]^{n-2}(x+1+y)\alpha(x+1+y)
\end{aligned}
$$

$$= [(x+1+y)\alpha]^{n-1}(x+1+y)$$
$$= 1.$$

Hence, the result follows by mathematical induction. □

4 Completely regular $\Gamma-$ semirings

In this section, we proved many results on different structures of $\Gamma-$ semirings like anti-inverse, quasi-separative, and distributive. Boolean $\Gamma-$ semiring is proved through a completely regular $\Gamma-$ semiring application. We determine the additive and multiplicative structures of completely regular $\Gamma-$ semiring by assuming different properties on the additive and multiplicative structures. Finally, by using the concept of a simple and completely regular $\Gamma-$ semiring, we prove that R is a partially ordered $\Gamma-$ semiring.

Definition 3. *An element x of a $\Gamma-$ semiring R is said to be completely regular if there exists an element y in R and $\alpha \in \Gamma$ satisfying $(i) x = x+y+x, (ii) x+y = y+x, (iii) x\alpha(x+y) = x+y$. If every element of R is completely regular, then R is said to be a completely regular $\Gamma-$ semiring.*

Definition 4. *An element x of a $\Gamma-$ semiring R is said to be completely $\Gamma-$regular if there exists an element $y \in R$ and $\alpha \in \Gamma$ satisfying $(i) x = x\alpha y\alpha x, (ii) x\alpha y = y\alpha x, (iii) x\alpha(x+y) = x+y$. If every element of R is completely $\Gamma-$regular, then R is said to be completely $\Gamma-$regular $\Gamma-$ semiring.*

Definition 5. *Let S be a completely regular semigroup. If for each element $x \in S$ there exists an element $y \in S$ such that $x+y+x = y$ and $y+x+y = x$, then the elements x, y are called anti-inverse elements in S.*

Definition 6. *Let S be a completely $\Gamma-$regular semigroup. If for each element $a \in S$ and $\alpha \in \Gamma$ there exists an element $b \in S$ such that $a\alpha b\alpha a = b$ and $b\alpha a\alpha b = a$ and a, b have their own anti-inverses, then the elements a, b are called anti-inverse elements in S.*

Example 2. Let S be a completely $\Gamma-$regular semigroup. We define operations with the following tables.

α	x	y	z
x	x	x	z
y	x	y	z
z	z	z	x

Since x and y are their own anti-inverses, that is, clearly $x\alpha x\alpha x = x, y\alpha y\alpha y = y$ and $z\alpha z\alpha z = x\alpha z = z$. Also, $z\alpha x\alpha z = z\alpha z = x, x\alpha z\alpha x = x\alpha z = z$. Hence, x and z are anti-inverses. If every element of R has an anti-inverse, then R is called an anti-inverse $\Gamma-$ semiring.

Definition 7. *A $\Gamma-$ semiring R is called quasi-seprative if for any $x, y \in R, \alpha \in \Gamma$, we have $x\alpha x = x\alpha y = y\alpha y$ implies that $x = y$.*

Definition 8. *An element x of a $\Gamma-$ semiring R is infinite if $x+r=x=r+x$ for all $r \in R$. Such an element is necessarily unique, since if x and y are infinite elements of R, then $x=x+y=y+x=y$.*

Note that 0 can never be infinite, since $0+1=1+0 \neq 0$.

Definition 9. *A $\Gamma-$ semiring R is simple if and only if $x+1=1=1+x$ for all $x \in R$.*

Simple $\Gamma-$ semirings are additive idempotent, but the converse is not true.

Example 3. Let $R=R^{+} \cup\{\infty\}$. Define a mapping $R \times \Gamma \times R \rightarrow R$ by $x \alpha y=x y$ for all $x, y \in R$ and $\alpha \in \Gamma$. Then R is a $\Gamma-$semiring. The $\Gamma-$ semiring $(R, min,+)$ is additive idempotent but not simple, where R^{+} is the set of nonnegative real numbers.

Theorem 6. *Let R be a multiplicative $\Gamma-$ idempotent and completely regular $\Gamma-$ semiring with strong identity. Then R is a simple $\Gamma-$ semiring.*

Proof. Let R be a completely regular $\Gamma-$ semiring. Then $x+y+x=x$. This implies that $y+x+x=x$. Therefore, $y+x=x$ so $y+x+1=x+1$ gives that $y+x+1+1=x+1+1$. This implies that $y+(1+x+1)=1+x+1$. Thus, $y+1=1$. Again, for some $\alpha \in \Gamma$, $x \alpha(x+y)=x+y$. This implies that $x \alpha x+x \alpha y+1=x+y+1$. Therefore, $x+x \alpha y+1+1=x+y+1+1= x+(1+y+1)=x+1$. Thus, $x+1=x \alpha 1+x \alpha y+1+1=x \alpha(1+y)+1=1+y+1=1$. Hence, $x+1=1$ for all $x \in R$. So R is a simple $\Gamma-$ semiring. □

Theorem 7. *Let R be a completely regular simple $\Gamma-$ semiring. Then R is a quasi-separative.*

Proof. Let R be a completely regular $\Gamma-$ semiring and $x, y \in R$.
Let $x \alpha x=x \alpha y$ for some $\alpha \in \Gamma$. Then $x \alpha x+y=x \alpha y+y$. Therefore, $x \alpha(x+y+x)+y=x \alpha(y+x+y)+y$ gives that $x \alpha(x+y)+x \alpha x+y=x \alpha(y+x)+x \alpha y+y$. So by multiplicative $\Gamma-$ idempotent and 3rd property of completely regular $\Gamma-$ semiring, we have $x+y+x+y=y+x+x \alpha y+y$. Therefore, $x+y+x+y=y+x \alpha(1+y)+y$ gives that $x+y+x+y=y+x+y$, since R is simple. This implies that $x+x+y+y=y+x+y$. Thus, $x+x+y=y$ implies that $x+y+x=y$. Hence, $x=y$. Again, let $x \alpha y=y \alpha y$ for some $x, y \in R, \alpha \in \Gamma$. Then

$$
\begin{aligned}
x+x \alpha y & =x+y \alpha y \\
x+x \alpha(y+x+y) & =x+y \alpha(y+x+y) \\
x+x \alpha(y+x)+x \alpha y & =x+y \alpha(y+x)+y \alpha y \\
x+y+x+x \alpha y & =(x+y+x)+y \\
x+y+x \alpha 1+x \alpha y & =x+y \\
x+y+x \alpha(1+y) & =x+y+(x+y) \\
x+y+x & =x+y+y \\
x+y+x & =y+x+y \\
x & =y .
\end{aligned}
$$

□

Theorem 8. *Let R be a strong multiplicative $\Gamma-$ idempotent and completely $\Gamma-$regular $\Gamma-$ semiring. Then $(x+z)\alpha(y+z) = x\alpha y + z$. That is, R is distributive.*

Proof. Let R be a completely $\Gamma-$regular $\Gamma-$ semiring. Then $x\alpha y\alpha x = x$ for all $x, y \in R, \alpha \in \Gamma$. Consider,

$$\begin{aligned}
(x+z)\alpha(y+z) &= (x+z)\alpha y + (x+z)\alpha z \\
&= x\alpha y + z\alpha y + x\alpha z + z\alpha z \\
&= x\alpha y + z\alpha y + x\alpha z + z \\
&= x\alpha y + z\alpha y + (x\beta z\beta x)\alpha(z\alpha x\alpha z) + z \\
&= x\alpha y + z\alpha y + (x\beta z)\beta(x\alpha z)\alpha x\alpha z + z \\
&= x\alpha y + z\alpha y + (x\beta z)\beta(z\alpha x)\alpha x\alpha z + z \\
&= x\alpha y + z\alpha y + (x\beta z\beta z)\alpha(x\alpha x)\alpha z + z \\
&= x\alpha y + z\alpha y + (x\beta z)\beta(z\alpha x\alpha z) + z \\
&= x\alpha y + z\alpha y + (x\beta z)\beta z + z \\
&= x\alpha y + z\alpha y + (z\beta x\beta z) + z \\
&= x\alpha y + z\alpha y + z + z \\
&= x\alpha y + z\alpha y + z \\
&= x\alpha y + (z\beta y\beta z)\alpha(y\alpha z\alpha y) + z \\
&= x\alpha y + (z\beta z)\beta(y\alpha y)\alpha(z\alpha y) + z \\
&= x\alpha y + (z\beta z)\beta(y\alpha z\alpha y) + z \\
&= x\alpha y + (z\beta z)\beta y + z \\
&= x\alpha y + (z\beta z\beta y) + z \\
&= x\alpha y + (z\beta y\beta z) + z \\
&= x\alpha y + z + z \\
&= x\alpha y + z \quad \text{for all} \quad x, y, z \in R, \quad \alpha, \beta, \delta \in \Gamma.
\end{aligned}$$

Similarly, we can prove all distributive conditions, that is, $z + x\alpha y = (z+x)\alpha(z+y)$ for all $x, y, z \in R, \alpha \in \Gamma$. □

The Proof of the following theorem is simple.

Theorem 9. *Let R be a multiplicative $\Gamma-$ idempotent $\Gamma-$semiring. Then R is completely regular if and only if $x + y = x = y + x$.*

Theorem 10. *Let R be a band and a completely regular $\Gamma-$ semiring. Then R is an anti-inverse $\Gamma-$ semiring.*

Proof. Let R be a completely regular $\Gamma-$ semiring. Then for any $x \in R$, there exists $y \in R$ satisfying the condition $x + y + x = x, x + y = y + x$ and $x\alpha(x+y) = x + y, \alpha \in \Gamma$, we will prove that $x + y + x = y$ and $y + x + y = x$. Now, Since R is completely regular $\Gamma-$ semiring so $x + y + x = x$ and $x + y = y + x$. This implies that $x + x + y = x$. Again, R is band and

$x + y = x$ therefore $x + (y + y) = x$. This implies that $y + x + y = x$. Similarly, we can prove $x + y + x = y$. □

Definition 10. *A $\Gamma-$ semiring R is said to be mono $\Gamma-$ semiring if $x + y = x\alpha y$ for all $x, y \in R, \alpha \in \Gamma$.*

Theorem 11. *Let R be a simple, $\Gamma-$multiplicatively cancellative, and completely regular $\Gamma-$ semiring with a strong identity. Then R is anti-inverse if and only if R is a mono $\Gamma-$ semiring.*

Proof. Let R be an anti-inverse and completely regular $\Gamma-$ semiring. Let $x \in R, \alpha \in \Gamma$ and $x + y + x = y$. Then $x\alpha(x + y + x) = x\alpha y$. This implies that $x\alpha(x + y) + x\alpha x = x\alpha y$. This implies that $x + y + x\alpha x = x\alpha y$. Therefore, $x\alpha(1 + x) + y = x\alpha y$. Thus, $x + y = x\alpha y$. Hence, R is a mono $\Gamma-$ semiring. Conversely, let $x + y = x\alpha y$. This implies that
$x\alpha y = x + y = x\alpha(x + y) = x\alpha x + x\alpha y = x\alpha(x + y + x) + x\alpha y$
This implies that $x\alpha y = x\alpha(x + y + x + y)$, so $y = x + y + x + y = x + y + x$.
Again, $x\alpha y = x\alpha(x + y + x + y)$, therefore, $y = x + y + x + y$. Thus, $y = y + x + y$. Hence, R is anti-inverse.

□

Theorem 12. *Let R be a simple, completely regular mono $\Gamma-$ semiring. Then $x + y + 1 = x\alpha y$ for all $x, y \in R, \alpha \in \Gamma$.*

Proof. Let R be a completely regular $\Gamma-$ semiring and $x, y \in R, \alpha \in \Gamma$. Consider $x + y + x = x$. This implies that $x + y + x + y = x + y$. Therefore, $x + y + x + y + 1 = x + y + 1$. So, $x\alpha y + x\alpha y + 1 = x + y + 1$. This implies that $x\alpha y + 1 + x\alpha y = x + y + 1$. Thus, $x\alpha y = x + y + 1$. Again, consider, $x + y = x\alpha(x + y) = x\alpha x + x\alpha y$ This implies that $x + y + 1 = x\alpha x + x\alpha y + 1 = x + x\alpha y + 1 = x\alpha y + (x + 1) = x\alpha y + 1$. Therefore, $x\alpha y + 1 + x\alpha y = x + y + 1 + x\alpha y = x + y + x\alpha y + 1 = x + y\alpha(1 + x) + 1 = x + y + 1$. Thus, $x\alpha y = x + y + 1$.

□

Theorem 13. *Let R be a $\Gamma-$multiplicative cancellative and completely regular $\Gamma-$ semiring with a strong identity. Then R is a simple $\Gamma-$ semiring if and only if R is multiplicative $\Gamma-$ idempotent.*

Proof. Let R be a simple and completely regular $\Gamma-$ semiring. Then $x + 1 = 1$ for all $x \in R$. This implies that $x\alpha(x + 1) = x\alpha 1$, which implies $x\alpha x + x\alpha 1 = x$. Therefore, $x\alpha x + x = x$. By the second condition of complete regularity and since R is simple, we have $x = x\alpha x + (x + y + x) = x\alpha x + (x + y) + x = x\alpha x + x\alpha(x + y) + x = x\alpha x + x\alpha x + x\alpha y + x = x\alpha x + x\alpha x + x\alpha(y + 1) = x\alpha x + x\alpha x + x = x\alpha x + x + x\alpha x = x\alpha x$. for all $x \in R$. Thus, R is multiplicative $\Gamma-$ idempotent. Conversely, let R be multiplicative $\Gamma-$ idempotent. Therefore, $x = x\alpha x = x\alpha(x + y + x) = x\alpha(x + y) + x\alpha x = x + y + x\alpha x = x + x\alpha x = x\alpha(1 + x)$. Therefore, $x = x\alpha 1 = x\alpha(1 + x)$, gives that $1 = 1 + x$ for all $x \in R$. Thus, R is a simple $\Gamma-$ semiring. □

Theorem 14. *Let R be a completely $\Gamma-$ regular $\Gamma-$ semiring. Then $x\alpha y\alpha x = y\alpha x\alpha y$ for all $x, y \in R, \alpha \in \Gamma$.*

Proof. Let $x, y \in R$ and $\alpha \in \Gamma$. Then
$x\alpha y\alpha x = (x\alpha y\alpha x)\alpha(y\alpha x\alpha y)\alpha(x\alpha y\alpha x) = x\alpha(y\alpha x\alpha y)\alpha(x\alpha y\alpha x)\alpha(y\alpha x) = x\alpha y\alpha x\alpha y\alpha x$
$= y\alpha x\alpha x\alpha x\alpha y = y\alpha x\alpha y.$ □

The following definition is analogous to the definition in ([11], definition 1)

Definition 11. *Let* R *be a* $\Gamma-$ *semiring. Then* R *is called Boolean* $\Gamma-$ *semiring if*

(i) R *is an additive abelian group.*

(ii) R *is a* $\Gamma-$ *Semigroup*

(iii) $x\alpha(y + z) = x\alpha y + x\alpha z$

(iv) $x\alpha y\alpha z = y\alpha x\alpha z$ *for all* $x, y, z \in R$ *and* $\alpha \in \Gamma$.

Theorem 15. *Let* R *be a completely* $\Gamma-$ *regular* $\Gamma-$ *semiring. If* $x\alpha y = x\alpha x\alpha y$ *for all* $x, y \in R, \alpha \in \Gamma$*, then* R *is a Boolean* $\Gamma-$ *semiring.*

Proof. Let R be a completely regular $\Gamma-$ semiring. Then $x\alpha x = x\alpha x\alpha x = x$ and $x\alpha 0 = 0\alpha x = 0$ for all $x \in R, \alpha \in \Gamma$. For $x, y \in R$ and by the conditions of complete regularity, $x\alpha y = y\alpha x$ for all $x, y \in R, \alpha \in \Gamma$. Further, consider $x\alpha(y\alpha z) = x\alpha(y\alpha y\alpha z) = (x\alpha x)\alpha(y\alpha y\alpha z) = (x\alpha y)\alpha(y\alpha z\alpha z = (x\alpha y)\alpha(z\alpha y\alpha z = (x\alpha y)\alpha z$. Thus, R is a $\Gamma-$ semigroup.

Again, $x\alpha(y+z) = x\alpha(x\alpha(y+z)) = x\alpha(x\alpha y)+x\alpha(x\alpha z) = x\alpha y+x\alpha z$. Since R is commutative $\Gamma-$ semigroup so $(y + z)\alpha x = y\alpha x + z\alpha x = x\alpha y + x\alpha z$. Hence, $x\alpha(y + z) = (y + z)\alpha x$. Further, since $x + x = 0$ for all $x \in R$, every element of R has an additive inverse. Thus,

$$\begin{aligned}
(x\alpha y)\alpha z &= (x\alpha y\alpha x)\alpha z \\
&= (x\alpha x\alpha y)(x\alpha x\alpha y)\alpha z \\
&= (x\alpha y\alpha x)\alpha x\alpha(y\alpha x\alpha y)\alpha(x\alpha y\alpha x)\alpha(x\alpha y\alpha x)\alpha(y\alpha x\alpha y)\alpha z \\
&= x\alpha(y\alpha x\alpha x)\alpha y\alpha(x\alpha y\alpha x)\alpha(y\alpha x\alpha x)\alpha(y\alpha x\alpha y)\alpha(x\alpha z\alpha z) \\
&= y\alpha(x\alpha x\alpha x)\alpha y\alpha x\alpha x\alpha(y\alpha z) \\
&= y\alpha x\alpha(x\alpha y\alpha x)\alpha y\alpha z \\
&= y\alpha x\alpha x\alpha y\alpha z \\
&= y\alpha(x\alpha y\alpha x)\alpha z \\
&= y\alpha x\alpha z, \quad \text{for all} \quad x, y, z \in R, \quad \alpha \in \Gamma.
\end{aligned}$$

Hence, R is a Boolean $\Gamma-$ Semiring. □

Definition 12. *An element* x *of a* $\Gamma-$ *semiring* R *is quasi-completely regular if and only if for all* $x \in R$ *there exists an element* $y \in R$ *such that* $x + y + x = y$.

Theorem 16. *Let* R *be a band and quasi-completely* $\Gamma-$ *regular* $\Gamma-$*semiring. Then*

(i) $x\alpha(x + y) = (x\alpha y)\alpha(x + y)$;

(ii) $x\alpha(x + y) = y$;

(iii) $x\alpha(x+y) = x$;

(iv) $x\alpha(x+y) = y\alpha(x+y)$;

(v) $x\alpha(x+y) = x\alpha y$.

Proof. Let $x, y \in R, \alpha \in \Gamma$.

(i) $x\alpha(x+y) = x+y = x\alpha y\alpha x + y\alpha x\alpha y = x\alpha y\alpha x + x\alpha y\alpha y = (x\alpha y)\alpha(x+y)$.

(ii) $x\alpha(x+y) = x+y = x+y+x+y = y+x+x+y = y+x+y = y$.

(iii) $x\alpha(x+y) = x+y = x+y+x+y = x+x+y+y = x+x+y = x+y+x = x$.

(iv) $x\alpha(x+y) = x\alpha(x\alpha y\alpha x + y\alpha x\alpha y) = x\alpha(x\alpha y\alpha(x+y)) = x\alpha y\alpha(x\alpha(x+y) = x\alpha y\alpha(x+y) = y\alpha(x\alpha(x+y)) = y\alpha(x+y)$.

(v) $x\alpha(x+y) = x\alpha(x\alpha y\alpha x + y\alpha x\alpha y) = x\alpha(x\alpha y\alpha(x+y)) = x\alpha x\alpha y\alpha(x+y)) = x\alpha y\alpha(x\alpha(x+y) = x\alpha y\alpha(y\alpha(x+y) = x\alpha y\alpha y\alpha(x+y) = x\alpha(y\alpha y\alpha x + y\alpha y\alpha y) = x\alpha(y\alpha x\alpha y + y) = x\alpha(y+y) = x\alpha y$.

□

Definition 13. *Let R be a $\Gamma-$ semiring. Then R is called partially ordered $\Gamma-$ semiring if there exists a partial order relation $\leq$ on R satisfying the following conditions:*
If $x \leq y$ and $z \geq 0$, then

(i) $x + z \leq y + z$

(ii) $x\alpha z \leq y\alpha z$

(iii) $z\alpha x \leq z\alpha y$, *for all $x, y, z \in R$ and $\alpha \in \Gamma$.*

If the relation $\leq$ is a total order, then R is a totally ordered $\Gamma-$ semiring.

Example 4 ([8])**.** Let $R = [0, 1], \Gamma = \mathbb{N}$. Define $x + y = max\{x, y\}$ and $x\alpha y = min\{x, \alpha, y\}$ for all $x, y \in R$ and $\alpha \in \Gamma$. Then R is a partially ordered $\Gamma-$ semiring with respect to usual ordering.

Example 5. let R be a $\Gamma-$ semiring and let $\{\leq_i | i \in \Omega\}$ be a family of partial order relations on R each of which turns R into a partially ordered $\Gamma-$ semirings. Then R is a partially ordered $\Gamma-$ semiring concerning the relation $\leq$ defined by $x \leq y$ if and only if $x \leq_i y$ for all $i \in \Omega$.

Finally, we have

Theorem 17. *Let R be a simple, completely regular, and multiplicative $\Gamma-$ idempotent infinite $\Gamma-$ semiring with strong identity. Define a relation $\leq$ on R such that $x \leq y$ if and only if $x + y + 1 = x\alpha y$ for all $x, y \in R$. If e is an identity of R, then R is a partially ordered $\Gamma-$ semiring.*

Proof. Let R be a completely regular $\Gamma-$ semiring. Consider $x\alpha x = x+x+1$. Then $x = x+1+x$. This implies that $x = x$. Thus, the relation $\leq$ is reflexive. Again, let $x \leq y$ and $y \leq x$. Then $x = y$. That is, $x + y + 1 = x\alpha y$ and $y + x + 1 = y\alpha x$ implies that $x = y$. Consider, $x\alpha y = x + y + 1 = y + x + 1 = y\alpha x$. This implies that $x\alpha y = y + (x + 1) = y + 1$. Therefore, $x\alpha y + y = y + 1 + y = y$. So $y = y\alpha x + y\alpha e = y\alpha(x + e) = x\alpha(x + e)$ for all $x \in R$. Thus, $x+e = y$. That is, $x = y$. Hence, $\leq$ is antisymmetric. Let $x \leq y$ and $y \leq z$. Then $x \leq z$. That is $x+y+1 = x\alpha z$. Consider, $x\alpha y = x+y+1$. This implies that $x\alpha y+z = x+y+1+z = x+1+z$. Therefore, $x + z + y + 1 = x\alpha z$, so $x + z + 1 = x\alpha z$.Thus, $x \leq z$. Hence, $\leq$ is transitive.

Claim: $x + z \leq y + z$. That is, $x + z + y + z + 1 = (x + z)\alpha(y + z)$. Let $x \leq y$. Then $x\alpha y = x+y+1$. This implies that $x\alpha y+z = x+y+1+z$. Therefore, $x\alpha y+z+z = (x+z)+(y+z)+1$, so $x\alpha y+x\alpha z+y\alpha z = (x+z)+(y+z)+1$. Thus, $x\alpha y+x\alpha z+y\alpha z+z\alpha z = (x+z)+(y+z)+1+z\alpha z$. Hence, $(x + z)\alpha(y + z) = (x + z) + (y + z) + 1$. Similarly, we can prove that $z + x \leq z + y$. Hence, $(R, \leq))$ is a partially ordered $\Gamma-$ semiring.

Claim: $x\alpha z \leq y\alpha z$. That is, $x\alpha z + y\alpha z + 1 = (x\alpha z)\alpha(y\alpha z)$. Consider $x\alpha y = x + y + 1$. This implies that $(x\alpha z)\alpha y = x\alpha z + y\alpha z + z$. Therefore, $(x\alpha z)\alpha(y\alpha z) = x\alpha(z\alpha z) + y\alpha(z\alpha z) + (z\alpha z) = (x\alpha z)\alpha(y\alpha z) + z\alpha 1 = x\alpha z + y\alpha z + 1$. Therefore, $x\alpha z \leq y\alpha z$. Similarly, we can prove that $z\alpha x \leq z\alpha y$.

□

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