

Complemented and completely regular Γ – semirings

Tilak Raj Sharma^{†*}, Rajesh Kumar[‡]

[†] *Himachal Pradesh University Regional Centre, Dharamshala (H.P.), India*

[‡] *Government Degree College, Dharamshala (H.P.), India*

Emails: trpangotra@gmail.com, rajeshdari1@gmail.com

Abstract. This paper investigates the additive and multiplicative properties of complemented and completely regular Γ – semirings. We prove many results on different structures of Γ – semirings like anti-inverse, quasi-separative, distributive, and partial order. Boolean Γ – semiring is demonstrated by applying the concept of a completely regular Γ – semiring. Finally, by using the idea of simple and completely regular Γ – semiring, we define a relation $\leq$ on R such that $x \leq y$ if and only if $x + y + 1 = x\alpha y$ for all $x, y \in R, \alpha \in \Gamma$ and prove that R is a partially ordered Γ – semiring.

Keywords: Regular band, Γ –semigroup, Quasi-separative, Partially ordered Γ – semiring.

AMS Subject Classification 2010: 16Y60 .

1 Introduction

Nobusawa [6] first proposed the concept of Γ –ring in 1964 as a generalization of the ring. Sen [9, 10] first developed the idea of Γ –semigroup. Rao [7] introduced the notion of Γ –semiring as a generalization of Γ –semiring, ternary semiring, and semirings. Because of the significance of results on Γ –semirings, which were proved by Dutta and Sardar [2–4] and many others, we were inspired to study Γ –semirings. Later, Sharma and Gupta [13] examined the implications of a commutative, simple, additive idempotent and multiplicative Γ –idempotent on a Γ –semiring R and studied the consequences of these implications on R . Further, more restrictions were imposed on Γ –semirings by Sharma and Ranote [17], to generalize the work of Sharma and Gupta [13]. Furthermore, Sharma and Gupta [16] introduced the concept of complementation of Γ – semirings. Because the complemented elements play an important role in lattices, another major source of inspiration for the theory of Γ – semirings is lattice theory. Thus, Sharma [12] introduced the concept of lattices in Γ –semirings.

*Corresponding author

Received: 20 December 2024/ Revised: 05 July 2025/ Accepted: 07 July 2025

DOI: [10.22124/JART.2025.29305.1747](https://doi.org/10.22124/JART.2025.29305.1747)

This paper is divided into two sections. In the first part of this paper, we extend some of the results regarding complemented Γ -semirings. We determine the additive and multiplicative structure of complemented Γ -semiring from [5] by assuming different properties on the additive and multiplicative structures. In the second section, we generalize some of the results of [1] to completely regular Γ -semirings. We proved many criteria on different structures of Γ -semirings like, anti-inverse, quasi-separative, distributive, and Boolean Γ -semirings.

2 Preliminaries and examples

We recall from [1, 14–17] some basic notations, definitions, and examples of Γ -semirings needed for this paper.

Let R and Γ be two additive commutative semigroups. Then R is called a Γ -semiring if there exists a mapping $R \times \Gamma \times R \rightarrow R$ denoted by $x\alpha y$ for all $x, y \in R$ and $\alpha \in \Gamma$ satisfying (i) $(x + y)\alpha z = x\alpha z + y\alpha z$ (ii) $x(\alpha + \beta)z = x\alpha z + x\beta z$ (iii) $x\alpha(y + z) = x\alpha y + x\alpha z$ (iv) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.

Every semiring R is a Γ -semiring. Let R be a semiring and Γ be a commutative semigroup. Define a mapping $R \times \Gamma \times R \rightarrow R$ denoted by $x\gamma y = xy$ for all $x, y, z \in R$ and $\gamma \in \Gamma$. Then R is a Γ -semiring. Let A and B be semirings and let $R = \text{Hom}(A, B)$ and $\Gamma = \text{Hom}(B, A)$ denote the sets of homomorphisms from A to B and B to A , respectively. Then R is a Γ -semiring with operations of pointwise addition and composition of mappings. Let M be a Γ -ring and R the set of all ideals of M . Define addition in the natural way and if $A, B \in R$, $\gamma \in \Gamma$, let $A\gamma B$ denote the ideal generated by $\{x\gamma y | x, y \in M\}$. Then R is a Γ -semiring.

R is said to be commutative if $x\gamma y = y\gamma x$ and $x + y = y + x$ for all $x, y \in R$ and $\gamma \in \Gamma$. A Γ -semiring R is said to have a zero element if $0\gamma x = 0 = x\gamma 0$ and $x + 0 = x = 0 + x$ for all $x \in R$ and $\gamma \in \Gamma$. A Γ -semiring R is said to have an identity element if there exists $\gamma \in \Gamma$ such that $1\gamma x = x = x\gamma 1$ for all $x \in R$. A Γ -semiring R is said to have a strong identity element if for all $x \in R$, $1\alpha x = x = x\alpha 1$ for all $\alpha \in \Gamma$. A Γ -semiring R is said to be Γ -multiplicatively cancellative if for all $x, y, z \in R$, $x \neq 0$ and $\alpha \in \Gamma$ we have, $x\alpha y = x\alpha z$ and $y\alpha x = z\alpha x$ implies $y = z$. An element x of a Γ -semiring R is said to be additive idempotent if $x + x = x$. If every element of R is additive idempotent, then R is called additive idempotent(band). An element x of a Γ -semiring R is said to be multiplicative Γ -idempotent if there exists $\gamma \in \Gamma$ such that $x = x\gamma x$. If every element of R is multiplicative Γ -idempotent, then R is called multiplicative Γ -idempotent(Γ -band). An element x of R is said to be strongly multiplicative Γ -idempotent if $x = x\gamma x$ for all $\gamma \in \Gamma$.

An element x of a Γ -semiring R is multiplicatively sub Γ -idempotent if $x + x\alpha x = x$ for all $\alpha \in \Gamma$. R is multiplicatively sub Γ -idempotent if each of its element is multiplicatively sub Γ -idempotent. A semigroup S is said to be a rectangular band if $x + y + x = x$, for all $x, y \in S$. S is called a rectangular Γ -band if for every pair $x, y \in S$ there exists $\alpha, \beta \in \Gamma$ such that $x\alpha y\beta x = x$. R is a left(right) singular if $x + y = x(x + y = y)$ for all $x, y \in R$. R is singular if R is both left and right singular. R is Γ -singular if $x\alpha y = x(x\alpha y = y)$ for all $x, y \in R$, $\alpha \in \Gamma$.

Remark 1. Throughout this paper, R will denote a Γ -semiring with zero elements “0” and identity element “1” unless otherwise stated.

3 Complemented Γ - semirings

In this section, we discuss properties of complemented Γ - semirings. We determine the additive and multiplicative structures of complemented Γ - semirings by assuming different properties on the additive and multiplicative structures.

Recall the following definitions from [16].

Definition 1 ([16]). *Let x, y be elements of a Γ - semiring R . Then x is Γ - interior y denoted by $x \nabla y$ if there exists an element $z \in R, \alpha \in \Gamma$ such that $x\alpha z = z\alpha x = 0$, and $z + y = 1$.*

Definition 2 ([16]). *An element x is complemented if $x \nabla x$. That is, there exists an element $y \in R, \alpha \in \Gamma$ such that $x\alpha y = y\alpha x = 0$ and $x + y = 1$. This element $y \in R$ is the complement of x in R . We will denote the complement of x by $x^\perp$.*

The following theorem is easy to prove, so we omit the proof.

Theorem 1. *Let R be commutative and complemented Γ - semiring(semiring). Then $x + 1 = 1 + x = x$ if and only if R is a rectangular band.*

If R is complemented and a Γ -band, then by Theorem 1, R is a band.

Theorem 2. *Let R be a complemented Γ -semiring with strong identity. If R contains an additive identity zero, then R is a Γ -band.*

Proof. Let R be a complemented Γ - semiring. Then for all $x \in R$, there exists an element $y \in R, \alpha \in \Gamma$ such that $x\alpha y = y\alpha x = 0$ and $x + y = 1$. Since $x + y = 1$ for all $x, y \in R$. So by Theorem 1, we have $x = x + 1$. Therefore, $x + 1 + y = x + y$ gives that $x + 1 + y = 1$. Now, for all $\alpha \in \Gamma$ we have $x\alpha(x + 1 + y) = x\alpha 1$ implies that $x = x\alpha x + x\alpha 1 + x\alpha y = x\alpha x + x + x\alpha y = x\alpha x + x\alpha 1 = x\alpha(x + 1) = x\alpha x$. Hence, R is a Γ -band. $\square$

Example 1. Consider the set $R = \{0, 1, x, y\}$ and $\alpha \in \Gamma$. We define additive and multiplicative structure on R as

$+$	0	1	x	y	α	0	1	x	y
0	0	1	x	y	0	0	0	0	0
1	1	1	x	y	1	0	1	x	y
x	x	x	x	1	x	0	x	x	0
y	y	y	1	y	y	0	y	0	y

Then R is a Γ - band.

Theorem 3. *Let R be a complemented Γ - band with strong identity. Then R is a singular, commutative, and rectangular band.*

Proof. Let R be a complemented Γ - band. Then for all $x \in R, \alpha \in \Gamma, x\alpha x = x$ and there exists an element $y \in R, \alpha \in \Gamma$ such that $x\alpha y = y\alpha x = 0$ and $x + y = 1$.

First, we show that R is singular. By Theorem 1, we have $x + 1 = x = 1 + x$, for all $x \in R$. This implies that $x + 1 + y = x + y$ gives that $x + 1 + y = 1$. Now, for $\alpha \in \Gamma$ we have $x\alpha(x + 1 + y) = x\alpha 1$

implies that $x = x\alpha x + x\alpha 1 + x\alpha y = x\alpha x + x + x\alpha y = x\alpha x + x\alpha(1 + y) = x\alpha x + x\alpha y$. Therefore, $x + y = x\alpha x + x\alpha y + y = x\alpha x + (x + 1)\alpha y = x\alpha x + x\alpha y = x\alpha x = x$. Thus, R is left singular. Similarly, we can prove R is a right singular matrix. Hence, R is singular.

Now, we prove that R is commutative. Consider $x\alpha y + 1\alpha y + y\alpha y + x\alpha x + 1\alpha x + x\alpha y = (x + 1 + y)\alpha y + x\alpha(x + 1 + y)$. This implies that $0 + y + y\alpha y + x\alpha x + x = (x + 1 + y)\alpha(y + x)$. Therefore, $y\alpha(1 + y) + x\alpha(x + 1) = x\alpha(y + x) + 1\alpha(y + x) + y\alpha(y + x)$. Thus, $y\alpha y + x\alpha x = x\alpha y + x\alpha x + y + x + y\alpha y + y\alpha x$. Now, since R is a Γ -band so, $y + x = x\alpha y + x\alpha x + y + x + y\alpha y + y\alpha x = 0 + x + y + x + y + y\alpha x = x + y + x + y\alpha(1 + x) = x + y + x + y\alpha x = x + y + (1 + y)\alpha x = x + y + y\alpha x = x + y + 0 = x + y$. Hence, R is commutative.

Finally, we show that R is a rectangular band. Consider $x + 1 + y = 1$ for all $x, y \in R$. Now, $y = y\alpha 1 = y\alpha(x + 1 + y) = y\alpha x + y + y\alpha y = y\alpha x + y + y\alpha y = y\alpha(x + 1) + y\alpha y = y\alpha x + y\alpha y$. Again, $x + y = x + y\alpha x + y\alpha y = (1 + y)\alpha x + y\alpha y = y\alpha x + y\alpha y = y\alpha(x + 1) = y\alpha x$. This implies that $x + y + x = y\alpha x + x = 0 + x = x$. Hence, R is a rectangular band. $\square$

Theorem 4. *Let R be a complemented Γ -semiring with a strong identity. Then R is multiplicative sub Γ -idempotent.*

Proof. Let R be a complemented Γ -semiring. Then for all $x \in R, \alpha \in \Gamma$ there exists an element $y \in R$ such that $x\alpha y = y\alpha x = 0$ and $x + y = 1$ for all $x \in R$. So, by Theorem 1, $x + 1 = x$. This implies that $x + 1 + y = x + y = 1$ for all $x, y \in R$. Now, for all $\alpha \in \Gamma$, $x = x\alpha 1 = x\alpha(x + 1 + y) = x\alpha x + x\alpha 1 + x\alpha y = x\alpha x + x + x\alpha y = x\alpha x + x = x + x\alpha x$. Hence, R is a multiplicative sub- Γ -idempotent. $\square$

Theorem 5. *Let R be a rectangular band. If R is complemented Γ -semiring with strong identity, then $(x\alpha)^{n-1}x + (y\alpha)^{n-1}y = [(x + 1 + y)\alpha]^{n-1}(x + 1 + y) = 1$ for all n .*

Proof. Let R be a complemented Γ -semiring. Then for all $x \in R, \alpha \in \Gamma$ there exists an element $y \in R$ such that $x\alpha y = y\alpha x = 0$ and $x + y = 1$ for all $x \in R$. Now, by Theorem 1, $x + 1 = x = 1 + x$ for all $x \in R$. This implies that $x + 1 + y = x + y = 1$ for all $x, y \in R$. Now, we will prove the result by the induction process. Let $n = 1$. Then clearly, $x + y = x + 1 + y = 1$. Assume that the result is true for $n - 1$. That is, $(x\alpha)^{n-2}x + (y\alpha)^{n-2}y = [(x + 1 + y)\alpha]^{n-2}(x + 1 + y) = 1$ for all n . Now, we will prove that the result is true for n . That is, $(x\alpha)^{n-1}x + (y\alpha)^{n-1}y = [(x + 1 + y)\alpha]^{n-1}(x + 1 + y) = 1$ for all n . Now,

$$\begin{aligned}
 (x\alpha)^{n-1}x + (y\alpha)^{n-1}y &= (x\alpha)^{n-1}x + (x\alpha)^{n-3}x.0 + (y\alpha)^{n-3}y.0 + (y\alpha)^{n-1}y \\
 &= (x\alpha)^{n-1}x + (x\alpha)^{n-3}x\alpha(x\alpha y) + (y\alpha)^{n-3}y\alpha(y\alpha x) + (y\alpha)^{n-1}y \\
 &= (x\alpha)^{n-1}x + (x\alpha)^{n-2}x\alpha y + (y\alpha)^{n-2}y\alpha x + (y\alpha)^{n-1}y \\
 &= (x\alpha)^{n-1}x + (x\alpha)^{n-2}x\alpha(1 + y) + (y\alpha)^{n-2}y\alpha(x + 1) + (y\alpha)^{n-1}y \\
 &= (x\alpha)^{n-1}x + (x\alpha)^{n-2}x\alpha 1 + (x\alpha)^{n-2}x\alpha y + (y\alpha)^{n-2}y\alpha x + (y\alpha)^{n-2}y\alpha 1 \\
 &\quad + (y\alpha)^{n-1}y \\
 &= [(x\alpha)^{n-2}x + (y\alpha)^{n-2}y]\alpha[x + 1 + y] \\
 &= [(x + 1 + y)\alpha]^{n-2}(x + 1 + y)\alpha(x + 1 + y)
 \end{aligned}$$

$$\begin{aligned}
&= [(x + 1 + y)\alpha]^{n-1}(x + 1 + y) \\
&= 1.
\end{aligned}$$

Hence, the result follows by mathematical induction. $\square$

4 Completely regular Γ - semirings

In this section, we proved many results on different structures of Γ - semirings like anti-inverse, quasi-separative, and distributive. Boolean Γ - semiring is proved through a completely regular Γ - semiring application. We determine the additive and multiplicative structures of completely regular Γ - semiring by assuming different properties on the additive and multiplicative structures. Finally, by using the concept of a simple and completely regular Γ - semiring, we prove that R is a partially ordered Γ - semiring.

Definition 3. An element x of a Γ - semiring R is said to be completely regular if there exists an element y in R and $\alpha \in \Gamma$ satisfying (i) $x = x + y + x$, (ii) $x + y = y + x$, (iii) $x\alpha(x + y) = x + y$. If every element of R is completely regular, then R is said to be a completely regular Γ - semiring.

Definition 4. An element x of a Γ - semiring R is said to be completely Γ -regular if there exists an element $y \in R$ and $\alpha \in \Gamma$ satisfying (i) $x = x\alpha y\alpha x$, (ii) $x\alpha y = y\alpha x$, (iii) $x\alpha(x + y) = x + y$. If every element of R is completely Γ -regular, then R is said to be completely Γ -regular Γ -semiring.

Definition 5. Let S be a completely regular semigroup. If for each element $x \in S$ there exists an element $y \in S$ such that $x + y + x = y$ and $y + x + y = x$, then the elements x, y are called anti-inverse elements in S .

Definition 6. Let S be a completely Γ -regular semigroup. If for each element $a \in S$ and $\alpha \in \Gamma$ there exists an element $b \in S$ such that $a\alpha b\alpha a = b$ and $b\alpha a\alpha b = a$ and a, b have their own anti-inverses, then the elements a, b are called anti-inverse elements in S .

Example 2. Let S be a completely Γ -regular semigroup. We define operations with the following tables.

α	x	y	z
x	x	x	z
y	x	y	z
z	z	z	x

Since x and y are their own anti-inverses, that is, clearly $x\alpha x\alpha x = x$, $y\alpha y\alpha y = y$ and $z\alpha z\alpha z = x\alpha z = z$. Also, $z\alpha x\alpha z = z\alpha z = x$, $x\alpha z\alpha x = x\alpha z = z$. Hence, x and z are anti-inverses. If every element of R has an anti-inverse, then R is called an anti-inverse Γ - semiring.

Definition 7. A Γ - semiring R is called quasi-separative if for any $x, y \in R, \alpha \in \Gamma$, we have $x\alpha x = x\alpha y = y\alpha y$ implies that $x = y$.

Definition 8. An element x of a Γ -semiring R is infinite if $x + r = x = r + x$ for all $r \in R$. Such an element is necessarily unique, since if x and y are infinite elements of R , then $x = x + y = y + x = y$.

Note that 0 can never be infinite, since $0 + 1 = 1 + 0 \neq 0$.

Definition 9. A Γ -semiring R is simple if and only if $x + 1 = 1 = 1 + x$ for all $x \in R$.

Simple Γ -semirings are additive idempotent, but the converse is not true.

Example 3. Let $R = R^+ \cup \{\infty\}$. Define a mapping $R \times \Gamma \times R \rightarrow R$ by $x\alpha y = xy$ for all $x, y \in R$ and $\alpha \in \Gamma$. Then R is a Γ -semiring. The Γ -semiring $(R, \min, +)$ is additive idempotent but not simple, where R^+ is the set of nonnegative real numbers.

Theorem 6. Let R be a multiplicative Γ -idempotent and completely regular Γ -semiring with strong identity. Then R is a simple Γ -semiring.

Proof. Let R be a completely regular Γ -semiring. Then $x + y + x = x$. This implies that $y + x + x = x$. Therefore, $y + x = x$ so $y + x + 1 = x + 1$ gives that $y + x + 1 + 1 = x + 1 + 1$. This implies that $y + (1 + x + 1) = 1 + x + 1$. Thus, $y + 1 = 1$. Again, for some $\alpha \in \Gamma$, $x\alpha(x + y) = x + y$. This implies that $x\alpha x + x\alpha y + 1 = x + y + 1$. Therefore, $x + x\alpha y + 1 + 1 = x + y + 1 + 1 = x + (1 + y + 1) = x + 1$. Thus, $x + 1 = x\alpha 1 + x\alpha y + 1 + 1 = x\alpha(1 + y) + 1 = 1 + y + 1 = 1$. Hence, $x + 1 = 1$ for all $x \in R$. So R is a simple Γ -semiring. $\square$

Theorem 7. Let R be a completely regular simple Γ -semiring. Then R is a quasi-separative.

Proof. Let R be a completely regular Γ -semiring and $x, y \in R$. Let $x\alpha x = x\alpha y$ for some $\alpha \in \Gamma$. Then $x\alpha x + y = x\alpha y + y$. Therefore, $x\alpha(x + y + x) + y = x\alpha(y + x + y) + y$ gives that $x\alpha(x + y) + x\alpha x + y = x\alpha(y + x) + x\alpha y + y$. So by multiplicative Γ -idempotent and 3rd property of completely regular Γ -semiring, we have $x + y + x + y = y + x + x\alpha y + y$. Therefore, $x + y + x + y = y + x\alpha(1 + y) + y$ gives that $x + y + x + y = y + x + y$, since R is simple. This implies that $x + x + y + y = y + x + y$. Thus, $x + x + y = y$ implies that $x + y + x = y$. Hence, $x = y$. Again, let $x\alpha y = y\alpha y$ for some $x, y \in R, \alpha \in \Gamma$. Then

$$\begin{aligned}
 x + x\alpha y &= x + y\alpha y \\
 x + x\alpha(y + x + y) &= x + y\alpha(y + x + y) \\
 x + x\alpha(y + x) + x\alpha y &= x + y\alpha(y + x) + y\alpha y \\
 x + y + x + x\alpha y &= (x + y + x) + y \\
 x + y + x\alpha 1 + x\alpha y &= x + y \\
 x + y + x\alpha(1 + y) &= x + y + (x + y) \\
 x + y + x &= x + y + y \\
 x + y + x &= y + x + y \\
 x &= y.
 \end{aligned}$$

$\square$

Theorem 8. *Let R be a strong multiplicative Γ - idempotent and completely Γ -regular Γ - semiring. Then $(x + z)\alpha(y + z) = x\alpha y + z$. That is, R is distributive.*

Proof. Let R be a completely Γ -regular Γ - semiring. Then $x\alpha y\alpha x = x$ for all $x, y \in R, \alpha \in \Gamma$. Consider,

$$\begin{aligned}
 (x + z)\alpha(y + z) &= (x + z)\alpha y + (x + z)\alpha z \\
 &= x\alpha y + z\alpha y + x\alpha z + z\alpha z \\
 &= x\alpha y + z\alpha y + x\alpha z + z \\
 &= x\alpha y + z\alpha y + (x\beta z\beta x)\alpha(z\alpha x\alpha z) + z \\
 &= x\alpha y + z\alpha y + (x\beta z)\beta(x\alpha z)\alpha x\alpha z + z \\
 &= x\alpha y + z\alpha y + (x\beta z)\beta(z\alpha x)\alpha x\alpha z + z \\
 &= x\alpha y + z\alpha y + (x\beta z\beta z)\alpha(x\alpha x)\alpha z + z \\
 &= x\alpha y + z\alpha y + (x\beta z)\beta(z\alpha x\alpha z) + z \\
 &= x\alpha y + z\alpha y + (x\beta z)\beta z + z \\
 &= x\alpha y + z\alpha y + (z\beta x\beta z) + z \\
 &= x\alpha y + z\alpha y + z + z \\
 &= x\alpha y + z\alpha y + z \\
 &= x\alpha y + (z\beta y\beta z)\alpha(y\alpha z\alpha y) + z \\
 &= x\alpha y + (z\beta z)\beta(y\alpha y)\alpha(z\alpha y) + z \\
 &= x\alpha y + (z\beta z)\beta(y\alpha z\alpha y) + z \\
 &= x\alpha y + (z\beta z)\beta y + z \\
 &= x\alpha y + (z\beta z\beta y) + z \\
 &= x\alpha y + (z\beta y\beta z) + z \\
 &= x\alpha y + z + z \\
 &= x\alpha y + z \quad \text{for all } x, y, z \in R, \quad \alpha, \beta, \delta \in \Gamma.
 \end{aligned}$$

Similarly, we can prove all distributive conditions, that is, $z + x\alpha y = (z + x)\alpha(z + y)$ for all $x, y, z \in R, \alpha \in \Gamma$. $\square$

The Proof of the following theorem is simple.

Theorem 9. *Let R be a multiplicative Γ - idempotent Γ -semiring. Then R is completely regular if and only if $x + y = x = y + x$.*

Theorem 10. *Let R be a band and a completely regular Γ - semiring. Then R is an anti-inverse Γ - semiring.*

Proof. Let R be a completely regular Γ - semiring. Then for any $x \in R$, there exists $y \in R$ satisfying the condition $x + y + x = x, x + y = y + x$ and $x\alpha(x + y) = x + y, \alpha \in \Gamma$, we will prove that $x + y + x = y$ and $y + x + y = x$. Now, Since R is completely regular Γ - semiring so $x + y + x = x$ and $x + y = y + x$. This implies that $x + x + y = x$. Again, R is band and

$x + y = x$ therefore $x + (y + y) = x$. This implies that $y + x + y = x$. Similarly, we can prove $x + y + x = y$. $\square$

Definition 10. A Γ - semiring R is said to be mono Γ - semiring if $x + y = x\alpha y$ for all $x, y \in R, \alpha \in \Gamma$.

Theorem 11. Let R be a simple, Γ -multiplicatively cancellative, and completely regular Γ - semiring with a strong identity. Then R is anti-inverse if and only if R is a mono Γ - semiring.

Proof. Let R be an anti-inverse and completely regular Γ - semiring. Let $x \in R, \alpha \in \Gamma$ and $x + y + x = y$. Then $x\alpha(x + y + x) = x\alpha y$. This implies that $x\alpha(x + y) + x\alpha x = x\alpha y$. This implies that $x + y + x\alpha x = x\alpha y$. Therefore, $x\alpha(1 + x) + y = x\alpha y$. Thus, $x + y = x\alpha y$. Hence, R is a mono Γ - semiring. Conversely, let $x + y = x\alpha y$. This implies that

$$x\alpha y = x + y = x\alpha(x + y) = x\alpha x + x\alpha y = x\alpha(x + y + x) + x\alpha y$$

This implies that $x\alpha y = x\alpha(x + y + x + y)$, so $y = x + y + x + y = x + y + x$.

Again, $x\alpha y = x\alpha(x + y + x + y)$, therefore, $y = x + y + x + y$. Thus, $y = y + x + y$. Hence, R is anti-inverse. $\square$

Theorem 12. Let R be a simple, completely regular mono Γ - semiring. Then $x + y + 1 = x\alpha y$ for all $x, y \in R, \alpha \in \Gamma$.

Proof. Let R be a completely regular Γ - semiring and $x, y \in R, \alpha \in \Gamma$. Consider $x + y + x = x$. This implies that $x + y + x + y = x + y$. Therefore, $x + y + x + y + 1 = x + y + 1$. So, $x\alpha y + x\alpha y + 1 = x + y + 1$. This implies that $x\alpha y + 1 + x\alpha y = x + y + 1$. Thus, $x\alpha y = x + y + 1$. Again, consider, $x + y = x\alpha(x + y) = x\alpha x + x\alpha y$. This implies that $x + y + 1 = x\alpha x + x\alpha y + 1 = x + x\alpha y + 1 = x\alpha y + (x + 1) = x\alpha y + 1$. Therefore, $x\alpha y + 1 + x\alpha y = x + y + 1 + x\alpha y = x + y + x\alpha y + 1 = x + y\alpha(1 + x) + 1 = x + y + 1$. Thus, $x\alpha y = x + y + 1$. $\square$

Theorem 13. Let R be a Γ -multiplicative cancellative and completely regular Γ - semiring with a strong identity. Then R is a simple Γ - semiring if and only if R is multiplicative Γ - idempotent.

Proof. Let R be a simple and completely regular Γ - semiring. Then $x + 1 = 1$ for all $x \in R$. This implies that $x\alpha(x + 1) = x\alpha 1$, which implies $x\alpha x + x\alpha 1 = x$. Therefore, $x\alpha x + x = x$. By the second condition of complete regularity and since R is simple, we have $x = x\alpha x + (x + y + x) = x\alpha x + (x + y) + x = x\alpha x + x\alpha(x + y) + x = x\alpha x + x\alpha x + x\alpha y + x = x\alpha x + x\alpha x + x\alpha(y + 1) = x\alpha x + x\alpha x + x = x\alpha x + x + x\alpha x = x\alpha x$. for all $x \in R$. Thus, R is multiplicative Γ - idempotent. Conversely, let R be multiplicative Γ - idempotent. Therefore, $x = x\alpha x = x\alpha(x + y + x) = x\alpha(x + y) + x\alpha x = x + y + x\alpha x = x + x\alpha x = x\alpha(1 + x)$. Therefore, $x = x\alpha 1 = x\alpha(1 + x)$, gives that $1 = 1 + x$ for all $x \in R$. Thus, R is a simple Γ - semiring. $\square$

Theorem 14. Let R be a completely Γ - regular Γ - semiring. Then $x\alpha y\alpha x = y\alpha x\alpha y$ for all $x, y \in R, \alpha \in \Gamma$.

Proof. Let $x, y \in R$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} x\alpha y\alpha x &= (x\alpha y\alpha x)\alpha(y\alpha x\alpha y)\alpha(x\alpha y\alpha x) = x\alpha(y\alpha x\alpha y)\alpha(x\alpha y\alpha x)\alpha(y\alpha x) = x\alpha y\alpha x\alpha y\alpha x \\ &= y\alpha x\alpha x\alpha x\alpha y = y\alpha x\alpha y. \end{aligned}$$

□

The following definition is analogous to the definition in ([11], definition 1)

Definition 11. Let R be a Γ - semiring. Then R is called Boolean Γ - semiring if

- (i) R is an additive abelian group.
- (ii) R is a Γ - Semigroup
- (iii) $x\alpha(y + z) = x\alpha y + x\alpha z$
- (iv) $x\alpha y\alpha z = y\alpha x\alpha z$ for all $x, y, z \in R$ and $\alpha \in \Gamma$.

Theorem 15. Let R be a completely Γ - regular Γ - semiring. If $x\alpha y = x\alpha x\alpha y$ for all $x, y \in R, \alpha \in \Gamma$, then R is a Boolean Γ - semiring.

Proof. Let R be a completely regular Γ - semiring. Then $x\alpha x = x\alpha x\alpha x = x$ and $x\alpha 0 = 0\alpha x = 0$ for all $x \in R, \alpha \in \Gamma$. For $x, y \in R$ and by the conditions of complete regularity, $x\alpha y = y\alpha x$ for all $x, y \in R, \alpha \in \Gamma$. Further, consider $x\alpha(y\alpha z) = x\alpha(y\alpha y\alpha z) = (x\alpha x)\alpha(y\alpha y\alpha z) = (x\alpha y)\alpha(y\alpha z\alpha z) = (x\alpha y)\alpha(z\alpha y\alpha z) = (x\alpha y)\alpha z$. Thus, R is a Γ - semigroup.

Again, $x\alpha(y+z) = x\alpha(x\alpha(y+z)) = x\alpha(x\alpha y) + x\alpha(x\alpha z) = x\alpha y + x\alpha z$. Since R is commutative Γ - semigroup so $(y+z)\alpha x = y\alpha x + z\alpha x = x\alpha y + x\alpha z$. Hence, $x\alpha(y+z) = (y+z)\alpha x$. Further, since $x + x = 0$ for all $x \in R$, every element of R has an additive inverse. Thus,

$$\begin{aligned} (x\alpha y)\alpha z &= (x\alpha y\alpha x)\alpha z \\ &= (x\alpha x\alpha y)(x\alpha x\alpha y)\alpha z \\ &= (x\alpha y\alpha x)\alpha x\alpha(y\alpha x\alpha y)\alpha(x\alpha y\alpha x)\alpha(x\alpha y\alpha x)\alpha(y\alpha x\alpha y)\alpha z \\ &= x\alpha(y\alpha x\alpha x)\alpha y\alpha(x\alpha y\alpha x)\alpha(y\alpha x\alpha x)\alpha(y\alpha x\alpha y)\alpha(x\alpha z\alpha z) \\ &= y\alpha(x\alpha x\alpha x)\alpha y\alpha x\alpha x\alpha(y\alpha z) \\ &= y\alpha x\alpha(x\alpha y\alpha x)\alpha y\alpha z \\ &= y\alpha x\alpha x\alpha y\alpha z \\ &= y\alpha(x\alpha y\alpha x)\alpha z \\ &= y\alpha x\alpha z, \quad \text{for all } x, y, z \in R, \quad \alpha \in \Gamma. \end{aligned}$$

Hence, R is a Boolean Γ - Semiring. □

Definition 12. An element x of a Γ - semiring R is quasi-completely regular if and only if for all $x \in R$ there exists an element $y \in R$ such that $x + y + x = y$.

Theorem 16. Let R be a band and quasi-completely Γ - regular Γ -semiring. Then

- (i) $x\alpha(x + y) = (x\alpha y)\alpha(x + y)$;
- (ii) $x\alpha(x + y) = y$;

$$(iii) \quad x\alpha(x+y) = x;$$

$$(iv) \quad x\alpha(x+y) = y\alpha(x+y);$$

$$(v) \quad x\alpha(x+y) = x\alpha y.$$

Proof. Let $x, y \in R, \alpha \in \Gamma$.

$$(i) \quad x\alpha(x+y) = x+y = x\alpha y\alpha x + y\alpha x\alpha y = x\alpha y\alpha x + x\alpha y\alpha y = (x\alpha y)\alpha(x+y).$$

$$(ii) \quad x\alpha(x+y) = x+y = x+y+x+y = y+x+x+y = y+x+y = y.$$

$$(iii) \quad x\alpha(x+y) = x+y = x+y+x+y = x+x+y+y = x+x+y = x+y+x = x.$$

$$(iv) \quad x\alpha(x+y) = x\alpha(x\alpha y\alpha x + y\alpha x\alpha y) = x\alpha(x\alpha y\alpha(x+y)) = x\alpha y\alpha(x\alpha(x+y)) = x\alpha y\alpha(x+y) = y\alpha(x\alpha(x+y)) = y\alpha(x+y).$$

$$(v) \quad x\alpha(x+y) = x\alpha(x\alpha y\alpha x + y\alpha x\alpha y) = x\alpha(x\alpha y\alpha(x+y)) = x\alpha x\alpha y\alpha(x+y) = x\alpha y\alpha(x\alpha(x+y)) = x\alpha y\alpha(y\alpha(x+y)) = x\alpha y\alpha y\alpha(x+y) = x\alpha(y\alpha y\alpha x + y\alpha y\alpha y) = x\alpha(y\alpha x\alpha y + y) = x\alpha(y+y) = x\alpha y.$$

□

Definition 13. Let R be a Γ - semiring. Then R is called partially ordered Γ - semiring if there exists a partial order relation $\leq$ on R satisfying the following conditions:

If $x \leq y$ and $z \geq 0$, then

$$(i) \quad x+z \leq y+z$$

$$(ii) \quad x\alpha z \leq y\alpha z$$

$$(iii) \quad z\alpha x \leq z\alpha y, \text{ for all } x, y, z \in R \text{ and } \alpha \in \Gamma.$$

If the relation $\leq$ is a total order, then R is a totally ordered Γ - semiring.

Example 4 ([8]). Let $R = [0, 1], \Gamma = \mathbb{N}$. Define $x+y = \max\{x, y\}$ and $x\alpha y = \min\{x, \alpha, y\}$ for all $x, y \in R$ and $\alpha \in \Gamma$. Then R is a partially ordered Γ - semiring with respect to usual ordering.

Example 5. let R be a Γ - semiring and let $\{\leq_i \mid i \in \Omega\}$ be a family of partial order relations on R each of which turns R into a partially ordered Γ - semirings. Then R is a partially ordered Γ - semiring concerning the relation $\leq$ defined by $x \leq y$ if and only if $x \leq_i y$ for all $i \in \Omega$.

Finally, we have

Theorem 17. Let R be a simple, completely regular, and multiplicative Γ - idempotent infinite Γ - semiring with strong identity. Define a relation $\leq$ on R such that $x \leq y$ if and only if $x+y+1 = x\alpha y$ for all $x, y \in R$. If e is an identity of R , then R is a partially ordered Γ - semiring.

Proof. Let R be a completely regular Γ -semiring. Consider $x\alpha x = x+x+1$. Then $x = x+1+x$. This implies that $x = x$. Thus, the relation $\leq$ is reflexive. Again, let $x \leq y$ and $y \leq x$. Then $x = y$. That is, $x + y + 1 = x\alpha y$ and $y + x + 1 = y\alpha x$ implies that $x = y$. Consider, $x\alpha y = x + y + 1 = y + x + 1 = y\alpha x$. This implies that $x\alpha y = y + (x + 1) = y + 1$. Therefore, $x\alpha y + y = y + 1 + y = y$. So $y = y\alpha x + y\alpha e = y\alpha(x + e) = x\alpha(x + e)$ for all $x \in R$. Thus, $x + e = y$. That is, $x = y$. Hence, $\leq$ is antisymmetric. Let $x \leq y$ and $y \leq z$. Then $x \leq z$. That is $x + y + 1 = x\alpha z$. Consider, $x\alpha y = x + y + 1$. This implies that $x\alpha y + z = x + y + 1 + z = x + 1 + z$. Therefore, $x + z + y + 1 = x\alpha z$, so $x + z + 1 = x\alpha z$. Thus, $x \leq z$. Hence, $\leq$ is transitive.

Claim: $x + z \leq y + z$. That is, $x + z + y + z + 1 = (x + z)\alpha(y + z)$. Let $x \leq y$. Then $x\alpha y = x + y + 1$. This implies that $x\alpha y + z = x + y + 1 + z$. Therefore, $x\alpha y + z + z = (x + z) + (y + z) + 1$, so $x\alpha y + x\alpha z + y\alpha z = (x + z) + (y + z) + 1$. Thus, $x\alpha y + x\alpha z + y\alpha z + z\alpha z = (x + z) + (y + z) + 1 + z\alpha z$. Hence, $(x + z)\alpha(y + z) = (x + z) + (y + z) + 1$. Similarly, we can prove that $z + x \leq z + y$. Hence, $(R, \leq)$ is a partially ordered Γ -semiring.

Claim: $x\alpha z \leq y\alpha z$. That is, $x\alpha z + y\alpha z + 1 = (x\alpha z)\alpha(y\alpha z)$. Consider $x\alpha y = x + y + 1$. This implies that $(x\alpha z)\alpha y = x\alpha z + y\alpha z + z$. Therefore, $(x\alpha z)\alpha(y\alpha z) = x\alpha(z\alpha z) + y\alpha(z\alpha z) + (z\alpha z) = (x\alpha z)\alpha(y\alpha z) + z\alpha 1 = x\alpha z + y\alpha z + 1$. Therefore, $x\alpha z \leq y\alpha z$. Similarly, we can prove that $z\alpha x \leq z\alpha y$. □

Acknowledgments

The authors would like to thank the reviewer for their constructive review of the paper's quality.

References

- [1] H. K. Chandrakala and A. Rajeshwari, *Completely regular semiring*, International Journal of Science and Research, (4) **9** (2020), 1696-1699.
- [2] T. K. Dutta and S. K. Sardar, *Semiprime ideals and irreducible ideals of a Γ -semiring*, Novi Sad Journal of Mathematics, (1) **30** (2000), 97-108.
- [3] T. K. Dutta and S. K. Sardar, *Study of noetherian Γ -semiring via operator semirings*, Southeast Asian Bulletin of Mathematics, (4) **25** (2002), 599-608.
- [4] T. K. Dutta and S. K. Sardar, *On the operator semirings of a Γ -semiring*, Southeast Asian Bulletin of Mathematics, (2) **26** (2003), 203-213.
- [5] G. Mallikarjuna and G. Shobhalatha, *Properties of complemented semirings*, International Organization of Scientific Research, Journal of Mathematics, (5) **12** (2016), 107-112.
- [6] N. Nobusawa, *On a generalization of the ring theory*, Osaka Journal of Mathematics, **1** (1964), 81-89.
- [7] M. M. K. Rao, *Γ -semirings-I*, South East Asian Bulletin of Mathematics, (1) **19** (1995), 49-54.

- [8] M. M. K. Rao, *On field Γ - semiring and complemented Γ - semiring with identity*, Bulletin of International Mathematical Virtual Institute, **8** (2017), 85-99.
- [9] M. K. Sen, *Γ - semigroup*, Proceedings of International Conference of Algebra and its Applications, Decker Publication, New York, (1981), 301-308.
- [10] M. K. Sen, and N. K. Saha, *On Γ - semigroup*, Bulletin of Calcutta Mathematical Society, (1986), 180-186.
- [11] N. V. Subrahmanyam, *Boolean semirings*, Mathematische Annalen, **148** (1962), 395-401.
- [12] T. R. Sharma, *Lattices in Γ - semirings*, International Journal of Creative Research Thoughts, (11) **8** (2020), 3360-3366.
- [13] T. R. Sharma and S. Gupta, *Some conditions on Γ - semirings*, Journal of Combinatorics, Information and System Sciences, (1-3) **41** (2016), 79-87.
- [14] T. R. Sharma and S. Gupta, *Ideals in Γ - semirings*, Journal of Combinatorics, Information and System Sciences, (1-4) **42** (2017), 165-177.
- [15] T. R. Sharma and S. Gupta, *Ideals of a bourne factor Γ - semirings*, Proceedings of National Conference on Advances in Mathematical Sciences, Research Journal of Science and Technology, (1) **9** (2017), 171-174.
- [16] T. R. Sharma and S. Gupta, *Complementation of Γ - semirings*, JP Journal of Algebra, Number theory and Applications, **50** (2021), 101-112.
- [17] T. R. Sharma and H. K. Ranote, *Some properties of a Γ - semiring*, JP Journal of Algebra, Number theory and Applications, (2) **52** (2021), 163-177.