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Monoid rings and the McCoy's theorem

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Abstract. Let M be a nilpotent quotient of a free monoid, satisfying the monoid ring $R[M]$ in the McCoy's theorem for any semiprime or right APP ring R is proven. Also, it is shown that $R[M]$ is right McCoy for any reduced ring R.

Keywords: APP ring, Free monoid, McCoy ring, McCoy's Theorem, Monoid ring, Triangular matrix ring.
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1 Introduction

Let R be an associative unitary ring. In [16] Liu and Zhao introduced right APP rings as a common generalization of right p.q.-Baer and left p.p.-rings. A ring R is called *right APP* if $r_R(aR)$ is left s-unital for any $a \in R$ (i.e. for each $b \in r_R(aR)$ there is an $x \in r_R(aR)$ such that $xb = b$). Clearly, every right p.q.-Baer ring is right APP, and so the class of right APP rings includes all biregular and all quasi-Baer rings. Recall that R is *quasi-Baer* if the right (equiv. left) annihilator of every right (equiv. left) ideal of R is generated by an idempotent of R as a right (equiv. left) ideal. Clark [2] defined quasi-Baer rings and used them to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. Next, Birkenmeier et al [1] introduced the concept of principally quasi-Baer rings. A ring R is called *right principally quasi-Baer* (simply *right p.q.-Baer*) if the right annihilator of a principal right ideal of R is generated by an idempotent of R as a right ideal. A ring R is called *Baer* (resp. *left p.p.*) if the left annihilator of every non-empty subset (with cardinal 1) of R is generated by an idempotent of R as a left ideal. Kaplansky [11] introduced Baer rings to abstract various properties of AW*-algebras and von-Neumann algebras. A left p.p.-ring (Baer ring) R is right APP, by [4, Proposition 1].

According to Corollary 2.10 of [16], If R is right APP, then R is semiprime if and only if $r_R(aR) \subseteq \ell_R(a)$ for all $a \in R$ ($r_R(A)$ and $\ell_R(A)$ denote the right and the left annihilators of a

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subset A in R, respectively). Also, commutative APP rings are reduced, by [16, Corollary 2.11]. Moreover, [16, Theorem 3.8] it is shown that the right APP condition is a Morita invariant property. Further, R is right APP if and only if the upper triangular matrix ring $T_n(R)$ is right APP if and only if the polynomial extension $R[X]$ is right APP if and only if the Laurent ring $R[x, x^{-1}]$ is right APP if and only if the monoid ring $R[S]$ is right APP, for any u.p.-monoid S and a non-empty set of not necessarily commuting indeterminates X (see Proposition 3.6 and Corollary 3.12 of [16]).

McCoy [17, Theorem 2] proved that if R is a commutative ring, then every polynomial has a non-zero annihilator in $R[x]$ if and only if it has a non-zero annihilator in R. But it was shown in [22] that this result is not necessarily true for non-commutative rings. According to Nielsen [19] a ring R is said to be *right McCoy*, if it satisfies the following condition.

$$\forall f(x), g(x) \in R[x] \qquad f(x)g(x) = 0 \quad \Rightarrow \quad \exists 0 \neq c \in R; \;\; f(x)c = 0.$$

In [19, Theorem 2] he proved that if R is *reversible* (i.e. for $a, b \in R$, $ab = 0$ implies $ba = 0$), then R is right McCoy. By changing his view from element to ideal, McCoy [18, Theorem 1] proved that for an arbitrary ring R and any right ideal I of $T = R[x_1, \ldots, x_m]$, if $r_T(I) \neq 0$, then $r_R(I) \neq 0$. This interesting result is known as the McCoy's theorem. Then it was shown in [3, Example 3] that McCoy's theorem does not hold for the formal power series ring $R[[x]]$, even if R is commutative. In [10], Hong et al under conditions and restrictions, extended McCoy's theorem to the Ore extensions and the skew power series rings over non-commutative rings. They also in [10, Theorem 3] proved that for a u.p.-monoid G, $r_{R[G]}(A) \neq 0$ implies that $r_R(A) \neq 0$ for any right ideal A of the monoid ring $R[G]$.

In 2014, the following structure was introduced by the first author and Moussavi that included many of the famous triangular matrix extensions. Due to the nilpotency of triangular matrices, whenever the elements of the original diameter were all nilpotent, nilpotent monoids were used in the proposed structure. Let $F' = \langle X \rangle$ be a free monoid with the basis $X = \{u_1, \ldots, u_t\}$ and $F = F' \cup \{0\}$. Let M be a factor of F which setting certain monomials in X to 0. We say M is *nilpotent* if there exists some positive integer n such that $\alpha_1 \cdots \alpha_n = 0$, for each α_i in M. In this article, we assume that n is the smallest number with the said property. The first author and Moussavi [6] considered nilpotent quotients of F and studied the monoid ring $R[M]$. The addition and multiplication operations of this ring are as usual. It will be useful to pay attention to the fact that according to the structure of $R[M]$, elements of this ring are as follows

$$ae + \sum_{1 \leq k \leq t} a_k u_k + \sum_{1 \leq k_1, k_2 \leq t} a_{k_1 k_2} u_{k_1} u_{k_2} + \cdots + \sum_{1 \leq k_1, \ldots, k_{n-1} \leq t} a_{k_1 \cdots k_{n-1}} u_{k_1} \cdots u_{k_{n-1}},$$

in which e is the identity element of F'. From now on, throughout the article, the subset $\mathcal{C}_\alpha$ of R represents the set of all coefficients of α in $R[M]$. For more information about this structure and the results obtained so far, see [5], [7–9], [12–15] and [20].

In this article, we introduce a wide range of monoid rings that satisfy the McCoy's theorem. In more detail, we prove that for any semiprime (hence prime) or APP (hence quasi-Baer) ring

R, the monoid rings $R[M]$ satisfy the McCoy's theorem. Also, we show that $R[M]$ is right McCoy for any reduced ring R.

2 Results

Before stating the results, let us mention the following four subrings of the upper triangular matrix ring $T_n(R)$ are important examples of $R[M]$ (for additional information see [20]). Therefore, the results stated in this article are also valid for them.

For a matrix $A = (a_{ij})$ in $T_n(R)$ we put:

$$D_i = \{a_{1i}, a_{2\,i+1}, \ldots, a_{n-i+1\,n}\} \qquad \forall i = 1, \ldots, n.$$

The four well-known and widely used subrings of $T_n(R)$ are:

$S(R, n) = \{A : |D_1| = 1\}$;
$A(R, n) = \{A : |D_i| = 1 \quad \forall i = 1, \ldots, [n/2]\}$;
$B(R, n) = \{A : |D_i| = 1\,, |D_m \backslash \{a_{1m}\}| = 0 \text{ or } 1 \quad \forall i = 1, \ldots, m-1\}\,(n = 2m)$ and
$T(R, n) = \{A : |D_i| = 1 \quad \forall i = 1, \ldots, n\}$.

Clearly, $T(R, n) \cong R[x]/\langle x^n \rangle$. Note that the common feature of all these subrings is that the elements of them have a constant main diagonal.

Now, as a first result we prove that the monoid rings $R[M]$ satisfy the McCoy's theorem, for any semiprime ring R.

Theorem 1. *Let R be a semiprime ring. If $r_{R[M]}(I) \neq 0$, then $r_R(I) \neq 0$, for any right ideal I of $R[M]$.*

Proof. Let I be a right ideal of $R[M]$ and

$$\beta = be + \sum_{1 \le k \le t} b_k u_k + \sum_{1 \le k_1, k_2 \le t} b_{k_1 k_2} u_{k_1} u_{k_2} + \cdots + \sum_{1 \le k_1, \ldots, k_{n-1} \le t} b_{k_1 \cdots k_{n-1}} u_{k_1} \cdots u_{k_{n-1}}$$

be a non-zero element in $r_{R[M]}(I)$. Suppose

$$\alpha = ae + \sum_{1 \le k \le t} a_k u_k + \sum_{1 \le k_1, k_2 \le t} a_{k_1 k_2} u_{k_1} u_{k_2} + \cdots + \sum_{1 \le k_1, \ldots, k_{n-1} \le t} a_{k_1 \cdots k_{n-1}} u_{k_1} \cdots u_{k_{n-1}}$$

be an arbitrary element of I. So, $\alpha(R[M])\beta = 0$, and hence for each $r \in R$ we have $\alpha(re)\beta = 0$. Thus, we obtain the following equations:

$$arb = 0; \tag{2.1}$$
$$arb_k + a_k rb = 0; \qquad 1 \le k \le t \tag{2.2}$$
$$arb_{k_1 k_2} + a_{k_1} rb_{k_2} + a_{k_1 k_2} rb = 0; \qquad 1 \le k_1, k_2 \le t \tag{2.3}$$
$$arb_{k_1 k_2 k_3} + a_{k_1} rb_{k_2 k_3} + a_{k_1 k_2} rb_{k_3} + a_{k_1 k_2 k_3} rb = 0; \qquad 1 \le k_1, k_2, k_3 \le t$$
$$\vdots$$

Considering Eq. (2.1), we obtain $aRb = 0$. Now, by replacing r with rbr' for any $r' \in R$ in Eq. (2.2), we get $a(rbr')b_k + a_k(rbr')b = 0$ and consequently $a_k(rbr')b = 0$. Thus $a_kRbRb = 0$ and hence $a_kRb(Ra_kR)b = 0$. This implies that $a_kRb = 0$, since R is semiprime and hence from Eq. (2.2) we find that $aRb_k = 0$ for all $1 \leq k \leq t$. Next, by replacing r with rbr' in Eq. (2.3), we have $a(rbr')b_{k_1k_2} + a_{k_1}(rbr')b_{k_2} + a_{k_1k_2}(rbr')b = 0$, and so $a_{k_1k_2}rbr'b = 0$. Therefore, $a_{k_1k_2}RbRb = 0$. Hence $a_{k_1k_2}Rb(Ra_{k_1k_2}R)b \subseteq a_{k_1k_2}RbRb = 0$. It concludes that $a_{k_1k_2}Rb = 0$, since R is semiprime and consequently

$$arb_{k_1k_2} + a_{k_1}rb_{k_2} = 0, \qquad 1 \leq k_1, k_2 \leq t. \tag{2.4}$$

Now, by replacing r with $rb_{k_2}r'$ in Eq. (2.4), we obtain

$$a(rb_{k_2}r')b_{k_1k_2} + a_{k_1}(rb_{k_2}r')b_{k_2} = 0,$$

and thus $a_{k_1}(rb_{k_2}r')b_{k_2} = 0$. Similar to the technique used so far, we obtain $aRb_{k_1k_2} = a_{k_1}Rb_{k_2} = 0$ for all $1 \leq k_1, k_2 \leq t$. By continuing in this way, one can see that $pRq = 0$, for each $p \in \mathcal{C}_\alpha$ and $q \in \mathcal{C}_\beta$. Now, let c be a non-zero element in $\mathcal{C}_\beta$. Clearly $\alpha c = 0$. Therefore $Ic = 0$ and the result follows. □

Corollary 1. *Let R be a prime ring. If $r_{R[M]}(I) \neq 0$, then $r_R(I) \neq 0$ for any right ideal I of $R[M]$.*

Next, we prove that for a right APP ring R, the monoid rings $R[M]$ satisfy the McCoy's theorem. The following lemma that stated in [5], plays a key role in the proof.

Lemma 1 ([5], Lemma 2.2)**.** *Let R be a right APP ring and $a_1, \ldots, a_m, b_1, \ldots, b_n$ belong to R. If $a_iRb_j = 0$, for all i and j, then there exists $c \in R$ such that $b_j = cb_j$ and $a_iRc = 0$, for all i and j.*

Theorem 2. *Let R be a right APP ring. If $r_{R[M]}(I) \neq 0$, then $r_R(I) \neq 0$, for any right ideal I of $R[M]$.*

Proof. Let I be a right ideal of $R[M]$ and

$$\beta = be + \sum_{1\leq k\leq t} b_ku_k + \sum_{1\leq k_1,k_2\leq t} b_{k_1k_2}u_{k_1}u_{k_2} + \cdots + \sum_{1\leq k_1,\ldots,k_{n-1}\leq t} b_{k_1\cdots k_{n-1}}u_{k_1}\cdots u_{k_{n-1}}$$

be a non-zero element in $r_{R[M]}(I)$. Suppose

$$\alpha = ae + \sum_{1\leq k\leq t} a_ku_k + \sum_{1\leq k_1,k_2\leq t} a_{k_1k_2}u_{k_1}u_{k_2} + \cdots + \sum_{1\leq k_1,\ldots,k_{n-1}\leq t} a_{k_1\cdots k_{n-1}}u_{k_1}\cdots u_{k_{n-1}}$$

be an arbitrary element of I. So, $\alpha(R[M])\beta = 0$ and consequently for each $r, r_k, r_{k_1k_2}, \ldots, r_{k_1\cdots k_{t-1}}$ in R:

$$(ae + \sum_{1\leq k\leq t} a_ku_k + \sum_{1\leq k_1,k_2\leq t} a_{k_1k_2}u_{k_1}u_{k_2} + \cdots + \sum_{1\leq k_1,\ldots,k_{n-1}\leq t} a_{k_1\cdots k_{n-1}}u_{k_1}\cdots u_{k_{n-1}})\times$$

$$(re + \sum_{1\le k\le t} r_k u_k + \sum_{1\le k_1,k_2\le t} r_{k_1k_2} u_{k_1} u_{k_2} + \cdots + \sum_{1\le k_1,\ldots,k_{n-1}\le t} r_{k_1\cdots k_{n-1}} u_{k_1}\cdots u_{k_{n-1}})\times$$
$$(be + \sum_{1\le k\le t} b_k u_k + \sum_{1\le k_1,k_2\le t} b_{k_1k_2} u_{k_1} u_{k_2} + \cdots + \sum_{1\le k_1,\ldots,k_{n-1}\le t} b_{k_1\cdots k_{n-1}} u_{k_1}\cdots u_{k_{n-1}}) = 0.$$

Therefore, we have the following equations:

$$arb = 0; \qquad (2.5)$$
$$arb_k + ar_k b + a_k rb = 0; \qquad 1\le k\le t \qquad (2.6)$$
$$arb_{k_1k_2} + ar_{k_1}b_{k_2} + a_{k_1}rb_{k_2} + ar_{k_1k_2}b + a_{k_1}r_{k_2}b + a_{k_1k_2}rb = 0; \qquad 1\le k_1,k_2\le t \qquad (2.7)$$
$$arb_{k_1k_2k_3} + ar_{k_1}b_{k_2k_3} + a_{k_1}rb_{k_2k_3} + ar_{k_1k_2}b_{k_3} + a_{k_1}r_{k_2}b_{k_3} + a_{k_1k_2}rb_{k_3} + ar_{k_1k_2k_3}b + a_{k_1}r_{k_2k_3}b + a_{k_1k_2}r_{k_3}b + a_{k_1k_2k_3}rb = 0; \qquad 1\le k_1,k_2,k_3\le t$$
$$\vdots$$

Considering Eq.(2.5), we obtain $aRb = 0$. Hence, there exists $c \in R$ such that $cb = b$ and $aRc = 0$, since R is right APP. On the other hand, applying the equation $\alpha(R[M])\beta = 0$, we obtain:

$$(ae + \sum_{1\le k\le t} a_k u_k + \sum_{1\le k_1,k_2\le t} a_{k_1k_2} u_{k_1} u_{k_2} + \cdots + \sum_{1\le k_1,\ldots,k_{n-1}\le t} a_{k_1\cdots k_{n-1}} u_{k_1}\cdots u_{k_{n-1}})\times$$
$$(re + \sum_{1\le k\le t} r_k u_k + \sum_{1\le k_1,k_2\le t} r_{k_1k_2} u_{k_1} u_{k_2} + \cdots + \sum_{1\le k_1,\ldots,k_{n-1}\le t} r_{k_1\cdots k_{n-1}} u_{k_1}\cdots u_{k_{n-1}})(ce)\times$$
$$(be + \sum_{1\le k\le t} b_k u_k + \sum_{1\le k_1,k_2\le t} b_{k_1k_2} u_{k_1} u_{k_2} + \cdots + \sum_{1\le k_1,\ldots,k_{n-1}\le t} b_{k_1\cdots k_{n-1}} u_{k_1}\cdots u_{k_{n-1}}) = 0.$$

So we get the following equation:

$$arcb_k + ar_k cb + a_k rcb = 0. \qquad \forall k \in \{1,\ldots,t\}$$

Thus, for each $1 \le k \le t$, we have $a_k rb = 0$, since $cb = b$, $aRc = 0$ and $arb = 0$. Therefore $a_k Rb = 0$ and consequently $aRb_k = 0$, by Eq. (2.6). Now, let I be a right ideal generated by $a, a_1, \ldots, a_t$. Clearly, $r_R(I) = r_R(a) \cap r_R(a_1) \cdots \cap r_R(a_t)$. Thus $b \in r_R(I)$ implies that

$$\exists c_0 \in R, \qquad aRc_0 = 0 \quad \text{and} \quad c_0 b = b$$
$$\forall k,\ \exists c_k \in R, \qquad a_k Rc_k = 0 \quad \text{and} \quad c_k b = b.$$

Get $c' = c_0c_1\cdots c_t$. One can easily check that $c' \in r_R(I)$ (i.e. $aRc' = a_kRc' = 0$) and $b = c'b$. Next, by using the equation $\alpha(R[M])\beta = 0$, we obtain:

$$(ae + \sum_{1\le k\le t} a_k u_k + \sum_{1\le k_1,k_2\le t} a_{k_1k_2} u_{k_1} u_{k_2} + \cdots + \sum_{1\le k_1,\ldots,k_{n-1}\le t} a_{k_1\cdots k_{n-1}} u_{k_1}\cdots u_{k_{n-1}})\times$$

$$(re+\sum_{1\leq k\leq t} r_k u_k+\sum_{1\leq k_1,k_2\leq t} r_{k_1k_2}u_{k_1}u_{k_2}+\cdots+\sum_{1\leq k_1,\ldots,k_{n-1}\leq t} r_{k_1\cdots k_{n-1}}u_{k_1}\cdots u_{k_{n-1}})(c'e)\times$$

$$(be+\sum_{1\leq k\leq t} b_k u_k+\sum_{1\leq k_1,k_2\leq t} b_{k_1k_2}u_{k_1}u_{k_2}+\cdots+\sum_{1\leq k_1,\ldots,k_{n-1}\leq t} b_{k_1\cdots k_{n-1}}u_{k_1}\cdots u_{k_{n-1}})=0$$

and so we get the following equation:

$$arc'b_{k_1k_2}+ar_{k_1}c'b_{k_2}+a_{k_1}rcb_{k_2}+ar_{k_1k_2}c'b+a_{k_1}r_{k_2}c'b+a_{k_1k_2}rc'b=0,$$

for all $k_1,k_2\in\{1,\ldots,t\}$. According to the property mentioned above for c', we have $a_{k_1k_2}rb=0$ and therefore $a_{k_1k_2}Rb=0$ for all $k_1,k_2\in\{1,\ldots,t\}$, since r is an arbitrary element of R. Furthermore, $aRb_k=0$ implies that

$$\exists c''\in R,\qquad aRc''=0\quad\text{and}\quad c''b_k=b_k,\quad\forall k=1,\ldots t$$

by [21, Theorem 1]. By reusing equation $\alpha(R[M])\beta=0$ we have

$$(ae+\sum_{1\leq k\leq t} a_k u_k+\sum_{1\leq k_1,k_2\leq t} a_{k_1k_2}u_{k_1}u_{k_2}+\cdots+\sum_{1\leq k_1,\ldots,k_{n-1}\leq t} a_{k_1\cdots k_{n-1}}u_{k_1}\cdots u_{k_{n-1}})\times$$

$$(re+\sum_{1\leq k\leq t} r_k u_k+\sum_{1\leq k_1,k_2\leq t} r_{k_1k_2}u_{k_1}u_{k_2}+\cdots+\sum_{1\leq k_1,\ldots,k_{n-1}\leq t} r_{k_1\cdots k_{n-1}}u_{k_1}\cdots u_{k_{n-1}})(c''e)\times$$

$$(be+\sum_{1\leq k\leq t} b_k u_k+\sum_{1\leq k_1,k_2\leq t} b_{k_1k_2}u_{k_1}u_{k_2}+\cdots+\sum_{1\leq k_1,\ldots,k_{n-1}\leq t} b_{k_1\cdots k_{n-1}}u_{k_1}\cdots u_{k_{n-1}})=0.$$

Therefore, we get the following equation:

$$arc''b_{k_1k_2}+ar_{k_1}c''b_{k_2}+a_{k_1}rc''b_{k_2}+ar_{k_1k_2}c''b+a_{k_1}r_{k_2}c''b+a_{k_1k_2}rc''b=0,$$

for all $k_1,k_2\in\{1,\ldots,t\}$. By a similar argument as above, we have $a_{k_1}Rb_{k_2}=0$. Now, by considering Eq. (2.7), we get $aRb_{k_1k_2}=0$. Thus

$$aRb_{k_1k_2}=a_{k_1}Rb_{k_2}=a_{k_1k_2}Rb,$$

for each k_1 and k_2 in $\{1,\ldots,t\}$. By continuing in this way, we obtain $pRq=0$, for each $p\in\mathcal{C}_\alpha$ and $q\in\mathcal{C}_\beta$. Now, let c be a non-zero element in $\mathcal{C}_\beta$. Clearly $\alpha c=0$. Therefore, $Ic=0$ and the proof is complete. □

Corollary 2. *Let R be a quasi-Baer ring. If $r_{R[M]}(I)\neq 0$, then $r_R(I)\neq 0$ for any right ideal I of $R[M]$.*

Note that if $R[M]$ is right McCoy, then clearly so is R. In the following, we show that the property of being right McCoy is transferred from R to the monoid ring $R[M]$, for any reduced ring R. To demonstrate this, we need the following lemma.

Lemma 2. *Let R be a ring. Then $R[M][x]\cong R[x][M]$.*

Proof. Let $\alpha_0 + \cdots + \alpha_m x^m$ be an element of $R[M][x]$ and

$$\alpha_0 = a^{(0)}e + \sum_{1\le k\le t} a_k^{(0)} u_k + \sum_{1\le k_1,k_2\le t} a_{k_1k_2}^{(0)} u_{k_1}u_{k_2} + \cdots + \sum_{1\le k_1,\ldots,k_{n-1}\le t} a_{k_1\cdots k_{n-1}}^{(0)} u_{k_1}\cdots u_{k_{n-1}},$$

$$\vdots$$

$$\alpha_m = a^{(m)}e + \sum_{1\le k\le t} a_k^{(m)} u_k + \sum_{1\le k_1,k_2\le t} a_{k_1k_2}^{(m)} u_{k_1}u_{k_2} + \cdots + \sum_{1\le k_1,\ldots,k_{n-1}\le t} a_{k_1\cdots k_{n-1}}^{(m)} u_{k_1}\cdots u_{k_{n-1}}.$$

It is easy to show that $\varphi : R[M][x] \to R[x][M]$ given by

$$\sum_{i=0}^{m} \alpha_i x^i \to fe + \sum_{1\le k\le t} f_k u_k + \sum_{1\le k_1,k_2\le t} f_{k_1k_2} u_{k_1}u_{k_2} + \cdots + \sum_{1\le k_1,\ldots,k_{n-1}\le t} f_{k_1\cdots k_{n-1}} u_{k_1}\cdots u_{k_{n-1}},$$

where,

$$\begin{aligned}
&f = a^{(0)} + a^{(1)}x + \cdots + a^{(m)}x^m, && \\
&f_k = a_k^{(0)} + a_k^{(1)}x + \cdots + a_k^{(m)}x^m, && 1\le k\le t,\\
&f_{k_1k_2} = a_{k_1k_2}^{(0)} + a_{k_1k_2}^{(1)}x + \cdots + a_{k_1k_2}^{(m)}x^m, && 1\le k_1,k_2\le t,\\
&\quad\vdots && \\
&f_{k_1\cdots k_{n-1}} = a_{k_1\cdots k_{n-1}}^{(0)} + a_{k_1\cdots k_{n-1}}^{(1)}x + \cdots + a_{k_1\cdots k_{n-1}}^{(m)}x^m, && 1\le k_1,\ldots,k_{n-1}\le t,
\end{aligned}$$

is an isomorphism. □

Theorem 3. *Let R be a reduced ring. Then $R[M]$ is right McCoy.*

Proof. Let $F(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_r x^r$ and $G(x) = \beta_0 + \beta_1 x + \cdots + \beta_s x^s$ be two non-zero polynomials in $R[M][x]$ such that $F(x)G(x) = 0$. Suppose

$$\alpha_i = a^{(i)}e + \sum_{1\le k\le t} a_k^{(i)} u_k + \sum_{1\le k_1,k_2\le t} a_{k_1k_2}^{(i)} u_{k_1}u_{k_2} + \cdots + \sum_{1\le k_1,\ldots,k_{n-1}\le t} a_{k_1\cdots k_{n-1}}^{(i)} u_{k_1}\cdots u_{k_{n-1}},$$

and

$$\beta_j = b^{(j)}e + \sum_{1\le k\le t} b_k^{(j)} u_k + \sum_{1\le k_1,k_2\le t} b_{k_1k_2}^{(j)} u_{k_1}u_{k_2} + \cdots + \sum_{1\le k_1,\ldots,k_{n-1}\le t} b_{k_1\cdots k_{n-1}}^{(j)} u_{k_1}\cdots u_{k_{n-1}},$$

for each $0\le i\le r$ and $0\le j\le s$. We consider two following cases:

Case (1). Let $a^{(i)} = 0$ for all $i = 0,\ldots,r$. Since $M^{n-1}\ne 0$, there exist $p_1,\ldots,p_{n-1}$ in X such that $\varpi = u_{p_1}\cdots u_{p_{n-1}}$ is non-zero. Notice that some p_i's may be equal. Clearly $\alpha_i\varpi = 0$ for all $0\le i\le r$ and hence $F(x)\varpi = 0$.

Case (2). Let $a^{(j)} = 0$ for some j. According to Lemma 2, $F(x)G(x) = 0$ implies that

$$(fe + \sum_{1\le k\le t} f_k u_k + \sum_{1\le k_1,k_2\le t} f_{k_1k_2} u_{k_1}u_{k_2} + \cdots + \sum_{1\le k_1,\ldots,k_{n-1}\le t} f_{k_1\cdots k_{n-1}} u_{k_1}\cdots u_{k_{n-1}})\times$$

$$(ge+\sum_{1\le k\le t} g_k u_k+\sum_{1\le k_1,k_2\le t} g_{k_1k_2}u_{k_1}u_{k_2}+\cdots+\sum_{1\le k_1,\dots,k_{n-1}\le t} g_{k_1\cdots k_{n-1}}u_{k_1}\cdots u_{k_{n-1}})=0$$

where,

$$\begin{array}{ll}
f=a^{(0)}+a^{(1)}x+\cdots+a^{(r)}x^r, & g=b^{(0)}+b^{(1)}x+\cdots+b^{(s)}x^s,\\
f_k=a_k^{(0)}+a_k^{(1)}x+\cdots+a_k^{(r)}x^r, & g_k=b_k^{(0)}+b_k^{(1)}x+\cdots+b_k^{(s)}x^s,\\
f_{k_1k_2}=a_{k_1k_2}^{(0)}+a_{k_1k_2}^{(1)}x+\cdots+a_{k_1k_2}^{(r)}x^r, & g_{k_1k_2}=g_{k_1k_2}^{(0)}+g_{k_1k_2}^{(1)}x+\cdots+g_{k_1k_2}^{(s)}x^s,\\
\vdots & \vdots\\
f_{k_1\cdots k_{n-1}}=a_{k_1\cdots k_{n-1}}^{(0)}+\cdots+a_{k_1\cdots k_{n-1}}^{(r)}x^r, & g_{k_1\cdots k_{n-1}}=b_{k_1\cdots k_{n-1}}^{(0)}+\cdots+b_{k_1\cdots k_{n-1}}^{(s)}x^s,
\end{array}$$

Thus, we have:

$$\begin{array}{lll}
(1) & fg=0, & \\
(2) & fg_k+f_kg=0, & 1\le k\le t,\\
(3) & fg_{k_1k_2}+f_{k_1}g_{k_2}+f_{k_1k_2}g=0, & 1\le k_1,k_2\le t,\\
(4) & fg_{k_1k_2k_3}+f_{k_1}g_{k_2k_3}+f_{k_1k_2}g_{k_3}+f_{k_1k_2k_3}g=0, & 1\le k_1,k_2,k_3\le t,\\
 & \vdots &
\end{array}$$

Since $R[x]$ is reduced, by multiplying g from the right-hand side of above equations we obtain $fg=fg_k=fg_{k_1k_2}=\cdots=0$. On the other hand, $\beta\neq 0$ implies that at least one of the elements of $\mathbb{G}=\{g,g_k,\dots,g_{k_1\cdots k_{n-1}}\,|\,1\le k,k_i\le t\}$ is non-zero. Thus there exists a non-zero g' in $\mathbb{G}$ such that $fg'=0$. It concludes that $fc=0$ for some non-zero $c\in R$ and consequently $a^{(0)}c=a^{(1)}c=\cdots=a^{(r)}c=0$. This implies that $\alpha_i(c\varpi)=0$ for all $0\le i\le r$ and the proof is complete. □

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References

[1] G. F. Birkenmeier, J. Y. Kim and J. K. Park, *Principally quasi-Baer rings*, Comm. Algebra, **29** (2001), 639-660.

[2] W. E. Clark, *Twistedmatrix units semigroup algebras*, DukeMath. J., **34** (1967), 417-423.

[3] D. E. Fields, *Zero divisors and nilpotent elements in power series rings*, Proc. Amer. Math. Soc., (3) **27** (1971), 427-433.

[4] J. A. Fraser and W. K. Nicholson, *Reduced p.p.-rings*, Math. Japonica, **34** (1989), 715-725.

[5] M. Habibi, *A new class of non-semiprime quasi-Armendariz rings*, Stud. Sci. Math. Hung., (2) **51** (2014), 165-171.

[6] M. Habibi and A. Moussavi, *Annihilator properties of skew monoid rings*, Comm. Algebra, (2) **42** (2014), 842-852.

[7] M. Habibi, A. Moussavi and R. Manaviyat, *On quasi-Armendariz skew monoid rings*, J. New Researches Math., (24) **6** (2020), 45-52.

[8] M. Habibi and K. Paykan, *On skew generalized triangular matrix rings, J. Algebraic Systems*, (1) **13** (2025), 151-162.

[9] M. Habibi, K. Paykan and H. Arianpoor, *On generalized quasi Baer skew monoid rings*, J. Algebra Appl., (6) **23** (2024).

[10] C. Y. Hong, N. K. Kim and Y. Lee, *Extensions of McCoy's theorem*, Glasg. Math. J., (1) **52** (2010), 155-159.

[11] I. Kaplansky, *Rings of Operators*, W. A. Benjamin, 1968.

[12] A. Karimimansoub and A. Moussavi, *Idempotent elements and uniquely clean property of skew monoid rings*, Stud. Sci. Math. Hung., (1) **55** (2018), 23-40.

[13] A. Karimimansoub, A. Moussavi and M. Habibi, *Strongly clean elements of a skew monoid rings*, Int. Electron.J. Algebra, **21** (2017), 164-179.

[14] A. Karimimansoub and M. R. Sadeghi, *Strongly clean matrix rings over a skew monoid ring*, Algebra Colloq., (3) **30** (2023), 361-370.

[15] A. Leroy and J. Matczuk, *Zip property of certain ring extensions*, J. Pure Appl. Algebra, (1) **220** (2016), 335-345.

[16] Z. K. Liu and R. Y. Zhao, *A generalization of PP-rings and p.q.-Baer rings*, Glasg. Math. J., (2) **48** (2006), 217-229.

[17] N. H. McCoy, *Remarks on divisors of zero*, Amer. Math. Monthly, **49** (1942), 286-295.

[18] N. H. McCoy, *Annihilators in polynomial rings*, Amer. Math. Monthly, **64** (1957), 28-29.

[19] P. P. Nielsen, *Semi-commutativity and the McCoy condition*, J. Algebra , **298** (2006), 134-141.

[20] K. Paykan and M. Habibi, *Further results on skew monoid rings of a certain free monoid*, Cogent math. stat., **5** (2018).

[21] H. Tominaga, *On s-unital rings*, Math. J. Okayama Univ., **18** (1976), 117-134.

[22] L. Weiner, *Concerning a theorem of McCoy*, Amer. Math. Monthly, **59** (1952), 336-337.