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On Hom-Jacobi algebra structures

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Abstract. We define Hom-Jacobi algebras as an extension of Hom-Poisson algebras and we give some examples. We describe the universal property of first-order Hom-differential operators as well as the universal property of first-order Hom-differential multi-operators. By using these universal properties, we prove the existence and uniqueness of a canonical purely Hom-Jacobi form associated to purely Hom-Jacobi algebra.

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1 Introduction

Jacobi algebras are abstract algebraic counterparts of Jacobi manifolds. Jacobi manifolds as generalizations of symplectic or more generally Poisson manifolds, were introduced independently by Kirillov [6] and Lichnerowicz [9]. Both Poisson and Jacobi algebras are commutative algebras endowed with a Lie bracket. However, while the Poisson bracket is a derivation of the underlying commutative algebra, the Jacobi bracket is a first-order differential operator on the commutative algebra [1, 3, 4].

The notion of Hom-Lie algebras was introduced by Hartwig, Larsson, and Silvestrov in [5] as apart of a study of deformations of the Witt and the Virasoro algebras. In a Hom-Lie algebra, the Jacobi identity is twisted by a linear map, called the Hom-Jacobi identity. A Hom-type generalization of Poisson algebras, called Hom-Poisson algebras, was introduced by Makhlouf and Silvestrov in [11], which combines a commutative Hom-associative algebra and a Hom-Lie algebra such that a Hom-Leibniz identity is satisfied. In [7], the authors introduced the notion of a purely Hom-Poisson algebra, which combines a commutative associative algebra and a Hom-Lie algebra such that a Hom-Leibniz identity is satisfied.

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The aim of this paper is to generalize the notion of Hom-Poisson algebras by using the first-order Hom-differential operators on the commutative Hom-associative algebra with unit. The paper is organized as follows. In Section 2, we give the concept of first-order Hom-differential operator and we show that the space of first-order Hom-differential operators is a Hom-Lie algebra. In Section 3, we introduce the notion of Hom-Jacobi algebras, the notion of purely Hom-Jacobi algebras and we give some examples. Finally, in Section 4, we establish the universal property of first-order Hom-differential operators as well as the universal property of first-order hom-differential multi-operators. By using the universal property of first-order Hom-differential operators, we describe the necessary and sufficient condition of existence of purely Hom-Jacobi algebra structure in terms of canonical purely Hom-Jacobi form. We give some properties associated with a canonocal purely Hom-Jacobi form.

2 First-order Hom-differential operators

Throughout this paper, $\mathbb{K}$ will be an algebraically closed field of characteristic 0. Let A be a commutative associative algebra with unit 1_A over $\mathbb{K}$. Recall that an A-module M is an abelian group $(M,+)$ together with a scalar multiplication, $A\times\ M \longrightarrow M, (a,m)\longmapsto am$, such that the following hold:

(1) $a(m+n)=am+an$ and $(a+b)m=am+bm$, for all $a,b\in A$ and $m,n\in M$;

(2) $a(bm)=(ab)m$, for all $a,b\in A$ and $m\in M$;

(3) $1_Am=m$, for all $m\in M$.

An A-submodule N of an A-module M is a subgroup of $(M,+)$ which is closed under multiplication, that is, $an\in N$, for all $a\in A$ and $n\in N$.

Definition 1 ([5], Definition 2)**.** *Given a commutative associative algebra A with unit 1_A, an A-module M and an algebra endomorphism $\phi : A\longrightarrow A$, a $\mathbb{K}$-linear map $d : A\longrightarrow M$ is an ϕ-derivation of A into M if $d(ab)=d(a)\phi(b)+\phi(a)d(b)$, for all $a,b\in A$.*

Let us denote by $Der_\phi(A,M)$ the set of all ϕ-derivations of A into M and $Der_\phi(A)$ the set of all ϕ-derivations of A.

Let us now make the following definition.

Definition 2. *Let A be a commutative associative algebra with unit 1_A and M be an A-module. A first-order Hom-differential operator of A into M is a pair (D,ϕ) where $D : A\longrightarrow M$ is a $\mathbb{K}$-linear map and $\phi : A\longrightarrow A$ an algebra endomorphism such that*

$$D(ab)=D(a)\phi(b)+\phi(a)D(b)-\phi(a)\phi(b)D(1_A), \tag{1}$$

for all $a,b\in A$.

In this case, the $\mathbb{K}$-linear map $D : A\longrightarrow M$ is called first-order ϕ-differential operator of A into M. The space of all first-order ϕ-differential operators of A into M is denoted by $Diff_\phi(A,M)$ and the space of all first-order ϕ-differential operators of A is denoted by $Diff_\phi(A)$.

For all $a,b,c\in A$, $D_1,D_2,D\in Diff_\phi(A)$, we have

$$(aD)(bc)=(aD)(b)\phi(c)+\phi(b)(aD)(c)-\phi(b)\phi(c)(aD)(1_A), \tag{2}$$

$$\begin{aligned} (D_1 - D_2)(ab) &= (D_1 - D_2)(a)\phi(b) + \phi(a)(D_1 - D_2)(b) \\ &\quad -\phi(a)\phi(b)(D_1 - D_2)(1_A). \end{aligned} \tag{3}$$

Relations (2) and (3) respectively mean that for all $a, b, c \in A$, $D_1, D_2, D \in Diff_\phi(A)$, then $aD \in Diff_\phi(A)$ and $D_1 - D_2 \in Diff_\phi(A)$. Thus, $Diff_\phi(A)$ is an A-module and

$$Der_\phi(A) = \{D \in Diff_\phi(A) / D(1_A) = 0\}$$

is an A-submodule of $Diff_\phi(A)$.

For $x \in M$, let us define the linear map $L_x : A \longrightarrow M, a \longmapsto \phi(a) \cdot x$.

Proposition 1. *A linear map $D : A \longrightarrow M$ is a first-order ϕ-differential operator of A into M if and only if the linear map $D - L_{D(1_A)} : A \longrightarrow M$ is an ϕ-derivation.*

Proof. Let $D : A \longrightarrow M$ be a first-order ϕ-differential operator of A into M. For all $a, b \in A$,

$$\begin{aligned} \left(D - L_{D(1_A)}\right)(ab) &= D(a)\phi(b) + \phi(a)D(b) - \phi(a)\phi(b)D(1_A) \\ &\quad -\phi(a)\phi(b)D(1_A) \\ &= [D(a) - \phi(a)D(1_A)]\phi(b) \\ &\quad +\phi(a)[D(b) - \phi(b)D(1_A)] \\ &= \left(D - L_{D(1_A)}\right)(a)\phi(b) + \phi(a)\left(D - L_{D(1_A)}\right)(b). \end{aligned}$$

Conversely, if $D - L_{D(1_A)} : A \longrightarrow M$ is an ϕ-derivation, for all $a, b \in A$,

$$\left(D - L_{D(1_A)}\right)(ab) = \left(D - L_{D(1_A)}\right)(a)\phi(b) + \phi(a)\left(D - L_{D(1_A)}\right)(b).$$

That is,

$$\begin{aligned} D(ab) - \phi(a)\phi(b)D(1_A) &= D(a)\phi(b) - \phi(a)\phi(b)D(1_A) + \phi(a)D(b) \\ &\quad -\phi(a)\phi(b)D(1_A). \end{aligned}$$

Thus, we get (1). □

Proposition 2. *Let $\phi : A \longrightarrow A$ be an algebra automorphism. Then for $D \in Diff_\phi(A)$, the linear map $\alpha_\phi(D) : A \longrightarrow A$ such that*

$$\alpha_\phi(D) = \phi \circ D \circ \phi^{-1} \tag{4}$$

is a first-order ϕ-differential operator of A and the linear map
$\alpha_\phi : Diff_\phi(A) \longrightarrow Diff_\phi(A), D \longmapsto \phi \circ D \circ \phi^{-1}$ satisfies

$$\alpha_\phi(a \cdot D) = \phi(a) \cdot \alpha_\phi(D). \tag{5}$$

Proof. For all $a, b \in A$, $D \in Diff_\phi(A)$,

$$\begin{aligned} \alpha_\phi(D)(ab) &= \phi\left[D\left(\phi^{-1}(a)\phi^{-1}(b)\right)\right] \\ &= \phi\left[D(\phi^{-1}(a))b + aD(\phi^{-1}(b)) - abD(1_A)\right] \end{aligned}$$

$$\begin{aligned} &= \left(\phi\circ D\circ\phi^{-1}\right)(a)\,\phi(b)+\phi(a)\left(\phi\circ D\circ\phi^{-1}\right)(b) \\ &\quad -\phi(a)\phi(b)\phi\left[D\left(1_A\right)\right]. \end{aligned}$$

Since $1_A=\phi^{-1}\left(1_A\right)$, then

$$\alpha_\phi\left(D\right)(ab)=\alpha_\phi\left(D\right)(a)\phi(b)+\phi(a)\alpha_\phi\left(D\right)(b)-\phi(a)\phi(b)\alpha_\phi\left(D\right)\left(1_A\right).$$

For all $a,b\in A$, $D\in Diff_\phi\left(A\right)$,

$$\begin{aligned} \left[\alpha_\phi\left(a\cdot D\right)\right](b) &= \phi\left[aD(\phi^{-1}\left(b\right))\right] \\ & \phi(a)\cdot\left(\phi\circ D\circ\phi^{-1}\right)(b). \end{aligned}$$

Therefore, for all $a\in A$, $D\in Diff_\phi\left(A\right)$,

$$\alpha_\phi(a\cdot D)=\phi(a)\cdot\alpha_\phi(D).$$

□

Proposition 3. *Let R be a commutative ring with unit, A be an associative commutative R-algebra with unit 1_A, $\phi : A\longrightarrow A$ be an algebra automorphism and $Diff_\phi\left(A\right)$ be the space of first-order ϕ-differential operators of A. Then the linear map $\left[D_1,D_2\right]_\phi : A\longrightarrow A$ such that*

$$\left[D_1,D_2\right]_\phi=\phi\circ D_1\circ\phi^{-1}\circ D_2\circ\phi^{-1}-\phi\circ D_2\circ\phi^{-1}\circ D_1\circ\phi^{-1} \tag{6}$$

is a first-order ϕ-differential operator of A, for all $D_1,D_2\in Diff_\phi\left(A\right)$.

Proof. For all $a,b\in A$ and $D_1,D_2\in Diff_\phi\left(A\right)$, we have

$$\begin{aligned} & \left[D_1,D_2\right]_\phi(ab) \\ &= \left(\phi\circ D_1\circ\phi^{-1}\right)\left[D_2(\phi^{-1}\left(a\right))b+aD_2(\phi^{-1}\left(b\right))-abD_2\left(1_A\right)\right] \\ &\quad -\left(\phi\circ D_2\circ\phi^{-1}\right)\left[D_1(\phi^{-1}\left(a\right))b+aD_1(\phi^{-1}\left(b\right))-abD_1\left(1_A\right)\right]. \end{aligned}$$

By straightforward computations, we get,

$$\begin{aligned} & \left[D_1,D_2\right]_\phi(ab) \\ &= \left(\phi\circ D_1\circ\phi^{-1}\circ D_2\circ\phi^{-1}\right)(a)\phi(b)+\phi(a)\left(\phi\circ D_1\circ\phi^{-1}\circ D_2\circ\phi^{-1}\right)(b) \\ &\quad -\left(\phi\circ D_2\circ\phi^{-1}\circ D_1\circ\phi^{-1}\right)(a)\phi(b)-\phi(a)\left(\phi\circ D_2\circ\phi^{-1}\circ D_1\circ\phi^{-1}\right)(b) \\ &\quad -\phi(a)\phi(b)\left(\phi\circ D_1\circ\phi^{-1}\circ D_2\circ\phi^{-1}-\phi\circ D_2\circ\phi^{-1}\circ D_1\circ\phi^{-1}\right)\left(1_A\right). \end{aligned}$$

That is,

$$\begin{aligned} & \left[D_1,D_2\right]_\phi(ab) \\ &= \left[D_1,D_2\right]_\phi(a)\phi(b)+\phi(a)\left[D_1,D_2\right]_\phi(b)-\phi(a)\phi(b)\left[D_1,D_2\right]_\phi\left(1_A\right), \end{aligned}$$

for all $a,b\in A$. □

Definition 3 ([5], Definition 14)**.** *A Hom-Lie algebra is a triple* $(\mathcal{G}, [-,-], \alpha)$ *where* $\mathcal{G}$ *a vector space equipped with a skew-symmetric bilinear map* $[-,-] : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ *and a linear map* $\alpha : \mathcal{G} \longrightarrow \mathcal{G}$ *satisfying the following Hom-Jacobi identity:*

$$[\alpha(x), [y,z]] + [\alpha(y), [z,x]] + [\alpha(z), [x,y]] = 0,$$

for all $x, y, z \in \mathcal{G}$.

Proposition 4. *Let* $\phi : A \longrightarrow A$ *be an algebra automorphism. Define a bilinear map* $[-,-]_\phi : Diff_\phi(A) \times Diff_\phi(A) \longrightarrow Diff_\phi(A)$ *as*

$$[D_1, D_2]_\phi = \phi \circ D_1 \circ \phi^{-1} \circ D_2 \circ \phi^{-1} - \phi \circ D_2 \circ \phi^{-1} \circ D_1 \circ \phi^{-1}$$

and define a linear map $\alpha_\phi : Diff_\phi(A) \longrightarrow Diff_\phi(A)$ *as*

$$\alpha_\phi(D) = \phi \circ D \circ \phi^{-1}.$$

Then, the triple $\left(Diff_\phi(A), [-,-]_\phi, \alpha_\phi\right)$ *is a Hom-Lie algebra. Furthermore,* $Diff_\phi(A)$ *is an* A*-module satisfying following identities:*

(1)

$$\alpha_\phi\left([D_1, D_2]_\phi\right) = [\alpha_\phi(D_1), \alpha_\phi(D_2)]_\phi, \tag{7}$$

for all $a \in A$, *and* $D \in Diff_\phi(A)$;

(2)

$$[D_1, a \cdot D_2]_\phi = \phi(a) \cdot [D_1, D_2]_\phi + [\alpha_\phi(D_1)(a) - \phi(a) \cdot \alpha_\phi(D_1)(1_A)] \cdot \alpha_\phi(D_2), \tag{8}$$

for all $a \in A$, *and* $D_1, D_2 \in Diff_\phi(A)$.

Proof. From (6), for all $D \in Diff_\phi(A)$, we obtain $[D, D]_\phi = 0$. Thus the bracket is skew-symmetric. From (4) and (6), for all $D_1, D_2, D_3 \in Diff_\phi(A)$, we obtain

$$\begin{aligned}
&\left[[D_1, D_2]_\phi, \alpha_\phi(D_3)\right]_\phi \\
= \;& \phi^2 \circ D_1 \circ \phi^{-1} \circ D_2 \circ \phi^{-1} \circ D_3 \circ \left(\phi^{-1}\right)^2 \\
& -\phi^2 \circ D_3 \circ \phi^{-1} \circ D_1 \circ \phi^{-1} \circ D_2 \circ \left(\phi^{-1}\right)^2 \\
& -\phi^2 \circ D_2 \circ \phi^{-1} \circ D_1 \circ \phi^{-1} \circ D_3 \circ \left(\phi^{-1}\right)^2 \\
& +\phi^2 \circ D_3 \circ \phi^{-1} \circ D_2 \circ \phi^{-1} \circ D_1 \circ \left(\phi^{-1}\right)^2.
\end{aligned}$$

Thus, grouping relevant terms together, we get

$$\left[[D_1, D_2]_\phi, \alpha_\phi(D_3)\right]_\phi + \left[[D_2, D_3]_\phi, \alpha_\phi(D_1)\right]_\phi + \left[[D_3, D_1]_\phi, \alpha_\phi(D_2)\right]_\phi = 0.$$

(1) From (4) and (6), for all $D_1, D_2 \in Diff_\phi(A)$,

$$[\alpha_\phi(D_1), \alpha_\phi(D_2)]_\phi \;=\; \phi^2 \circ D_1 \circ \phi^{-1} \circ D_2 \circ \left(\phi^{-1}\right)^2$$

$$-\phi^2 \circ D_2 \circ \phi^{-1} \circ D_1 \circ \left(\phi^{-1}\right)^2 .$$

Thus, we obtain (7).

(2) For all $a, b \in A$, and $D_1, D_2 \in Diff_\phi(A)$, we have

$$\begin{aligned}
[D_1, a \cdot D_2]_\phi (b) &= \left(\phi \circ D_1 \circ \phi^{-1} \circ (a \cdot D_2) \circ \phi^{-1}\right)(b) \\
&\quad - \left(\phi \circ (a \cdot D_2) \circ \phi^{-1} \circ D_1 \circ \phi^{-1}\right)(b) \\
&= \phi \circ D_1 \left[\phi^{-1}(a) \cdot \phi^{-1}\left(D_2\left(\phi^{-1}(b)\right)\right)\right] \\
&\quad -\phi(a) \cdot \phi\left(D_2 \circ \phi^{-1} \circ D_1 \circ \phi^{-1}(b)\right).
\end{aligned}$$

Since $D_1 \in Diff_\phi(A)$, then

$$\begin{aligned}
[D_1, a \cdot D_2]_\phi (b) &= \phi \circ \left[D_1\left(\phi^{-1}(a)\right) \cdot \phi\left(\phi^{-1}\left(D_2\left(\phi^{-1}(b)\right)\right)\right)\right] \\
&\quad +\phi \circ \left[\phi\left(\phi^{-1}(a)\right) \cdot D_1\left(\phi^{-1}\left(D_2\left(\phi^{-1}(b)\right)\right)\right)\right] \\
&\quad -\phi \circ \left[\phi\left(\phi^{-1}(a)\right) \cdot \phi\left(\phi^{-1}\left(D_2\left(\phi^{-1}(b)\right)\right)\right) D_1(1_A)\right] \\
&\quad -\phi(a) \cdot \phi\left(D_2 \circ \phi^{-1} \circ D_1 \circ \phi^{-1}(b)\right).
\end{aligned}$$

That is

$$\begin{aligned}
[D_1, a \cdot D_2]_\phi (b) &= \phi(a) \cdot [D_1, D_2]_\phi (b) \\
&\quad + \left(\phi \circ D_1 \circ \phi^{-1}\right)(a) \cdot \left(\phi \circ D_2 \circ \phi^{-1}\right)(b) \\
&\quad -\phi(a) \cdot \phi\left[D_1(1_A)\right] \cdot \left(\phi \circ D_2 \circ \phi^{-1}\right)(b),
\end{aligned}$$

Since $1_A = \phi^{-1}(1_A)$ and $\alpha_\phi(D) = \phi \circ D \circ \phi^{-1}$, then

$$\begin{aligned}
& [D_1, a \cdot D_2]_\phi (b) \\
=\ & \phi(a) \cdot [D_1, D_2]_\phi (b) + [\alpha_\phi(D_1)(a) - \phi(a) \cdot \alpha_\phi(D_1)(1_A)] \cdot \alpha_\phi(D_2)(b),
\end{aligned}$$

for all $b \in A$. Thus, we obtain (8). □

3 Hom-Jacobi algebras

Definition 4 ([10], Definition 1.1)**.** *Let A be a vector space over an algebraically closed field $\mathbb{K}$ of characteristic 0. A Hom-associative algebra is a triple $(A, \cdot, \alpha)$ consisting of a vector space A, a bilinear map $\cdot : A \times A \longrightarrow A$ and a vector space homomorphism $\alpha : A \longrightarrow A$ satisfying*

$$\alpha(a) \cdot (b \cdot c) = (a \cdot b) \cdot \alpha(c),$$

for all $a, b, c \in A$. A Hom-associative algebra for which $x \cdot y = y \cdot x$ is called a commutative Hom-associative algebra. A commutative Hom-associative algebra $(A, \cdot, \alpha)$ is said to be regular if α is invertible.

Let us now make the following definition.

Definition 5. *Let* $(A,\cdot,\alpha)$ *be a commutative Hom-associative algebra. A skew-symmetric A-multilinear map* $Q : A \times A \times \cdots \times A \longrightarrow A$ *of degree* k *is a skew-symmetric* α*-derivation of degree* k *if*

$$Q\left(a\cdot b,\alpha\left(a_2\right),\ldots,\alpha\left(a_k\right)\right)=\alpha\left(a\right)\ldots Q\left(b,a_2,\ldots,a_k\right)+\alpha\left(b\right)\cdot Q\left(a,a_2,\ldots,a_k\right),$$

for all $a,b,a_i\in A$*,* $i\in\{2,\ldots,k\}$*.*

Example 1. If $(A,\cdot,\{-,-\},\alpha)$ is a Hom-Poisson algebra (see [7, Definition 1.4]), then the bracket $\{-,-\}: A\times A\longrightarrow A$ is a skew-symmetric α-derivation of degree 2, that is,

$$\{\alpha\left(a\right),b\cdot c\}=\alpha\left(b\right)\cdot\{a,c\}+\{a,b\}\cdot\alpha\left(c\right)$$

and

$$\{b\cdot c,\alpha\left(a\right)\}=\alpha\left(b\right)\cdot\{a,c\}+\{a,b\}\cdot\alpha\left(c\right),$$

for all $a,b,c\in A$.

Definition 6. *Let* $(A,\cdot,\alpha)$ *be a commutative Hom-associative algebra. A skew-symmetric A-multilinear map* $\varphi : A\times A\times\cdots\times A\longrightarrow A$ *of degree* k *is a skew-symmetric first-order* α*-differential operator of degree* k *if*

$$\begin{aligned}\varphi\left(a\cdot b,\alpha\left(a_2\right),\ldots,\alpha\left(a_k\right)\right) &= \alpha\left(a\right)\cdot\varphi\left(b,a_2,\ldots,a_k\right)\\ &\quad+\alpha\left(b\right)\cdot\varphi\left(a,a_2,\ldots,a_k\right)\\ &\quad-\alpha\left(a\right)\cdot\alpha\left(b\right)\cdot\varphi\left(1_A,a_2,\ldots,a_k\right),\end{aligned}$$

for all $a,b,a_i\in A$*,* $i\in\{2,\ldots,k\}$*.*

We introduce the notion of Hom-Jacobi algebra structure:

Definition 7. *A Hom-Jacobi algebra is a quadruple* $(A,\cdot,\{-,-\},\alpha)$*, where* $(A,\cdot,\alpha)$ *is a commutative Hom-associative algebra and* $(A,\{-,-\},\alpha)$ *is a Hom-Lie algebra, satisfying the generalized Hom-Leibniz identity:*

$$\{\alpha\left(a\right),b\cdot c\}=\alpha\left(b\right)\cdot\{a,c\}+\{a,b\}\cdot\alpha\left(c\right)-\alpha\left(b\right)\cdot\alpha\left(c\right)\cdot\{a,1_A\},$$

for all $a,b,c\in A$*.*

When $1_A\in Z(A)$, that is, $\{a,1_A\}=0$, we recover the notion of Hom-Poisson algebras given in [11, Definition 1.4]. When $\alpha=id_A$, we recover the notion of Jacobi algebras given in [4,6,9,13].

Example 2. Let $(A,\cdot_A,\{,\}_A,\alpha_A)$ and $(B,\cdot_B,\{,\}_B,\alpha_B)$ be two Hom-Jacobi algebras. Then $(A\oplus B,\cdot_{A\oplus B},\{-,-\}_{A\oplus B},\alpha_A\oplus\alpha_B)$ is a Hom-Jacobi algebra, where the bracket $\{-,-\}_{A\oplus B}$, the product $\cdot_{A\oplus B}$ and $\alpha_A\oplus\alpha_B$ are respectively given by

$$\{a+x,b+y\}_{A\oplus B}=\{a,b\}_A+\{x,y\}_B,$$

$$\left(a+x\right)\cdot_{A\oplus B}\left(b+y\right)=a\cdot_A b+x\cdot_B y,$$

$$(\alpha_A \oplus \alpha_B)(a+x) = \alpha_A(a) + \alpha_B(x),$$

for all $a, b, c \in A$ and $x, y, z \in B$, Indeed, the commutative Hom-associative algebra structure of $(A \oplus B, \cdot_{A\oplus B}, \alpha_A \oplus \alpha_B)$ follows immediately from the Hom-associative algebra structure of $(A, \cdot_A, \alpha_A)$ and $(B, \cdot_B, \alpha_B)$.

Likewise, the Hom-Lie algebra structure of $(A \oplus B, \{-,-\}_{A\oplus B}, \alpha_A \oplus \alpha_B)$ follows immediately from the Hom-associative algebra structure of $(A, \{,\}_A, \alpha_A)$ and $(B, \{,\}_B, \alpha_B)$.

For the generalized Hom-Leibniz identity, for all $a, b, c \in A$ and for all $x, y, z \in A$, we have

$$\begin{aligned} & \{(\alpha_A \oplus \alpha_B)(a+x), (b+y) \cdot_{A\oplus B} (c+z)\}_{A\oplus B} \\ = \ & \{\alpha_A(a) + \alpha_B(x), b \cdot_A c + y \cdot_B z\}_{A\oplus B} \\ = \ & \{\alpha_A(a), b \cdot_A c\}_A + \{\alpha_B(x), y \cdot_B z\}_B. \end{aligned}$$

On the one hand, since $(A, \cdot_A, \{,\}_A, \alpha_A)$ and $(B, \cdot_B, \{,\}_B, \alpha_B)$ are two Hom-Jacobi algebras, for all $a, b, c \in A$ and $x, y, z \in B$, we have

$$\begin{aligned} & \{(\alpha_A \oplus \alpha_B)(a+x), (b+y) \cdot_{A\oplus B} (c+z)\}_{A\oplus B} \\ = \ & \alpha_A(b) \cdot_A \{a, c\}_A + \alpha_A(c) \cdot_A \{a, b\}_A - \alpha_A(b) \cdot_A \alpha_A(c) \cdot_A \{a, 1_A\}_A \\ & + \alpha_B(y) \cdot_B \{x, z\}_B + \alpha_B(z) \cdot_B \{x, y\}_B \\ & - \alpha_B(y) \cdot_B \alpha_B(z) \cdot_B \{x, 1_B\}_B. \end{aligned}$$

On the other hand, for all $a, b, c \in A$ and $x, y, z \in B$, we have

$$\begin{aligned} & (\alpha_A \oplus \alpha_B)(b+y) \cdot_{A\oplus B} \{a+x, c+z\}_{A\oplus B} \\ = \ & \alpha_A(b) \cdot_A \{a, c\}_A + \alpha_B(y) \cdot_B \{x, z\}_B, \end{aligned}$$

$$\begin{aligned} & (\alpha_A \oplus \alpha_B)(c+z) \cdot_{A\oplus B} \{a+x, b+y\}_{A\oplus B} \\ = \ & \alpha_A(c) \cdot_A \{a, b\}_A + \alpha_B(z) \cdot_B \{x, y\}_B, \end{aligned}$$

and

$$\begin{aligned} & (\alpha_A \oplus \alpha_B)(b+y) \cdot_{A\oplus B} (\alpha_A \oplus \alpha_B)(c+z) \cdot_{A\oplus B} \{a+x, 1_A + 1_B\}_{A\oplus B} \\ = \ & \alpha_A(b) \cdot_A \alpha_A(c) \cdot_A \{a, 1_A\}_A + \alpha_B(y) \cdot_B \alpha_B(z) \cdot_B \{x, 1_B\}_B. \end{aligned}$$

Thus, the bracket $\{-,-\}_{A\oplus B}$ satisfies the generalized Hom-Leibniz identity.
Therefore, $(A \oplus B, \cdot_{A\oplus B}, \{-,-\}_{A\oplus B}, \alpha_A \oplus \alpha_B)$ is a Hom-Jacobi algebra.

Definition 8. *Let $(A, \cdot, \alpha)$ be a Hom-associative algebra. A submodule $I \subseteq A$ is called a Hom-associative ideal of A if $x \cdot y \in I$, $y \cdot x \in I$ for all $x \in I$, $y \in A$, and $\alpha(I) \subseteq I$.*

Definition 9 ([2], Definition 2.9)**.** *Let $(\mathcal{G}, [-,-], \alpha_{\mathcal{G}})$ be a Hom-Lie algebra. A Hom-Lie subalgebra $(\mathcal{H}, \alpha_{\mathcal{H}})$ of $(\mathcal{G}, [-,-], \alpha_{\mathcal{G}})$ is a linear subspace $\mathcal{H}$ of $\mathcal{G}$, which is closed for the bracket and invariant by $\alpha_{\mathcal{G}}$, that is,*

a) $[x, y] \in \mathcal{H}$, for all $x, y \in \mathcal{H}$,

b) $\alpha_{\mathcal{G}}(x) \in \mathcal{H}$, for all $x \in \mathcal{H}$, $(\alpha_{\mathcal{H}} = \alpha_{|\mathcal{G}})$.

A Hom-Lie subalgebra $(\mathcal{H}, \alpha_{\mathcal{H}})$ of $(\mathcal{G}, [-,-], \alpha_{\mathcal{G}})$ is said to be a Hom-ideal if $[x, y] \in \mathcal{H}$, for all $x \in \mathcal{H}$, $y \in \mathcal{G}$.

Definition 10. *A Hom-Jacobi ideal of a Hom-Jacobi algebra* $(A, \cdot, \{-,-\}, \alpha)$ *is a linear subspace* I *which is both an ideal with respect to the Hom-associative product and Hom-Lie bracket.*

Proposition 5. *If* I *is a Hom-Jacobi ideal of* $(A, \cdot, \{-,-\}, \alpha)$*, then* A/I *inherits a Hom-Jacobi algebra structure.*

Proof. Let us define the product $\cdot_{A/I} : A/I \longrightarrow A/I$, the bracket $\{-,-\}_{A/I} : A/I \times A/I \longrightarrow A/I$ and $\alpha_{A/I} : A/I \longrightarrow A/I$ respectively by

$$(a+I) \cdot_{A/I} (b+I) = a \cdot b + I, \tag{9}$$

$$\{a+I, b+I\}_{A/I} = \{a,b\} + I, \tag{10}$$

$$\alpha_{A/I}(a+I) = \alpha(a) + I. \tag{11}$$

For all $a, b, c \in A$,

$$\begin{aligned} \alpha_{A/I}(a+I) \cdot_{A/I} \left[(b+I) \cdot_{A/I} (c+I)\right] &= (\alpha(a)+I)(b \cdot c + I) \\ &= \alpha(a) \cdot (b \cdot c) + I. \end{aligned}$$

By Hom-associativity, (9) and (11), for all $a, b, c \in A$, we have

$$\begin{aligned} & \alpha_{A/I}(a+I) \cdot_{A/I} \left[(b+I) \cdot_{A/I} (c+I)\right] \\ = \; & (a \cdot b) \cdot \alpha(c) + I \\ = \; & \left[(a+I) \cdot_{A/I} (b+I)\right] \cdot_{A/I} \alpha_{A/I}(c+I). \end{aligned}$$

Thus, $\left(A/I, \cdot_{A/I}, \alpha_{A/I}\right)$ is a commutative Hom-associative algebra.

By (10), for all $a \in A$, we have

$$\{a+I, a+I\}_{A/I} = \{a,a\} + I.$$

Since the bracket $\{-,-\}$ is skew-symmetric, then $\{a+I, a+I\}_{A/I} = I = 0_{A/I}$. Thus, the bracket $\{-,-\}_{A/I}$ is skew-symmetric.

By (10) and (11), for all $a, b, c \in A$, we have

$$\left\{\alpha_{A/I}(a+I), \{b+I, c+I\}_{A/I}\right\}_{A/I} = \{\alpha(a), \{b,c\}\} + I.$$

Thus, for all $a, b, c \in A$,

$$\circlearrowleft_{a,b,c} \left\{\alpha_{A/I}(a+I), \{b+I, c+I\}_{A/I}\right\}_{A/I} = \circlearrowleft_{a,b,c} \{\alpha(a), \{b,c\}\} + I.$$

where $\circlearrowleft_{a,b,c}$ denotes summation over the cyclic permutation on a, b, c. By Hom-Jacobi identity, we have $\circlearrowleft_{a,b,c} \{\alpha(a), \{b,c\}\} = 0$. Which implies that $\circlearrowleft_{a,b,c} \left\{\alpha_{A/I}(a+I), \{b+I, c+I\}_{A/I}\right\}_{A/I} \in I = 0_{A/I}$. Thus, $\left(A/I, \{-,-\}_{A/I}, \alpha_{A/I}\right)$ is a Hom-Lie algebra.

For the generalized Hom-Leibniz identity, for all $a, b, c \in A$, we have

$$\begin{aligned}\left\{\alpha_{A/I}(a+I),(b+I)\cdot_{A/I}(c+I)\right\}_{A/I} &= \{\alpha(a)+I, b\cdot c+I\}_{A/I}\\ &= \{\alpha(a), b\cdot c\}+I.\end{aligned}$$

By Hom-Leibniz identity, for all $a, b, c \in A$, we have

$$\begin{aligned}\left\{\alpha_{A/I}(a+I),(b+I)\cdot_{A/I}(c+I)\right\}_{A/I} &= \alpha(b)\cdot\{a,c\}+\{a,b\}\cdot\alpha(c)\\ &\quad -\alpha(b)\cdot\alpha(c)\cdot\{a,1_A\}+I.\end{aligned}$$

That is,

$$\begin{aligned}&\left\{\alpha_{A/I}(a+I),(b+I)\cdot_{A/I}(c+I)\right\}_{A/I}\\ =\ &(\alpha(b)+I)\cdot_{A/I}(\{a,c\}+I)\\ &+(\{a,b\}+I)\cdot_{A/I}(\alpha(c)+I)\\ &-(\alpha(b)+I)\cdot_{A/I}(\alpha(c)+I)\cdot_{A/I}(\{a,1_A\}+I).\end{aligned}$$

Which implies that

$$\begin{aligned}&\left\{\alpha_{A/I}(a+I),(b+I)\cdot_{A/I}(c+I)\right\}_{A/I}\\ =\ &\alpha_{A/I}(b+I)\cdot_{A/I}\{a+I,c+I\}_{A/I}\\ &+\alpha_{A/I}(c+I)\cdot_{A/I}\{a+I,b+I\}_{A/I}\\ &-\alpha_{A/I}(b+I)\cdot_{A/I}\alpha_{A/I}(c+I)\cdot_{A/I}\left\{a+I,1_{A/I}\right\}_{A/I}.\end{aligned}$$

□

Definition 11. *Let $(A, \cdot, \{,\}_A, \alpha_A)$ and $(B, *, \{,\}_B, \alpha_B)$ be two Hom-Jacobi algebras. A linear map $f : A \longrightarrow B$ is called a homomorphism of Hom-Jacobi algebras if $\alpha_B(f(a)) = f(\alpha_A(a))$, $f(a\cdot b) = f(a) * f(b)$ and $f(\{a,b\}_A) = \{f(a), f(b)\}_B$, for all $a, b \in A$.*

If $\alpha_A = id_A$ and $\alpha_B = id_B$, we recover the notion of Jacobi homomorphisms given in [1,9].

Definition 12. *A generalized Hom-Poisson algebra is a quadruple $(A, \cdot, \{-,-\}, \alpha)$, where $(A, \cdot, \alpha)$ is a Hom-associaitve algebra and $(A, \{,\}, \alpha)$ is a Hom-Lie algebra, satisfying:*

$$\{a, b\cdot c\} = \alpha(b)\cdot\{a,c\}+\{a,b\}\cdot\alpha(c)+\alpha(b)\cdot\alpha(c)\cdot D(a),$$

where D is an α-derivation of $(A, \cdot)$ and α is a homomorphism, for all $a, b, c \in A$.

Example 3. Any commutative Hom-associative algebra $(A, \cdot, \alpha)$ equipped with an α-derivation D is a generalized Hom-Poisson algebra with respect to

$$\{a,b\} = \alpha(a)\cdot D(b) - \alpha(b)\cdot D(a). \tag{12}$$

Indeed, for all $a, b, c \in A$, we have

$$\{a, b\cdot c\} = \alpha(a)\cdot D(b\cdot c) - \alpha(b\cdot c)\cdot D(a)$$

$$\begin{aligned} &= \alpha(a)\cdot\alpha(b)\cdot D(c)+\alpha(a)\cdot\alpha(c)\cdot D(b) \\ &\quad -\alpha(b)\cdot\alpha(c)\cdot D(a). \end{aligned}$$

On the other hand, by (12), for all $a,b,c\in A$, we get

$$\begin{aligned} &\alpha(b)\cdot\{a,c\}+\{a,b\}\cdot\alpha(c)+\alpha(b)\cdot\alpha(c)\cdot D(a) \\ =\ &\alpha(a)\cdot\alpha(b)\cdot D(c)+\alpha(a)\cdot\alpha(c)\cdot D(b)-\alpha(b)\cdot\alpha(c)\cdot D(a). \end{aligned}$$

For the Hom-Jacobi identity, by (12), for all $a,b,c\in A$, we have

$$\begin{aligned} &\{\alpha(a),\{b,c\}\} \\ =\ &\alpha(a)\left(\alpha(\alpha(b))D(D(c))+\alpha(D(c))D(\alpha(b))\right) \\ &-\alpha(a)\left(\alpha(\alpha(c))D(D(b))+\alpha(D(b))D(\alpha(c))\right) \\ &-\alpha(\alpha(b))\alpha(D(c))D(\alpha(a))+\alpha(\alpha(c))\alpha(D(b))D(\alpha(a)). \end{aligned}$$

By commutative Hom-associative algebra of $(A,\cdot,\alpha)$, we get $\{\alpha(a),\{b,c\}\}+\{\alpha(b),\{c,a\}\}+\{\alpha(c),\{a,b\}\}=0$.

By (12), for all $a\in A$, $\{a,a\}=0$. Thus, the bracket $\{-,-\}$ is skew-symmetric.

We introduce the notion of purely Hom-Jacobi algebra structure:

Definition 13. *A purely Hom-Jacobi algebra is a quadruple $(A,\cdot,\{,\},\alpha)$, where $(A,\cdot)$ is a commutative associative algebra, $(A,\{,\},\alpha)$ is a Hom-Lie algebra and if, for all $a\in A$, the inner derivation $ad(a):A\longrightarrow A, b\longmapsto\{a,b\}$ is a first-order α-differential operator of commutative associative algebra $(A,\cdot)$, that is,*

$$\{a,b\cdot c\}=\alpha(b)\cdot\{a,c\}+\{a,b\}\cdot\alpha(c)-\alpha(b)\cdot\alpha(c)\cdot\{a,1_A\},$$

for all $b,c\in A$.

When $1_A\in Z(A)$, that is, $\{a,1_A\}=0$ and we recover the notion of purely Hom-Poisson algebras given in [7, Definition 2.7]. When $\alpha=id_A$, we recover the notion of Jacobi algebras given in [6].

Any purely Hom-Jacobi algebra is a generalized Hom-Poisson algebra, where $D=ad(-1_A)$.

Proposition 6. *Let $(A,\cdot,\{-,-\})$ be a Jacobi algebra and $\alpha:A\longrightarrow A$ a Jacobi endomorphism, then $(A,\cdot,\{-,-\}_\alpha=\alpha\circ\{-,-\},\alpha)$ is a purely Hom-Jacobi algebra.*

Proof. For all $a,b,c\in A$, we have $\{a,b\cdot c\}_\alpha=\alpha(\{a,b\cdot c\})$. Since $(A,\cdot,\{-,-\})$ is a Jacobi algebra,

$$\begin{aligned} \{a,b\cdot c\}_\alpha &= \alpha(b\cdot\{a,c\}+\{a,b\}\cdot c-b\cdot c\cdot\{a,1_A\}) \\ &= \alpha(b)\cdot\alpha(\{a,c\})+\alpha(\{a,b\})\cdot\alpha(c) \\ &\quad -\alpha(b)\cdot\alpha(c)\cdot\alpha(\{a,1_A\}). \end{aligned}$$

Thus,

$$\{a,b\cdot c\}_\alpha=\alpha(b)\cdot\{a,c\}_\alpha+\{a,b\}_\alpha\cdot\alpha(c)-\alpha(b)\cdot\alpha(c)\cdot\{a,1_A\}_\alpha,$$

for all $b,c\in A$. □

Example 4. Let M be a Jacobi manifold. Then $(C^{\infty}(M), \cdot, \{-,-\})$ is a Jacobi algebra where $\cdot$ is the usual product on $C^{\infty}(M)$ (see [6, 9]). Let $\varphi : M \longrightarrow M$ be a smooth map. Then the pullback map $\varphi^{*} : C^{\infty}(M) \longrightarrow C^{\infty}(M), f \longmapsto \varphi^{*}(f) = f \circ \varphi$ is a homomorphism of the function ring $C^{\infty}(M)$, that is,

$$\varphi^{*}(f \cdot g) = \varphi^{*}(f) \cdot \varphi^{*}(g),$$

for all $f, g \in C^{\infty}(M)$. Indeed, for all $p \in M$, we have

$$[\varphi^{*}(f \cdot g)](p) = [(f \cdot g) \circ \varphi](p) = (f \cdot g)[\varphi(p)] = f[\varphi(p)] \cdot g[\varphi(p)].$$

That is,

$$[\varphi^{*}(f \cdot g)](p) = [(f \circ \varphi) \cdot (g \circ \varphi)](p) = [\varphi^{*}(f) \cdot \varphi^{*}(g)](p),$$

for all $p \in M$. Under these conditions, $\left(C^{\infty}(M), \cdot, \{-,-\}_{\varphi^{*}} = \varphi^{*} \circ \{-,-\}, \varphi^{*}\right)$ is a purely Hom-Jacobi algebra.

4 Canonical form of purely Hom-Jacobi algebra

4.1 Universal property of first-order Hom-differential operators

Let A be a commutative associative algebra with unity 1_A, and let $\alpha : A \longrightarrow A$ be an algebra automorphism. The map $f : A \longrightarrow A \bigotimes_{\mathbb{K}} A, a \longmapsto a \otimes 1_A$ is a homomorphism of algebras. Thus, $A \bigotimes_{\mathbb{K}} A$ is equipped with the structure of A-module defined by f. We consider the A-submodule I of $A \bigotimes_{\mathbb{K}} A$ generated by the elements of the form $\alpha(a) \otimes 1_A - 1_A \otimes \alpha(a)$, that is, for all $z \in I$,

$$z = \sum_{i \in J: finite} a_i (\alpha(b_i) \otimes 1_A - 1_A \otimes \alpha(b_i)),$$

with $a_i, b_i \in A$. We denote by $\Omega_{\alpha}(A)$ the module of Kähler differentials of commutative algebra A, that is, the quotient space $\Omega_{\alpha}(A) = I/I^2$. For $a \in A$, we denote by $\overline{\alpha(a) \otimes 1_A - 1_A \otimes \alpha(a)}$ the class of $\alpha(a) \otimes 1_A - 1_A \otimes \alpha(a)$ in $\Omega_{\alpha}(A)$.

Proposition 7. *The map $d_{A/\mathbb{K}} : A \longrightarrow \Omega_{\alpha}(A)$ such that, for all $a \in A$,*

$$d_{A/\mathbb{K}}(a) = \overline{\alpha(a) \otimes 1_A - 1_A \otimes \alpha(a)}$$

is an α-derivation. Moreover, the image of $d_{A/\mathbb{K}}$ generates the A-module $\Omega_{\alpha}(A)$.

Proof. For all $a, b \in A$, we have

$$\begin{aligned} & (\alpha(a) \otimes 1_A - 1_A \otimes \alpha(a))(\alpha(b) \otimes 1_A - 1_A \otimes \alpha(b)) \\ = \; & (\alpha(a) \cdot \alpha(b)) \otimes 1_A - \alpha(a) \otimes \alpha(b) \\ & - \alpha(b) \otimes \alpha(a) + 1_A \otimes (\alpha(a) \cdot \alpha(b)). \end{aligned}$$

Since I is generated by the elements of the form $\alpha(a)\otimes 1_A - 1_A\otimes\alpha(a)$, for all $a,b\in A$, we have $(\alpha(a)\otimes 1_A - 1_A\otimes\alpha(a))(\alpha(b)\otimes 1_A - 1_A\otimes\alpha(b))\in I^2$, that is,

$$\overline{(\alpha(a)\cdot\alpha(b))\otimes 1_A - \alpha(a)\otimes\alpha(b) - \alpha(b)\otimes\alpha(a) + 1_A\otimes(\alpha(a)\cdot\alpha(b))} = 0.$$

Thus, for all $a,b\in A$,

$$\begin{aligned} d_{A/K}(a\cdot b) &= \overline{\alpha(a\cdot b)\otimes 1_A - 1_A\otimes\alpha(a\cdot b)} \\ &= \overline{(\alpha(a)\cdot\alpha(b))\otimes 1_A - 1_A\otimes(\alpha(a)\cdot\alpha(b))} \\ &\quad +\overline{(\alpha(a)\cdot\alpha(b))\otimes 1_A - \alpha(a)\otimes\alpha(b)} \\ &\quad +\overline{1_A\otimes(\alpha(a)\cdot\alpha(b)) - \alpha(b)\otimes\alpha(a)}. \end{aligned}$$

Which implies that

$$\begin{aligned} d_{A/K}(a\cdot b) &= \overline{(\alpha(a)\cdot\alpha(b))\otimes 1_A - 1_A\otimes(\alpha(a)\cdot\alpha(b))} \\ &\quad +\overline{(\alpha(a)\cdot\alpha(b))\otimes 1_A - \alpha(a)\otimes\alpha(b)} \\ &\quad +\overline{1_A\otimes(\alpha(a)\cdot\alpha(b)) - \alpha(b)\otimes\alpha(a)}. \end{aligned}$$

Therefore, for all $a,b\in A$,

$$\begin{aligned} & d_{A/K}(a\cdot b) \\ = \; & \overline{(\alpha(a)\cdot\alpha(b))\otimes 1_A - \alpha(a)\otimes\alpha(b)} + \overline{(\alpha(a)\cdot\alpha(b))\otimes 1_A - \alpha(b)\otimes\alpha(a)} \\ = \; & \alpha(a)\cdot\overline{(\alpha(b)\otimes 1_A - 1_A\otimes\alpha(b))} + \overline{(\alpha(a)\otimes 1_A - 1_A\otimes\alpha(a))}\cdot\alpha(b) \\ = \; & \overline{(\alpha(a)\otimes 1_A - 1_A\otimes\alpha(a))}\cdot\alpha(b) + \alpha(a)\cdot\overline{(\alpha(b)\otimes 1_A - 1_A\otimes\alpha(b))} \\ = \; & d_{A/K}(a)\cdot\alpha(b) + \alpha(a)\cdot d_{A/K}(b). \end{aligned}$$

As $d_{A/K}$ is clearly $\mathbb{K}$-linear, we conclude that $d_{A/K}$ is an α-derivation. The map $d_{A/\mathbb{K}}$ is surjective by definition, that is, $d_{A/\mathbb{K}}(A) = \Omega_\alpha(A) = I/I^2$. Since I is generated by the elements of the form $\alpha(a)\otimes 1_A - 1_A\otimes\alpha(a)$, then for all $z\in I$, $z = \sum\limits_{i\in J:finite} a_i(\alpha(b_i)\otimes 1_A - 1_A\otimes\alpha(b_i))$, with $a_i, b_i\in A$. Thus, $\dot{z}\in\Omega_\alpha(A) = I/I^2$,

$$\dot{z} = \sum_{i\in J:finite} a_i\dot{\overline{(\alpha(b_i)\otimes 1_A - 1_A\otimes\alpha(b_i))}} = \sum_{i\in J:finite} a_i d_{A/\mathbb{K}}(b_i).$$

□

Lemma 1. *Let M be an A-module, $A\ltimes M$ be the semi-direct product and $d: A\longrightarrow M$ be an α-derivation, then the map $\tilde{d}: A\longrightarrow A\ltimes M$ given by $\tilde{d}(a) = \left(a, d\left(\alpha^{-1}(a)\right)\right)$, for $a\in A$, is an algebra homomorphism with $\pi_1\circ\tilde{d} = Id_A$, where $\pi_1(a,m) = a$.*

Conversely, for an algebra homomorphism $h: A\longrightarrow A\ltimes M$ satisfying $\pi_1\circ\tilde{d} = Id_A$, there is a unique α-derivation $d: A\longrightarrow M$ with
$h = \tilde{d}$.

Proof. Let $a, b \in A$. From the definition of $\tilde{d}$ and π_1 we get

$$\tilde{d}(ab) = \left(ab, d\left(\alpha^{-1}(ab)\right)\right) = \left(a, d\left(\alpha^{-1}(a)\right)\right)\left(b, d\left(\alpha^{-1}(b)\right)\right) = \tilde{d}(a)\tilde{d}(b),$$

and $\pi_1 \circ \tilde{d} = Id_A$.

Conversely, let $h : A \longrightarrow A \ltimes M$ be an algebra homomorphism satisfying $\pi_1 \circ \tilde{d} = Id_A$. Then for $a \in A$, we can write $h(a) = (a, h_1(a))$ where $h_1 : A \longrightarrow M$ is a $\mathbb{K}$ -linear map. Define $d : A \longrightarrow M$ by $d(a) = h_1(\varphi(a))$ for $a \in A$. It follows that d is an α-derivation and the map $\tilde{d} = h$. □

In the following theorem, we give the universal property of the couple $\left(\Omega_\alpha(A), d_{A/\mathbb{K}}\right)$.

Theorem 1 ([12])**.** *For any A-module M and for any α-derivation $\varphi : A \longrightarrow M$, there exists a unique homomorphism $\tilde{\varphi} : \Omega_\alpha(A) \longrightarrow M$ of A-modules such that $\tilde{\varphi} \circ d_{A/\mathbb{K}} = \varphi$.*

Proof. Let M be an A-module and $\varphi : A \longrightarrow M$ be an α-derivation. According to Lemma 1, we have an algebra homomorphism $\psi : A \longrightarrow A \ltimes M$ given by $\psi(a) = \left(a, \varphi\left(\alpha^{-1}(a)\right)\right)$ for $a \in A$. If we consider the map $i : A \longrightarrow A \ltimes M$ defined by $i(a) = (a, 0)$, then we get an R-algebra homomorphism $h : A \otimes A \longrightarrow A \ltimes M$ defined by

$$h(a \otimes b) = \psi(a)i(b) = \left(ab, b\varphi\left(\alpha^{-1}(a)\right)\right);$$

for all $a, b \in A$. So, for $x = \alpha(a) \otimes 1 - 1 \otimes \alpha(a) \in I$, we have $h(x) = (0, \varphi(a))$. Thus h vanishes on $I^2 \subset A \otimes A$. So it induces an A-module homomorphism $\tilde{\varphi} : I/I^2 = \Omega_\alpha(A) \longrightarrow M$ such that

$$\tilde{\varphi}(d_{A/\mathbb{K}}(a)) = \varphi(a),$$

for all $a \in A$. □

Therefore, the map

$$Hom_A(\Omega_\alpha(A), M) \longrightarrow Der_\alpha(A, M), \psi \longmapsto \psi \circ d_{A/\mathbb{K}}$$

is an isomorphism of A-modules. In particular, $\Omega_\alpha(A)^* \simeq Der_\alpha(A)$.

Proposition 8. *The map $D_{A/\mathbb{K}} : A \longrightarrow A \oplus \Omega_\alpha(A)$ such that, for all $a \in A$,*

$$D_{A/\mathbb{K}}(a) = \alpha(a) + d_{A/\mathbb{K}}(a)$$

is a first-order α-differential operator.

Proof. For all $a, b \in A$, we have

$$\begin{aligned} & D_{A/\mathbb{K}}(ab) - \alpha(a) D_{A/\mathbb{K}}(b) - \alpha(b) D_{A/\mathbb{K}}(a) + \alpha(a)\alpha(b) D_{A/\mathbb{K}}(1_A) \\ = \; & \alpha(a)\alpha(b) + d_{A/\mathbb{K}}(ab) - \alpha(a)\left(\alpha(b) + d_{A/\mathbb{K}}(b)\right) \\ & -\alpha(b)\left(\alpha(a) + d_{A/\mathbb{K}}(a)\right) \\ & +\alpha(a)\alpha(b)\left(\alpha(1_A) + d_{A/\mathbb{K}}(1_A)\right) \\ = \; & d_{A/\mathbb{K}}(ab) - \alpha(a) d_{A/\mathbb{K}}(b) - \alpha(b) d_{A/\mathbb{K}}(a). \end{aligned}$$

Since $d_{A/\mathbb{K}}(a)$ is an α-derivation, we get

$$D_{A/\mathbb{K}}(ab) = \alpha(a) D_{A/\mathbb{K}}(b) + \alpha(b) D_{A/\mathbb{K}}(a) - \alpha(a)\alpha(b) D_{A/\mathbb{K}}(1_A),$$

for all $a, b \in A$. Thus, $D_{A/\mathbb{K}} \in Diff_\alpha(A, A \oplus \Omega_\alpha(A))$. □

For all $x \in A \oplus \Omega_\alpha(A)$, $x = \sum_{i \in J: finite} a_i D_{A/\mathbb{K}}(b_i)$, with $a_i, b_i \in A$, that is, the image of $D_{A/\mathbb{K}}$ generates the A-module $A \oplus \Omega_\alpha(A)$.

In the following theorem, we give the universal property of the couple $\left(A \oplus \Omega_\alpha(A), D_{A/\mathbb{K}}\right)$.

Theorem 2. *For any A-module M and for any first-order α-differential operator $\varphi : A \longrightarrow M$, there exists a unique homomorphism $\tilde{\varphi} : A \oplus \Omega_\alpha(A) \longrightarrow M$ of A-modules such that $\tilde{\varphi} \circ D_{A/\mathbb{K}} = \varphi$.*

Proof. Since $\varphi : A \longrightarrow M$ is first-order α-differential operator, according to Proposition 1, the map $\varphi - L_{\varphi(1_A)} : A \longrightarrow M$ is an α-derivation. By the Theorem 1, there exists a unique homomorphism $\psi : \Omega_\alpha(A) \longrightarrow M$ such that $\psi \circ d_{A/\mathbb{K}} = \varphi - L_{\varphi(1_A)}$. The map $\tilde{\varphi} : A \oplus \Omega_\alpha(A) \longrightarrow M$ such that, for all $a \in A$ and $x \in \Omega_\alpha(A)$,

$$\tilde{\varphi}(a + x) = a\varphi(1_A) + \psi(x)$$

satisfies

$$\begin{aligned} \tilde{\varphi} \circ D_{A/\mathbb{K}}(a) &= \tilde{\varphi}\left(\alpha(a) + d_{A/\mathbb{K}}(a)\right) \\ &= \alpha(a)\varphi(1_A) + \psi \circ d_{A/\mathbb{K}}(a) \\ &= \alpha(a)\varphi(1_A) + \left(\varphi - L_{\varphi(1_A)}\right)(a). \end{aligned}$$

It follows that

$$\tilde{\varphi} \circ D_{A/\mathbb{K}}(a) = \alpha(a)\varphi(1_A) + \varphi(a) - \alpha(a)\varphi(1_A) = \varphi(a),$$

for all $a \in A$. Thus, $\tilde{\varphi} \circ D_{A/\mathbb{K}} = \varphi$. □

Definition 14 ([10]). *Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and A be a vector space over $\mathbb{K}$. A Hom-algebra is a triple $(A, \cdot, \alpha)$ consisting of a vector space A, a bilinear map $\cdot : A \times A \longrightarrow A$ and a vector space homomorphism $\alpha : A \longrightarrow A$.*

We introduce the following definition.

Definition 15. *Let $(A, \cdot, \alpha)$ be a Hom-algebra and let M be an A-module. A skew-symmetric A-multilinear map $Q : A \times A \times \cdots \times A \longrightarrow M$ of degree k is a skew-symmetric α-derivation of degree k if*

$$Q(a_1 b, a_2, \ldots, a_k) = \alpha(a_1) Q(b, a_2, \ldots, a_k) + \alpha(b) Q(a_1, a_2, \ldots, a_k)$$

for all $b, a_i \in A$, $i \in \{1, \ldots, k\}$.

Example 5. If $(A, \cdot, \{-,-\}, \alpha)$ is a purely Hom-Poisson algebra ([7, Definition 2.7]), then the bracket $\{-,-\} : A \times A \longrightarrow A$ is a skew-symmetric α-derivation of degree 2, that is,

$$\{b \cdot c, a\} = \alpha(b) \cdot \{a, c\} + \{a, b\} \cdot \alpha(c),$$

for all $a, b, c \in A$. When $\alpha = id_A$, we recover the notion of Poisson algebras given in [6, 8, 9, 13].

Definition 16. *Let $(A,\cdot,\alpha)$ be a Hom-algebra and let M be an A-module. A skew-symmetric A-multilinear map $\varphi : A\times A\times\cdots\times A\longrightarrow M$ of degree k is a skew-symmetric first-order α-differential operator of degree k if*

$$\begin{aligned}\varphi\left(a_1 b, a_2,\ldots,a_k\right) &= \alpha\left(a_1\right)\varphi\left(b,a_2,\ldots,a_k\right)+\alpha\left(b\right)\varphi\left(a_1,a_2,\ldots,a_k\right)\\ &\quad -\alpha\left(a_1\right)\alpha\left(b\right)\varphi\left(1_A,a_2,\ldots,a_k\right),\end{aligned}$$

for all $b, a_i\in A$, $i\in\{1,\ldots,k\}$.

We denote the A-module of all skew-symmetric first-order α-differential operators of degree k on A with coefficients in M by $Diff_{\alpha}^{k}\left(A,M\right)$ and we denote by $\mathfrak{L}_{sks}^{k}\left(A\oplus\Omega_{\alpha}\left(A\right),M\right)$ the A-module of all skew-symmetric A-multilinear maps of degree k on A with coefficients in M.

Consider the map $D_{A/\mathbb{K}}^{(k)}=D_{A/\mathbb{K}}\times D_{A/\mathbb{K}}\times\cdots\times D_{A/\mathbb{K}}: A^k\longrightarrow\left(A\oplus\Omega_{\alpha}\left(A\right)\right)^k$ such that

$$D_{A/\mathbb{K}}^{(k)}\left(a_1,a_2,\ldots,a_k\right)=D_{A/\mathbb{K}}\left(a_1\right)\times D_{A/\mathbb{K}}\left(a_2\right)\times\cdots\times D_{A/\mathbb{K}}\left(a_k\right),$$

for all $a_1,a_2,\ldots,a_k\in A$.

Theorem 3. *For any A-module M and for any skew-symmetric first-order α-differential operator $\varphi : A^k\longrightarrow M$ of degree k, there exists a unique skew-symmetric A-multilinear map $\widetilde{\varphi}:\left(A\oplus\Omega_{\alpha}\left(A\right)\right)^k\longrightarrow M$ of degree k such that*

$$\widetilde{\varphi}\left(D_{A/\mathbb{K}}\left(a_1\right),\ldots,D_{A/\mathbb{K}}\left(a_k\right)\right)=\varphi\left(a_1,\ldots,a_k\right),$$

for all $a_1,a_2,\ldots,a_k\in A$.

Proof. By definition of skew-symmetric first-order α-differential operator $\varphi : A^k\longrightarrow M$ of degree k, the map $\varphi^i : A\longrightarrow M$, such that

$$\varphi^i(a_i)=\varphi\left(a_1,a_2,\ldots,a_{i-1},a_i,a_{i+1},\ldots,a_k\right)$$

is a first-order α-differential operator, for all, $a_i\in A$, $i\in\{1,\ldots,k\}$. According to Theorem 2, there exists a unique homomorphism $\widetilde{\varphi^i}: A\oplus\Omega_{\alpha}\left(A\right)\longrightarrow M$ of A-modules such that $\widetilde{\varphi^i}\circ D_{A/\mathbb{K}}=\varphi^i$, that is,

$$\widetilde{\varphi^i}\left[D_{A/\mathbb{K}}\left(a_i\right)\right]=\varphi^i(a_i)=\varphi\left(a_1,a_2,\ldots,a_{i-1},a_i,a_{i+1},\ldots,a_k\right)$$

We deduce the existence and uniqueness of the skew-symmetric A-multilinear map $\widetilde{\varphi}:\left(A\oplus\Omega_{\alpha}\left(A\right)\right)^k\longrightarrow M$ of degree k such that

$$\widetilde{\varphi}\left(D_{A/\mathbb{K}}\left(a_1\right),\ldots,D_{A/\mathbb{K}}\left(a_k\right)\right)=\varphi\left(a_1,\ldots,a_k\right)$$

for all $a_1,a_2,\ldots,a_k\in A$. □

4.2 Canonical form of purely Hom-Jacobi algebra

If $(A, \cdot, \{-,-\}, \alpha)$ is a purely Hom-Jacobi algebra, then the map $\xi : A \longrightarrow A, a \longmapsto \{1_A, a\}$ is an α-derivation. Indeed, for all $a, b \in A$, $\xi(a \cdot b) = \{1_A, a \cdot b\}$. By the definition of a purely Hom-Jacobi algebra, for all $a, b \in A$, we have

$$\begin{aligned} \xi(a \cdot b) &= \alpha(a) \cdot \{1_A, b\} + \{1_A, a\} \cdot \alpha(b) - \alpha(b) \cdot \alpha(b) \cdot \{1_A, 1_A\} \\ &= \alpha(a) \cdot \xi(b) + \alpha(b) \cdot \xi(a). \end{aligned}$$

Theorem 4. *The following statements are equivalent:*

(1) $(A, \cdot, \{-,-\}, \alpha)$ is a purely Hom-Jacobi algebra.

(2) There exists a unique skew-symmetric 2-form

$$\omega_A : [A \oplus \Omega_\alpha(A)] \times [A \oplus \Omega_\alpha(A)] \longrightarrow A$$

such that, for all $a, b \in A$,

$$\{a, b\} = \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(b)) \tag{13}$$

defines a Hom-Lie algebra structure on A.

(3) There exists a skew-symmetric 2-form

$$\pi : \Omega_\alpha(A) \times \Omega_\alpha(A) \longrightarrow A$$

and there exists an α-derivation ξ of A such that, for all $a, b \in A$,

$$\{a, b\} = \pi(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)) + \alpha(a)\,\xi(b) - \alpha(b)\,\xi(a) \tag{14}$$

defines a Hom-Lie algebra structure on A.

Proof. (1) $\Rightarrow$ (2) If $(A, \cdot, \{-,-\}, \alpha)$ is a purely Hom-Jacobi algebra, the bracket $\{-,-\}$ is a skew-symmetric first-order α-differential operator of degree 2. According to Theorem 3, there exists a unique $\omega_A \in \mathfrak{L}^2_{sks}(A \oplus \Omega_\alpha(A), A)$ such that $\{a, b\} = \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(b))$, for all $a, b \in A$.

(2) $\Rightarrow$ (3) If there exists a skew-symmetric 2-form ω_A such that, for all $a, b \in A$, $\{a, b\} = \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(b))$, then

$$\begin{aligned} \{a, b\} &= \omega_A(\alpha(a) + d_{A/\mathbb{K}}(a), \alpha(b) + d_{A/\mathbb{K}}(b)) \\ &= \alpha(a) \cdot \omega_A(1_A, d_{A/\mathbb{K}}(b)) + \alpha(b) \cdot \omega_A(d_{A/\mathbb{K}}(a), 1_A)) \\ &\quad + \omega_A(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)). \end{aligned}$$

Since for all $a \in A$,

$$\omega_A(1_A, \alpha(a)) + \omega_A(1_A, d_{A/\mathbb{K}}(a)) = \omega_A(1_A, D_{A/\mathbb{K}}(a)) = \{1_A, a\} = ad(1_A)(a),$$

then there exists an α-derivation $\xi = ad(1_A)$ such that, for all $a \in A$,

$$\xi(a) = ad(1_A)(a) = \{1_A, a\} = \omega_A(1_A, d_{A/\mathbb{K}}(a)), \tag{15}$$

and there exists a skew-symmetric 2-form $\pi = \omega_{A|\Omega_\alpha(A)\times\Omega_\alpha(A)} : \Omega_\alpha(A) \times \Omega_\alpha(A) \longrightarrow A$, such that, for all $a, b \in A$,

$$\begin{aligned} \{a, b\} &= \pi(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)) + \alpha(a) \cdot \omega_A(1_A, d_{A/\mathbb{K}}(b)) + \alpha(b) \cdot \omega_A(d_{A/\mathbb{K}}(a), 1_A)) \\ &= \pi(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)) + \alpha(a)\,\xi(b) - \alpha(b)\,\xi(a). \end{aligned}$$

Here, π is the restriction of $\omega_A : [A \oplus \Omega_\alpha(A)] \times [A \oplus \Omega_\alpha(A)] \longrightarrow A$ to $\Omega_\alpha(A)$, that is $\pi(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)) = \omega_A(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b))$.

(3) $\Rightarrow$ (1) If the bracket $\{-,-\}$, such that, for all $a, b \in A$,

$$\{a, b\} = \pi(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)) + \alpha(a)\,\xi(b) - \alpha(b)\,\xi(a) = \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(b)),$$

defines a Hom-Lie algebra structure on A, then the bracket satisfies the Hom-Jacobi identity. For all $a, b, c \in A$, we have

$$\{a, b \cdot c\} = \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(b \cdot c)).$$

Since $D_{A/\mathbb{K}}$ is a first-order α-differential operator, for all $a, b, c \in A$,

$$\begin{aligned} \{a, b \cdot c\} &= \alpha(b) \cdot \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(c)) + \alpha(c) \cdot \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(c)) \\ &\quad - \alpha(b) \cdot \alpha(c) \cdot \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(1_A)) \\ &= \alpha(b) \cdot \{a, c\} + \{a, b\} \cdot \alpha(c) - \alpha(b) \cdot \alpha(c) \cdot \{a, 1_A\}, \end{aligned}$$

that is, $ad(a)$ is a first-order α-differential operator. Therefore, $(A, \cdot, \{-,-\}, \alpha)$ is a purely Hom-Jacobi algebra. □

In this case, we say that ω_A or the pair (π, ξ) defines a purely Hom-Jacobi structure on A and the pair (A, ω_A) or the triple (A, π, ξ) is a purely Hom-Jacobi algebra. The pair (π, ξ) is called purely Hom-Jacobi pair. If $\xi = 0$, the pair (A, π) is called purely Hom-Poisson algebra. The skew-symmetric 2-form ω_A on $A \oplus \Omega_\alpha(A)$ is called canonical purely Hom-Jacobi 2-form of purely Hom-Jacobi algebra.

When ω_A is a purely Hom-Jacobi 2-form, the map $i_{1_A}\omega_A : A \oplus \Omega_\alpha(A) \longrightarrow A, x \longmapsto \omega_A(x, 1_A)$ is called 1-form of purely Hom-Jacobi algebra.

Proposition 9. *If $(A, \cdot, \{-,-\}, \alpha)$ is a purely Hom-Jacobi algebra, then for $a \in A$, the map $\Phi_a : A \longrightarrow A, b \longmapsto \{a, b\} - \alpha(b)\{a, 1_A\}$ is an α-derivation.*

Moreover, the map $X : A \longrightarrow Der_\alpha(A), a \longmapsto \Phi_a$ is a first-order α-differential operator.

Proof. For all $a, b, c \in A$,

$$\begin{aligned} &\Phi_a(b \cdot c) - \alpha(b) \cdot \Phi_a(c) - \alpha(c) \cdot \Phi_a(b) \\ = \;& \{a, b \cdot c\} + \alpha(b \cdot c)\{a, 1_A\} - \alpha(b) \cdot (\{a, c\} - \alpha(c)\{a, 1_A\}) \end{aligned}$$

$$-\alpha\left(c\right)\cdot\left(\left\{a,b\right\}-\alpha\left(b\right)\left\{a,1_A\right\}\right).$$

By generalized Hom-Leibniz identity, we get

$$\Phi_a\left(b\cdot c\right)-\alpha\left(b\right)\cdot\Phi_a\left(c\right)-\alpha\left(c\right)\cdot\Phi_a\left(b\right)=0.$$

For all $a,b\in A$, we have

$$X\left(a\right)\left(b\right)=\omega_A\left(D_{A/\mathbb{K}}(a),d_{A/\mathbb{K}}(b)\right). \tag{16}$$

Indeed, for all $a,b\in A$, $X\left(a\right)\left(b\right)=\Phi_a\left(b\right))=\left\{a,b\right\}-\alpha\left(b\right)\left\{a,1_A\right\}$. By (13), we have

$$\begin{aligned} X\left(a\right)\left(b\right) &= \omega_A\left(D_{A/\mathbb{K}}(a),D_{A/\mathbb{K}}(b)\right)-\alpha\left(b\right)\omega_A\left(D_{A/\mathbb{K}}(a),D_{A/\mathbb{K}}(1_A)\right)\\ &= \omega_A\left(D_{A/\mathbb{K}}(a),D_{A/\mathbb{K}}(b)-\alpha\left(b\right)\right). \end{aligned}$$

For all $a,b,c\in A$, we have $X\left(a\cdot b\right)\left(c\right)=\omega_A\left(D_{A/\mathbb{K}}(a\cdot b),d_{A/\mathbb{K}}(c)\right)$. Since $D_{A/\mathbb{K}}$ is a a first-order α-differential operator, then

$$\begin{aligned} & X\left(a\cdot b\right)\left(c\right)\\ = & \alpha\left(b\right)\cdot\omega_A\left(D_{A/\mathbb{K}}(a),d_{A/\mathbb{K}}(c)\right)+\alpha\left(a\right)\cdot\omega_A\left(D_{A/\mathbb{K}}(b),d_{A/\mathbb{K}}(c)\right)\\ & -\alpha\left(a\right)\cdot\alpha\left(b\right)\cdot\omega_A\left(D_{A/\mathbb{K}}(1_A),d_{A/\mathbb{K}}(c)\right). \end{aligned}$$

Thus, for all $a,b,c\in A$,

$$X\left(a\cdot b\right)\left(c\right)=\left(\alpha\left(b\right)\cdot X\left(a\right)+\alpha\left(a\right)\cdot X\left(b\right)-\alpha\left(a\right)\cdot\alpha\left(b\right)\cdot X\left(1_A\right)\right)\left(c\right)$$

□

Since the map $X:A\longrightarrow Der_\alpha\left(A\right),a\longmapsto\Phi_a$ is a first-order α-differential operator, then according to Theorem 2, there exists a unique A-linear map $\tilde{X}:A\oplus\Omega_\alpha\left(A\right)\longrightarrow Der_\alpha\left(A\right)$ such that

$$\widetilde{X}\circ D_{A/\mathbb{K}}=X. \tag{17}$$

Proposition 10. *If* (A,ω_A) *is a purely Hom-Jacobi algebra, then*

$$[\widetilde{X}(x)](a)=\omega_A(x,d_{A/\mathbb{K}}(a)), \tag{18}$$

$$\left[\widetilde{ad}(x)\right](a)=\omega_A\left(x,D_{A/\mathbb{K}}(a)\right), \tag{19}$$

$$\left[\widetilde{\widetilde{ad}(x)}\right](y)=\omega_A(x,y), \tag{20}$$

for all $a,b\in A$ *and* $x,y\in A\oplus\Omega_\alpha\left(A\right)$*.*

Proof. Since the image of $D_{A/\mathbb{K}}$ generates the A-module $A\oplus\Omega_\alpha\left(A\right)$, for $x\in A\oplus\Omega_\alpha\left(A\right)$, $x=bD_{A/\mathbb{K}}(c)$ with $b,c\in A$. We have $[\widetilde{X}(x)](a)=[b\cdot(\widetilde{X}\circ D_{A/\mathbb{K}})(c))](a)$. By (17), $[\widetilde{X}(x)](a)=[b\cdot X(c)](a)$, for all $a,b\in A$. By (16),

$$[\widetilde{X}(x)](a) = b\cdot\omega_A\left(D_{A/\mathbb{K}}(c),d_{A/\mathbb{K}}(a)\right)$$

$$= \omega_A(x, d_{A/\mathbb{K}}(a)).$$

For (19), for all $a \in A$ and for all $x \in A \oplus \Omega_\alpha(A)$, $x = bD_{A/\mathbb{K}}(c)$ with $b, c \in A$, we have

$$\left[\widetilde{ad}(x)\right](a) = \left[b \cdot (\widetilde{ad} \circ D_{A/\mathbb{K}}(c))\right](a) = b \cdot [ad(c))(a)].$$

That is,

$$\left[\widetilde{ad}(x)\right](a) = b \cdot \omega_A\left(D_{A/\mathbb{K}}(c), D_{A/\mathbb{K}}(a)\right) = \omega_A\left(x, D_{A/\mathbb{K}}(a)\right).$$

For (20), by (18) and (19), for all $x, y \in A \oplus \Omega_\alpha(A)$, $x = a_1 D_{A/\mathbb{K}}(b_1), y = a_2 D_{A/\mathbb{K}}(b_2)$ with $a_1, a_2, b_1, b_2 \in A$, we get $[\widetilde{ad(x)}](y) = a_1 a_2 \omega_A\left(D_{A/\mathbb{K}}(b_1), D_{A/\mathbb{K}}(b_2)\right)$, that is, $\left[\widetilde{ad}(x)\right](y) = \omega_A(x, y)$. □

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