

On Hom-Jacobi algebra structures

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Abstract. We define Hom-Jacobi algebras as an extension of Hom-Poisson algebras and we give some examples. We describe the universal property of first-order Hom-differential operators as well as the universal property of first-order Hom-differential multi-operators. By using these universal properties, we prove the existence and uniqueness of a canonical purely Hom-Jacobi form associated to purely Hom-Jacobi algebra.

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1 Introduction

Jacobi algebras are abstract algebraic counterparts of Jacobi manifolds. Jacobi manifolds as generalizations of symplectic or more generally Poisson manifolds, were introduced independently by Kirillov [6] and Lichnerowicz [9]. Both Poisson and Jacobi algebras are commutative algebras endowed with a Lie bracket. However, while the Poisson bracket is a derivation of the underlying commutative algebra, the Jacobi bracket is a first-order differential operator on the commutative algebra [1, 3, 4].

The notion of Hom-Lie algebras was introduced by Hartwig, Larsson, and Silvestrov in [5] as apart of a study of deformations of the Witt and the Virasoro algebras. In a Hom-Lie algebra, the Jacobi identity is twisted by a linear map, called the Hom-Jacobi identity. A Hom-type generalization of Poisson algebras, called Hom-Poisson algebras, was introduced by Makhlouf and Silvestrov in [11], which combines a commutative Hom-associative algebra and a Hom-Lie algebra such that a Hom-Leibniz identity is satisfied. In [7], the authors introduced the notion of a purely Hom-Poisson algebra, which combines a commutative associative algebra and a Hom-Lie algebra such that a Hom-Leibniz identity is satisfied.

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The aim of this paper is to generalize the notion of Hom-Poisson algebras by using the first-order Hom-differential operators on the commutative Hom-associative algebra with unit. The paper is organized as follows. In Section 2, we give the concept of first-order Hom-differential operator and we show that the space of first-order Hom-differential operators is a Hom-Lie algebra. In Section 3, we introduce the notion of Hom-Jacobi algebras, the notion of purely Hom-Jacobi algebras and we give some examples. Finally, in Section 4, we establish the universal property of first-order Hom-differential operators as well as the universal property of first-order hom-differential multi-operators. By using the universal property of first-order Hom-differential operators, we describe the necessary and sufficient condition of existence of purely Hom-Jacobi algebra structure in terms of canonical purely Hom-Jacobi form. We give some properties associated with a canonocal purely Hom-Jacobi form.

2 First-order Hom-differential operators

Throughout this paper, $\mathbb{K}$ will be an algebraically closed field of characteristic 0. Let A be a commutative associative algebra with unit 1_A over $\mathbb{K}$. Recall that an A -module M is an abelian group $(M, +)$ together with a scalar multiplication, $A \times M \longrightarrow M, (a, m) \longmapsto am$, such that the following hold:

- (1) $a(m + n) = am + an$ and $(a + b)m = am + bm$, for all $a, b \in A$ and $m, n \in M$;
- (2) $a(bm) = (ab)m$, for all $a, b \in A$ and $m \in M$;
- (3) $1_A m = m$, for all $m \in M$.

An A -submodule N of an A -module M is a subgroup of $(M, +)$ which is closed under multiplication, that is, $an \in N$, for all $a \in A$ and $n \in N$.

Definition 1 ([5], Definition 2). *Given a commutative associative algebra A with unit 1_A , an A -module M and an algebra endomorphism $\phi : A \longrightarrow A$, a $\mathbb{K}$ -linear map $d : A \longrightarrow M$ is an ϕ -derivation of A into M if $d(ab) = d(a)\phi(b) + \phi(a)d(b)$, for all $a, b \in A$.*

Let us denote by $Der_\phi(A, M)$ the set of all ϕ -derivations of A into M and $Der_\phi(A)$ the set of all ϕ -derivations of A .

Let us now make the following definition.

Definition 2. *Let A be a commutative associative algebra with unit 1_A and M be an A -module. A first-order Hom-differential operator of A into M is a pair (D, ϕ) where $D : A \longrightarrow M$ is a $\mathbb{K}$ -linear map and $\phi : A \longrightarrow A$ an algebra endomorphism such that*

$$D(ab) = D(a)\phi(b) + \phi(a)D(b) - \phi(a)\phi(b)D(1_A), \quad (1)$$

for all $a, b \in A$.

In this case, the $\mathbb{K}$ -linear map $D : A \longrightarrow M$ is called first-order ϕ -differential operator of A into M . The space of all first-order ϕ -differential operators of A into M is denoted by $Diff_\phi(A, M)$ and the space of all first-order ϕ -differential operators of A is denoted by $Diff_\phi(A)$.

For all $a, b, c \in A$, $D_1, D_2, D \in Diff_\phi(A)$, we have

$$(aD)(bc) = (aD)(b)\phi(c) + \phi(b)(aD)(c) - \phi(b)\phi(c)(aD)(1_A), \quad (2)$$

$$\begin{aligned} (D_1 - D_2)(ab) &= (D_1 - D_2)(a)\phi(b) + \phi(a)(D_1 - D_2)(b) \\ &\quad - \phi(a)\phi(b)(D_1 - D_2)(1_A). \end{aligned} \quad (3)$$

Relations (2) and (3) respectively mean that for all $a, b, c \in A$, $D_1, D_2, D \in \text{Diff}_\phi(A)$, then $aD \in \text{Diff}_\phi(A)$ and $D_1 - D_2 \in \text{Diff}_\phi(A)$. Thus, $\text{Diff}_\phi(A)$ is an A -module and

$$\text{Der}_\phi(A) = \{D \in \text{Diff}_\phi(A) / D(1_A) = 0\}$$

is an A -submodule of $\text{Diff}_\phi(A)$.

For $x \in M$, let us define the linear map $L_x : A \rightarrow M, a \mapsto \phi(a) \cdot x$.

Proposition 1. *A linear map $D : A \rightarrow M$ is a first-order ϕ -differential operator of A into M if and only if the linear map $D - L_{D(1_A)} : A \rightarrow M$ is an ϕ -derivation.*

Proof. Let $D : A \rightarrow M$ be a first-order ϕ -differential operator of A into M . For all $a, b \in A$,

$$\begin{aligned} (D - L_{D(1_A)})(ab) &= D(a)\phi(b) + \phi(a)D(b) - \phi(a)\phi(b)D(1_A) \\ &\quad - \phi(a)\phi(b)D(1_A) \\ &= [D(a) - \phi(a)D(1_A)]\phi(b) \\ &\quad + \phi(a)[D(b) - \phi(b)D(1_A)] \\ &= (D - L_{D(1_A)})(a)\phi(b) + \phi(a)(D - L_{D(1_A)})(b). \end{aligned}$$

Conversely, if $D - L_{D(1_A)} : A \rightarrow M$ is an ϕ -derivation, for all $a, b \in A$,

$$(D - L_{D(1_A)})(ab) = (D - L_{D(1_A)})(a)\phi(b) + \phi(a)(D - L_{D(1_A)})(b).$$

That is,

$$\begin{aligned} D(ab) - \phi(a)\phi(b)D(1_A) &= D(a)\phi(b) - \phi(a)\phi(b)D(1_A) + \phi(a)D(b) \\ &\quad - \phi(a)\phi(b)D(1_A). \end{aligned}$$

Thus, we get (1). □

Proposition 2. *Let $\phi : A \rightarrow A$ be an algebra automorphism. Then for $D \in \text{Diff}_\phi(A)$, the linear map $\alpha_\phi(D) : A \rightarrow A$ such that*

$$\alpha_\phi(D) = \phi \circ D \circ \phi^{-1} \quad (4)$$

is a first-order ϕ -differential operator of A and the linear map

$$\alpha_\phi : \text{Diff}_\phi(A) \rightarrow \text{Diff}_\phi(A), D \mapsto \phi \circ D \circ \phi^{-1} \text{ satisfies}$$

$$\alpha_\phi(a \cdot D) = \phi(a) \cdot \alpha_\phi(D). \quad (5)$$

Proof. For all $a, b \in A$, $D \in \text{Diff}_\phi(A)$,

$$\begin{aligned} \alpha_\phi(D)(ab) &= \phi[D(\phi^{-1}(a)\phi^{-1}(b))] \\ &= \phi[D(\phi^{-1}(a))b + aD(\phi^{-1}(b)) - abD(1_A)] \end{aligned}$$

$$= (\phi \circ D \circ \phi^{-1})(a)\phi(b) + \phi(a)(\phi \circ D \circ \phi^{-1})(b) - \phi(a)\phi(b)\phi[D(1_A)].$$

Since $1_A = \phi^{-1}(1_A)$, then

$$\alpha_\phi(D)(ab) = \alpha_\phi(D)(a)\phi(b) + \phi(a)\alpha_\phi(D)(b) - \phi(a)\phi(b)\alpha_\phi(D)(1_A).$$

For all $a, b \in A$, $D \in \text{Diff}_\phi(A)$,

$$\begin{aligned} [\alpha_\phi(a \cdot D)](b) &= \phi[aD(\phi^{-1}(b))] \\ &\quad \phi(a) \cdot (\phi \circ D \circ \phi^{-1})(b). \end{aligned}$$

Therefore, for all $a \in A$, $D \in \text{Diff}_\phi(A)$,

$$\alpha_\phi(a \cdot D) = \phi(a) \cdot \alpha_\phi(D).$$

□

Proposition 3. *Let R be a commutative ring with unit, A be an associative commutative R -algebra with unit 1_A , $\phi : A \longrightarrow A$ be an algebra automorphism and $\text{Diff}_\phi(A)$ be the space of first-order ϕ -differential operators of A . Then the linear map $[D_1, D_2]_\phi : A \longrightarrow A$ such that*

$$[D_1, D_2]_\phi = \phi \circ D_1 \circ \phi^{-1} \circ D_2 \circ \phi^{-1} - \phi \circ D_2 \circ \phi^{-1} \circ D_1 \circ \phi^{-1} \quad (6)$$

is a first-order ϕ -differential operator of A , for all $D_1, D_2 \in \text{Diff}_\phi(A)$.

Proof. For all $a, b \in A$ and $D_1, D_2 \in \text{Diff}_\phi(A)$, we have

$$\begin{aligned} [D_1, D_2]_\phi(ab) &= (\phi \circ D_1 \circ \phi^{-1})[D_2(\phi^{-1}(a))b + aD_2(\phi^{-1}(b)) - abD_2(1_A)] \\ &\quad - (\phi \circ D_2 \circ \phi^{-1})[D_1(\phi^{-1}(a))b + aD_1(\phi^{-1}(b)) - abD_1(1_A)]. \end{aligned}$$

By straightforward computations, we get,

$$\begin{aligned} [D_1, D_2]_\phi(ab) &= (\phi \circ D_1 \circ \phi^{-1} \circ D_2 \circ \phi^{-1})(a)\phi(b) + \phi(a)(\phi \circ D_1 \circ \phi^{-1} \circ D_2 \circ \phi^{-1})(b) \\ &\quad - (\phi \circ D_2 \circ \phi^{-1} \circ D_1 \circ \phi^{-1})(a)\phi(b) - \phi(a)(\phi \circ D_2 \circ \phi^{-1} \circ D_1 \circ \phi^{-1})(b) \\ &\quad - \phi(a)\phi(b)(\phi \circ D_1 \circ \phi^{-1} \circ D_2 \circ \phi^{-1} - \phi \circ D_2 \circ \phi^{-1} \circ D_1 \circ \phi^{-1})(1_A). \end{aligned}$$

That is,

$$[D_1, D_2]_\phi(ab) = [D_1, D_2]_\phi(a)\phi(b) + \phi(a)[D_1, D_2]_\phi(b) - \phi(a)\phi(b)[D_1, D_2]_\phi(1_A),$$

for all $a, b \in A$. □

Definition 3 ([5], Definition 14). *A Hom-Lie algebra is a triple $(\mathcal{G}, [-, -], \alpha)$ where $\mathcal{G}$ a vector space equipped with a skew-symmetric bilinear map $[-, -] : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ and a linear map $\alpha : \mathcal{G} \longrightarrow \mathcal{G}$ satisfying the following Hom-Jacobi identity:*

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0,$$

for all $x, y, z \in \mathcal{G}$.

Proposition 4. Let $\phi : A \longrightarrow A$ be an algebra automorphism. Define a bilinear map $[-, -]_\phi : \text{Diff}_\phi(A) \times \text{Diff}_\phi(A) \longrightarrow \text{Diff}_\phi(A)$ as

$$[D_1, D_2]_\phi = \phi \circ D_1 \circ \phi^{-1} \circ D_2 \circ \phi^{-1} - \phi \circ D_2 \circ \phi^{-1} \circ D_1 \circ \phi^{-1}$$

and define a linear map $\alpha_\phi : \text{Diff}_\phi(A) \longrightarrow \text{Diff}_\phi(A)$ as

$$\alpha_\phi(D) = \phi \circ D \circ \phi^{-1}.$$

Then, the triple $(\text{Diff}_\phi(A), [-, -]_\phi, \alpha_\phi)$ is a Hom-Lie algebra. Furthermore, $\text{Diff}_\phi(A)$ is an A -module satisfying following identities:

(1)

$$\alpha_\phi([D_1, D_2]_\phi) = [\alpha_\phi(D_1), \alpha_\phi(D_2)]_\phi, \quad (7)$$

for all $a \in A$, and $D \in \text{Diff}_\phi(A)$;

(2)

$$[D_1, a \cdot D_2]_\phi = \phi(a) \cdot [D_1, D_2]_\phi + [\alpha_\phi(D_1)(a) - \phi(a) \cdot \alpha_\phi(D_1)(1_A)] \cdot \alpha_\phi(D_2), \quad (8)$$

for all $a \in A$, and $D_1, D_2 \in \text{Diff}_\phi(A)$.

Proof. From (6), for all $D \in \text{Diff}_\phi(A)$, we obtain $[D, D]_\phi = 0$. Thus the bracket is skew-symmetric. From (4) and (6), for all $D_1, D_2, D_3 \in \text{Diff}_\phi(A)$, we obtain

$$\begin{aligned} [[D_1, D_2]_\phi, \alpha_\phi(D_3)]_\phi &= \phi^2 \circ D_1 \circ \phi^{-1} \circ D_2 \circ \phi^{-1} \circ D_3 \circ (\phi^{-1})^2 \\ &\quad - \phi^2 \circ D_3 \circ \phi^{-1} \circ D_1 \circ \phi^{-1} \circ D_2 \circ (\phi^{-1})^2 \\ &\quad - \phi^2 \circ D_2 \circ \phi^{-1} \circ D_1 \circ \phi^{-1} \circ D_3 \circ (\phi^{-1})^2 \\ &\quad + \phi^2 \circ D_3 \circ \phi^{-1} \circ D_2 \circ \phi^{-1} \circ D_1 \circ (\phi^{-1})^2. \end{aligned}$$

Thus, grouping relevant terms together, we get

$$[[D_1, D_2]_\phi, \alpha_\phi(D_3)]_\phi + [[D_2, D_3]_\phi, \alpha_\phi(D_1)]_\phi + [[D_3, D_1]_\phi, \alpha_\phi(D_2)]_\phi = 0.$$

(1) From (4) and (6), for all $D_1, D_2 \in \text{Diff}_\phi(A)$,

$$\begin{aligned} [\alpha_\phi(D_1), \alpha_\phi(D_2)]_\phi &= \phi^2 \circ D_1 \circ \phi^{-1} \circ D_2 \circ (\phi^{-1})^2 \\ &\quad - \phi^2 \circ D_2 \circ \phi^{-1} \circ D_1 \circ (\phi^{-1})^2. \end{aligned}$$

Thus, we obtain (7).

(2) For all $a, b \in A$, and $D_1, D_2 \in \text{Diff}_\phi(A)$, we have

$$\begin{aligned} [D_1, a \cdot D_2]_\phi(b) &= (\phi \circ D_1 \circ \phi^{-1} \circ (a \cdot D_2) \circ \phi^{-1})(b) \\ &\quad - (\phi \circ (a \cdot D_2) \circ \phi^{-1} \circ D_1 \circ \phi^{-1})(b) \end{aligned}$$

$$= \phi \circ D_1 [\phi^{-1}(a) \cdot \phi^{-1}(D_2(\phi^{-1}(b)))] \\ - \phi(a) \cdot \phi(D_2 \circ \phi^{-1} \circ D_1 \circ \phi^{-1}(b)).$$

Since $D_1 \in \text{Diff}_\phi(A)$, then

$$[D_1, a \cdot D_2]_\phi(b) = \phi \circ [D_1(\phi^{-1}(a)) \cdot \phi(\phi^{-1}(D_2(\phi^{-1}(b))))] \\ + \phi \circ [\phi(\phi^{-1}(a)) \cdot D_1(\phi^{-1}(D_2(\phi^{-1}(b))))] \\ - \phi \circ [\phi(\phi^{-1}(a)) \cdot \phi(\phi^{-1}(D_2(\phi^{-1}(b)))) D_1(1_A)] \\ - \phi(a) \cdot \phi(D_2 \circ \phi^{-1} \circ D_1 \circ \phi^{-1}(b)).$$

That is

$$[D_1, a \cdot D_2]_\phi(b) = \phi(a) \cdot [D_1, D_2]_\phi(b) \\ + (\phi \circ D_1 \circ \phi^{-1})(a) \cdot (\phi \circ D_2 \circ \phi^{-1})(b) \\ - \phi(a) \cdot \phi[D_1(1_A)] \cdot (\phi \circ D_2 \circ \phi^{-1})(b),$$

Since $1_A = \phi^{-1}(1_A)$ and $\alpha_\phi(D) = \phi \circ D \circ \phi^{-1}$, then

$$[D_1, a \cdot D_2]_\phi(b) = \phi(a) \cdot [D_1, D_2]_\phi(b) + [\alpha_\phi(D_1)(a) - \phi(a) \cdot \alpha_\phi(D_1)(1_A)] \cdot \alpha_\phi(D_2)(b),$$

for all $b \in A$. Thus, we obtain (8). $\square$

3 Hom-Jacobi algebras

Definition 4 ([10], Definition 1.1). *Let A be a vector space over an algebraically closed field $\mathbb{K}$ of characteristic 0. A Hom-associative algebra is a triple $(A, \cdot, \alpha)$ consisting of a vector space A , a bilinear map $\cdot : A \times A \rightarrow A$ and a vector space homomorphism $\alpha : A \rightarrow A$ satisfying*

$$\alpha(a) \cdot (b \cdot c) = (a \cdot b) \cdot \alpha(c),$$

for all $a, b, c \in A$. A Hom-associative algebra for which $x \cdot y = y \cdot x$ is called a commutative Hom-associative algebra. A commutative Hom-associative algebra $(A, \cdot, \alpha)$ is said to be regular if α is invertible.

Let us now make the following definition.

Definition 5. *Let $(A, \cdot, \alpha)$ be a commutative Hom-associative algebra. A skew-symmetric A -multilinear map $Q : A \times A \times \cdots \times A \rightarrow A$ of degree k is a skew-symmetric α -derivation of degree k if*

$$Q(a \cdot b, \alpha(a_2), \dots, \alpha(a_k)) = \alpha(a) \dots Q(b, a_2, \dots, a_k) + \alpha(b) \cdot Q(a, a_2, \dots, a_k),$$

for all $a, b, a_i \in A$, $i \in \{2, \dots, k\}$.

Example 1. If $(A, \cdot, \{-, -\}, \alpha)$ is a Hom-Poisson algebra (see [7, Definition 1.4]), then the bracket $\{-, -\} : A \times A \longrightarrow A$ is a skew-symmetric α -derivation of degree 2, that is,

$$\{\alpha(a), b \cdot c\} = \alpha(b) \cdot \{a, c\} + \{a, b\} \cdot \alpha(c)$$

and

$$\{b \cdot c, \alpha(a)\} = \alpha(b) \cdot \{a, c\} + \{a, b\} \cdot \alpha(c),$$

for all $a, b, c \in A$.

Definition 6. Let $(A, \cdot, \alpha)$ be a commutative Hom-associative algebra. A skew-symmetric A -multilinear map $\varphi : A \times A \times \cdots \times A \longrightarrow A$ of degree k is a skew-symmetric first-order α -differential operator of degree k if

$$\begin{aligned} \varphi(a \cdot b, \alpha(a_2), \dots, \alpha(a_k)) &= \alpha(a) \cdot \varphi(b, a_2, \dots, a_k) \\ &\quad + \alpha(b) \cdot \varphi(a, a_2, \dots, a_k) \\ &\quad - \alpha(a) \cdot \alpha(b) \cdot \varphi(1_A, a_2, \dots, a_k), \end{aligned}$$

for all $a, b, a_i \in A$, $i \in \{2, \dots, k\}$.

We introduce the notion of Hom-Jacobi algebra structure:

Definition 7. A Hom-Jacobi algebra is a quadruple $(A, \cdot, \{-, -\}, \alpha)$, where $(A, \cdot, \alpha)$ is a commutative Hom-associative algebra and $(A, \{-, -\}, \alpha)$ is a Hom-Lie algebra, satisfying the generalized Hom-Leibniz identity:

$$\{\alpha(a), b \cdot c\} = \alpha(b) \cdot \{a, c\} + \{a, b\} \cdot \alpha(c) - \alpha(b) \cdot \alpha(c) \cdot \{a, 1_A\},$$

for all $a, b, c \in A$.

When $1_A \in Z(A)$, that is, $\{a, 1_A\} = 0$, we recover the notion of Hom-Poisson algebras given in [11, Definition 1.4]. When $\alpha = id_A$, we recover the notion of Jacobi algebras given in [4, 6, 9, 13].

Example 2. Let $(A, \cdot_A, \{\cdot, \cdot\}_A, \alpha_A)$ and $(B, \cdot_B, \{\cdot, \cdot\}_B, \alpha_B)$ be two Hom-Jacobi algebras. Then $(A \oplus B, \cdot_{A \oplus B}, \{-, -\}_{A \oplus B}, \alpha_A \oplus \alpha_B)$ is a Hom-Jacobi algebra, where the bracket $\{-, -\}_{A \oplus B}$, the product $\cdot_{A \oplus B}$ and $\alpha_A \oplus \alpha_B$ are respectively given by

$$\{a + x, b + y\}_{A \oplus B} = \{a, b\}_A + \{x, y\}_B,$$

$$(a + x) \cdot_{A \oplus B} (b + y) = a \cdot_A b + x \cdot_B y,$$

$$(\alpha_A \oplus \alpha_B)(a + x) = \alpha_A(a) + \alpha_B(x),$$

for all $a, b, c \in A$ and $x, y, z \in B$. Indeed, the commutative Hom-associative algebra structure of $(A \oplus B, \cdot_{A \oplus B}, \alpha_A \oplus \alpha_B)$ follows immediately from the Hom-associative algebra structure of $(A, \cdot_A, \alpha_A)$ and $(B, \cdot_B, \alpha_B)$.

Likewise, the Hom-Lie algebra structure of $(A \oplus B, \{-, -\}_{A \oplus B}, \alpha_A \oplus \alpha_B)$ follows immediately from the Hom-associative algebra structure of $(A, \{\cdot, \cdot\}_A, \alpha_A)$ and $(B, \{\cdot, \cdot\}_B, \alpha_B)$.

For the generalized Hom-Leibniz identity, for all $a, b, c \in A$ and for all $x, y, z \in A$, we have

$$\begin{aligned} \{(\alpha_A \oplus \alpha_B)(a+x), (b+y) \cdot_{A \oplus B} (c+z)\}_{A \oplus B} &= \{\alpha_A(a) + \alpha_B(x), b \cdot_A c + y \cdot_B z\}_{A \oplus B} \\ &= \{\alpha_A(a), b \cdot_A c\}_A + \{\alpha_B(x), y \cdot_B z\}_B. \end{aligned}$$

On the one hand, since $(A, \cdot_A, \{\cdot, \cdot\}_A, \alpha_A)$ and $(B, \cdot_B, \{\cdot, \cdot\}_B, \alpha_B)$ are two Hom-Jacobi algebras, for all $a, b, c \in A$ and $x, y, z \in B$, we have

$$\begin{aligned} &\{(\alpha_A \oplus \alpha_B)(a+x), (b+y) \cdot_{A \oplus B} (c+z)\}_{A \oplus B} \\ &= \alpha_A(b) \cdot_A \{a, c\}_A + \alpha_A(c) \cdot_A \{a, b\}_A - \alpha_A(b) \cdot_A \alpha_A(c) \cdot_A \{a, 1_A\}_A \\ &\quad + \alpha_B(y) \cdot_B \{x, z\}_B + \alpha_B(z) \cdot_B \{x, y\}_B \\ &\quad - \alpha_B(y) \cdot_B \alpha_B(z) \cdot_B \{x, 1_B\}_B. \end{aligned}$$

On the other hand, for all $a, b, c \in A$ and $x, y, z \in B$, we have

$$\begin{aligned} (\alpha_A \oplus \alpha_B)(b+y) \cdot_{A \oplus B} \{a+x, c+z\}_{A \oplus B} &= \alpha_A(b) \cdot_A \{a, c\}_A + \alpha_B(y) \cdot_B \{x, z\}_B, \\ (\alpha_A \oplus \alpha_B)(c+z) \cdot_{A \oplus B} \{a+x, b+y\}_{A \oplus B} &= \alpha_A(c) \cdot_A \{a, b\}_A + \alpha_B(z) \cdot_B \{x, y\}_B, \end{aligned}$$

and

$$\begin{aligned} &(\alpha_A \oplus \alpha_B)(b+y) \cdot_{A \oplus B} (\alpha_A \oplus \alpha_B)(c+z) \cdot_{A \oplus B} \{a+x, 1_A + 1_B\}_{A \oplus B} \\ &= \alpha_A(b) \cdot_A \alpha_A(c) \cdot_A \{a, 1_A\}_A + \alpha_B(y) \cdot_B \alpha_B(z) \cdot_B \{x, 1_B\}_B. \end{aligned}$$

Thus, the bracket $\{-, -\}_{A \oplus B}$ satisfies the generalized Hom-Leibniz identity.

Therefore, $(A \oplus B, \cdot_{A \oplus B}, \{-, -\}_{A \oplus B}, \alpha_A \oplus \alpha_B)$ is a Hom-Jacobi algebra.

Definition 8. Let $(A, \cdot, \alpha)$ be a Hom-associative algebra. A submodule $I \subseteq A$ is called a Hom-associative ideal of A if $x \cdot y \in I$, $y \cdot x \in I$ for all $x \in I$, $y \in A$, and $\alpha(I) \subseteq I$.

Definition 9 ([2], Definition 2.9). Let $(\mathcal{G}, [-, -], \alpha_{\mathcal{G}})$ be a Hom-Lie algebra. A Hom-Lie subalgebra $(\mathcal{H}, \alpha_{\mathcal{H}})$ of $(\mathcal{G}, [-, -], \alpha_{\mathcal{G}})$ is a linear subspace $\mathcal{H}$ of $\mathcal{G}$, which is closed for the bracket and invariant by $\alpha_{\mathcal{G}}$, that is,

- a) $[x, y] \in \mathcal{H}$, for all $x, y \in \mathcal{H}$,
- b) $\alpha_{\mathcal{G}}(x) \in \mathcal{H}$, for all $x \in \mathcal{H}$, $(\alpha_{\mathcal{H}} = \alpha|_{\mathcal{G}})$.

A Hom-Lie subalgebra $(\mathcal{H}, \alpha_{\mathcal{H}})$ of $(\mathcal{G}, [-, -], \alpha_{\mathcal{G}})$ is said to be a Hom-ideal if $[x, y] \in \mathcal{H}$, for all $x \in \mathcal{H}$, $y \in \mathcal{G}$.

Definition 10. A Hom-Jacobi ideal of a Hom-Jacobi algebra $(A, \cdot, \{-, -\}, \alpha)$ is a linear subspace I which is both an ideal with respect to the Hom-associative product and Hom-Lie bracket.

Proposition 5. If I is a Hom-Jacobi ideal of $(A, \cdot, \{-, -\}, \alpha)$, then A/I inherits a Hom-Jacobi algebra structure.

Proof. Let us define the product $\cdot_{A/I} : A/I \longrightarrow A/I$, the bracket $\{-, -\}_{A/I} : A/I \times A/I \longrightarrow A/I$ and $\alpha_{A/I} : A/I \longrightarrow A/I$ respectively by

$$(a+I) \cdot_{A/I} (b+I) = a \cdot b + I, \tag{9}$$

$$\{a + I, b + I\}_{A/I} = \{a, b\} + I, \quad (10)$$

$$\alpha_{A/I}(a + I) = \alpha(a) + I. \quad (11)$$

For all $a, b, c \in A$,

$$\begin{aligned} \alpha_{A/I}(a + I) \cdot_{A/I} [(b + I) \cdot_{A/I} (c + I)] &= (\alpha(a) + I)(b \cdot c + I) \\ &= \alpha(a) \cdot (b \cdot c) + I. \end{aligned}$$

By Hom-associativity, (9) and (11), for all $a, b, c \in A$, we have

$$\begin{aligned} \alpha_{A/I}(a + I) \cdot_{A/I} [(b + I) \cdot_{A/I} (c + I)] &= (a \cdot b) \cdot \alpha(c) + I \\ &= [(a + I) \cdot_{A/I} (b + I)] \cdot_{A/I} \alpha_{A/I}(c + I). \end{aligned}$$

Thus, $(A/I, \cdot_{A/I}, \alpha_{A/I})$ is a commutative Hom-associative algebra.

By (10), for all $a \in A$, we have

$$\{a + I, a + I\}_{A/I} = \{a, a\} + I.$$

Since the bracket $\{-, -\}$ is skew-symmetric, then $\{a + I, a + I\}_{A/I} = I = 0_{A/I}$. Thus, the bracket $\{-, -\}_{A/I}$ is skew-symmetric.

By (10) and (11), for all $a, b, c \in A$, we have

$$\{\alpha_{A/I}(a + I), \{b + I, c + I\}_{A/I}\}_{A/I} = \{\alpha(a), \{b, c\}\} + I.$$

Thus, for all $a, b, c \in A$,

$$\circlearrowleft_{a,b,c} \{\alpha_{A/I}(a + I), \{b + I, c + I\}_{A/I}\}_{A/I} = \circlearrowleft_{a,b,c} \{\alpha(a), \{b, c\}\} + I.$$

where $\circlearrowleft_{a,b,c}$ denotes summation over the cyclic permutation on a, b, c . By Hom-Jacobi identity, we have $\circlearrowleft_{a,b,c} \{\alpha(a), \{b, c\}\} = 0$. Which implies that $\circlearrowleft_{a,b,c} \{\alpha_{A/I}(a + I), \{b + I, c + I\}_{A/I}\}_{A/I} \in$

$I = 0_{A/I}$. Thus, $(A/I, \{-, -\}_{A/I}, \alpha_{A/I})$ is a Hom-Lie algebra.

For the generalized Hom-Leibniz identity, for all $a, b, c \in A$, we have

$$\begin{aligned} \{\alpha_{A/I}(a + I), (b + I) \cdot_{A/I} (c + I)\}_{A/I} &= \{\alpha(a) + I, b \cdot c + I\}_{A/I} \\ &= \{\alpha(a), b \cdot c\} + I. \end{aligned}$$

By Hom-Leibniz identity, for all $a, b, c \in A$, we have

$$\begin{aligned} \{\alpha_{A/I}(a + I), (b + I) \cdot_{A/I} (c + I)\}_{A/I} &= \alpha(b) \cdot \{a, c\} + \{a, b\} \cdot \alpha(c) \\ &\quad - \alpha(b) \cdot \alpha(c) \cdot \{a, 1_A\} + I. \end{aligned}$$

That is,

$$\{\alpha_{A/I}(a + I), (b + I) \cdot_{A/I} (c + I)\}_{A/I} = (\alpha(b) + I) \cdot_{A/I} (\{a, c\} + I)$$

$$\begin{aligned}
& + (\{a, b\} + I) \cdot_{A/I} (\alpha(c) + I) \\
& - (\alpha(b) + I) \cdot_{A/I} (\alpha(c) + I) \cdot_{A/I} (\{a, 1_A\} + I).
\end{aligned}$$

Which implies that

$$\begin{aligned}
\{\alpha_{A/I}(a + I), (b + I) \cdot_{A/I} (c + I)\}_{A/I} &= \alpha_{A/I}(b + I) \cdot_{A/I} \{a + I, c + I\}_{A/I} \\
&+ \alpha_{A/I}(c + I) \cdot_{A/I} \{a + I, b + I\}_{A/I} \\
&- \alpha_{A/I}(b + I) \cdot_{A/I} \alpha_{A/I}(c + I) \cdot_{A/I} \{a + I, 1_{A/I}\}_{A/I}.
\end{aligned}$$

□

Definition 11. Let $(A, \cdot, \{\cdot, \cdot\}_A, \alpha_A)$ and $(B, *, \{\cdot, \cdot\}_B, \alpha_B)$ be two Hom-Jacobi algebras. A linear map $f : A \rightarrow B$ is called a homomorphism of Hom-Jacobi algebras if $\alpha_B(f(a)) = f(\alpha_A(a))$, $f(a \cdot b) = f(a) * f(b)$ and $f(\{a, b\}_A) = \{f(a), f(b)\}_B$, for all $a, b \in A$.

If $\alpha_A = id_A$ and $\alpha_B = id_B$, we recover the notion of Jacobi homomorphisms given in [1, 9].

Definition 12. A generalized Hom-Poisson algebra is a quadruple $(A, \cdot, \{-, -\}, \alpha)$, where $(A, \cdot, \alpha)$ is a Hom-associative algebra and $(A, \{\cdot, \cdot\}, \alpha)$ is a Hom-Lie algebra, satisfying:

$$\{a, b \cdot c\} = \alpha(b) \cdot \{a, c\} + \{a, b\} \cdot \alpha(c) + \alpha(b) \cdot \alpha(c) \cdot D(a),$$

where D is an α -derivation of $(A, \cdot)$ and α is a homomorphism, for all $a, b, c \in A$.

Example 3. Any commutative Hom-associative algebra $(A, \cdot, \alpha)$ equipped with an α -derivation D is a generalized Hom-Poisson algebra with respect to

$$\{a, b\} = \alpha(a) \cdot D(b) - \alpha(b) \cdot D(a). \quad (12)$$

Indeed, for all $a, b, c \in A$, we have

$$\begin{aligned}
\{a, b \cdot c\} &= \alpha(a) \cdot D(b \cdot c) - \alpha(b \cdot c) \cdot D(a) \\
&= \alpha(a) \cdot \alpha(b) \cdot D(c) + \alpha(a) \cdot \alpha(c) \cdot D(b) \\
&\quad - \alpha(b) \cdot \alpha(c) \cdot D(a).
\end{aligned}$$

On the other hand, by (12), for all $a, b, c \in A$, we get

$$\begin{aligned}
& \alpha(b) \cdot \{a, c\} + \{a, b\} \cdot \alpha(c) + \alpha(b) \cdot \alpha(c) \cdot D(a) \\
&= \alpha(a) \cdot \alpha(b) \cdot D(c) + \alpha(a) \cdot \alpha(c) \cdot D(b) - \alpha(b) \cdot \alpha(c) \cdot D(a).
\end{aligned}$$

For the Hom-Jacobi identity, by (12), for all $a, b, c \in A$, we have

$$\begin{aligned}
\{\alpha(a), \{b, c\}\} &= \alpha(a) (\alpha(\alpha(b)) D(D(c)) + \alpha(D(c)) D(\alpha(b))) \\
&\quad - \alpha(a) (\alpha(\alpha(c)) D(D(b)) + \alpha(D(b)) D(\alpha(c))) \\
&\quad - \alpha(\alpha(b)) \alpha(D(c)) D(\alpha(a)) + \alpha(\alpha(c)) \alpha(D(b)) D(\alpha(a)).
\end{aligned}$$

By commutative Hom-associative algebra of $(A, \cdot, \alpha)$, we get $\{\alpha(a), \{b, c\}\} + \{\alpha(b), \{c, a\}\} + \{\alpha(c), \{a, b\}\} = 0$.

By (12), for all $a \in A$, $\{a, a\} = 0$. Thus, the bracket $\{-, -\}$ is skew-symmetric.

We introduce the notion of purely Hom-Jacobi algebra structure:

Definition 13. A purely Hom-Jacobi algebra is a quadruple $(A, \cdot, \{\cdot, \cdot\}, \alpha)$, where $(A, \cdot)$ is a commutative associative algebra, $(A, \{\cdot, \cdot\}, \alpha)$ is a Hom-Lie algebra and if, for all $a \in A$, the inner derivation $ad(a) : A \rightarrow A, b \mapsto \{a, b\}$ is a first-order α -differential operator of commutative associative algebra $(A, \cdot)$, that is,

$$\{a, b \cdot c\} = \alpha(b) \cdot \{a, c\} + \{a, b\} \cdot \alpha(c) - \alpha(b) \cdot \alpha(c) \cdot \{a, 1_A\},$$

for all $b, c \in A$.

When $1_A \in Z(A)$, that is, $\{a, 1_A\} = 0$ and we recover the notion of purely Hom-Poisson algebras given in [7, Definition 2.7]. When $\alpha = id_A$, we recover the notion of Jacobi algebras given in [6].

Any purely Hom-Jacobi algebra is a generalized Hom-Poisson algebra, where $D = ad(-1_A)$.

Proposition 6. Let $(A, \cdot, \{-, -\})$ be a Jacobi algebra and $\alpha : A \rightarrow A$ a Jacobi endomorphism, then $(A, \cdot, \{-, -\}_\alpha = \alpha \circ \{-, -\}, \alpha)$ is a purely Hom-Jacobi algebra.

Proof. For all $a, b, c \in A$, we have $\{a, b \cdot c\}_\alpha = \alpha(\{a, b \cdot c\})$. Since $(A, \cdot, \{-, -\})$ is a Jacobi algebra,

$$\begin{aligned} \{a, b \cdot c\}_\alpha &= \alpha(b \cdot \{a, c\} + \{a, b\} \cdot c - b \cdot c \cdot \{a, 1_A\}) \\ &= \alpha(b) \cdot \alpha(\{a, c\}) + \alpha(\{a, b\}) \cdot \alpha(c) \\ &\quad - \alpha(b) \cdot \alpha(c) \cdot \alpha(\{a, 1_A\}). \end{aligned}$$

Thus,

$$\{a, b \cdot c\}_\alpha = \alpha(b) \cdot \{a, c\}_\alpha + \{a, b\}_\alpha \cdot \alpha(c) - \alpha(b) \cdot \alpha(c) \cdot \{a, 1_A\}_\alpha,$$

for all $b, c \in A$. □

Example 4. Let M be a Jacobi manifold. Then $(C^\infty(M), \cdot, \{-, -\})$ is a Jacobi algebra where $\cdot$ is the usual product on $C^\infty(M)$ (see [6, 9]). Let $\varphi : M \rightarrow M$ be a smooth map. Then the pullback map $\varphi^* : C^\infty(M) \rightarrow C^\infty(M), f \mapsto \varphi^*(f) = f \circ \varphi$ is a homomorphism of the function ring $C^\infty(M)$, that is,

$$\varphi^*(f \cdot g) = \varphi^*(f) \cdot \varphi^*(g),$$

for all $f, g \in C^\infty(M)$. Indeed, for all $p \in M$, we have

$$[\varphi^*(f \cdot g)](p) = [(f \cdot g) \circ \varphi](p) = (f \cdot g)[\varphi(p)] = f[\varphi(p)] \cdot g[\varphi(p)].$$

That is,

$$[\varphi^*(f \cdot g)](p) = [(f \circ \varphi) \cdot (g \circ \varphi)](p) = [\varphi^*(f) \cdot \varphi^*(g)](p),$$

for all $p \in M$. Under these conditions, $(C^\infty(M), \cdot, \{-, -\}_{\varphi^*} = \varphi^* \circ \{-, -\}, \varphi^*)$ is a purely Hom-Jacobi algebra.

4 Canonical form of purely Hom-Jacobi algebra

4.1 Universal property of first-order Hom-differential operators

Let A be a commutative associative algebra with unity 1_A , and let $\alpha : A \rightarrow A$ be an algebra automorphism. The map $f : A \rightarrow A \otimes_{\mathbb{K}} A, a \mapsto a \otimes 1_A$ is a homomorphism of algebras. Thus, $A \otimes_{\mathbb{K}} A$ is equipped with the structure of A -module defined by f . We consider the A -submodule I of $A \otimes_{\mathbb{K}} A$ generated by the elements of the form $\alpha(a) \otimes 1_A - 1_A \otimes \alpha(a)$, that is, for all $z \in I$,

$$z = \sum_{i \in J: \text{finite}} a_i (\alpha(b_i) \otimes 1_A - 1_A \otimes \alpha(b_i)),$$

with $a_i, b_i \in A$. We denote by $\Omega_{\alpha}(A)$ the module of Kähler differentials of commutative algebra A , that is, the quotient space $\Omega_{\alpha}(A) = I/I^2$. For $a \in A$, we denote by $\overline{\alpha(a) \otimes 1_A - 1_A \otimes \alpha(a)}$ the class of $\alpha(a) \otimes 1_A - 1_A \otimes \alpha(a)$ in $\Omega_{\alpha}(A)$.

Proposition 7. *The map $d_{A/\mathbb{K}} : A \rightarrow \Omega_{\alpha}(A)$ such that, for all $a \in A$,*

$$d_{A/\mathbb{K}}(a) = \overline{\alpha(a) \otimes 1_A - 1_A \otimes \alpha(a)}$$

is an α -derivation. Moreover, the image of $d_{A/\mathbb{K}}$ generates the A -module $\Omega_{\alpha}(A)$.

Proof. For all $a, b \in A$, we have

$$\begin{aligned} (\alpha(a) \otimes 1_A - 1_A \otimes \alpha(a))(\alpha(b) \otimes 1_A - 1_A \otimes \alpha(b)) &= (\alpha(a) \cdot \alpha(b)) \otimes 1_A - \alpha(a) \otimes \alpha(b) \\ &\quad - \alpha(b) \otimes \alpha(a) + 1_A \otimes (\alpha(a) \cdot \alpha(b)). \end{aligned}$$

Since I is generated by the elements of the form $\alpha(a) \otimes 1_A - 1_A \otimes \alpha(a)$, for all $a, b \in A$, we have $(\alpha(a) \otimes 1_A - 1_A \otimes \alpha(a))(\alpha(b) \otimes 1_A - 1_A \otimes \alpha(b)) \in I^2$, that is,

$$\overline{(\alpha(a) \cdot \alpha(b)) \otimes 1_A - \alpha(a) \otimes \alpha(b) - \alpha(b) \otimes \alpha(a) + 1_A \otimes (\alpha(a) \cdot \alpha(b))} = 0.$$

Thus, for all $a, b \in A$,

$$\begin{aligned} d_{A/K}(a \cdot b) &= \overline{\alpha(a \cdot b) \otimes 1_A - 1_A \otimes \alpha(a \cdot b)} \\ &= \overline{(\alpha(a) \cdot \alpha(b)) \otimes 1_A - 1_A \otimes (\alpha(a) \cdot \alpha(b))} \\ &\quad + \overline{(\alpha(a) \cdot \alpha(b)) \otimes 1_A - \alpha(a) \otimes \alpha(b)} \\ &\quad + \overline{1_A \otimes (\alpha(a) \cdot \alpha(b)) - \alpha(b) \otimes \alpha(a)}. \end{aligned}$$

Which implies that

$$\begin{aligned} d_{A/K}(a \cdot b) &= \overline{(\alpha(a) \cdot \alpha(b)) \otimes 1_A - 1_A \otimes (\alpha(a) \cdot \alpha(b))} \\ &\quad + \overline{(\alpha(a) \cdot \alpha(b)) \otimes 1_A - \alpha(a) \otimes \alpha(b)} \\ &\quad + \overline{1_A \otimes (\alpha(a) \cdot \alpha(b)) - \alpha(b) \otimes \alpha(a)}. \end{aligned}$$

Therefore, for all $a, b \in A$,

$$\begin{aligned} d_{A/K}(a \cdot b) &= \overline{(\alpha(a) \cdot \alpha(b)) \otimes 1_A - \alpha(a) \otimes \alpha(b)} + \overline{(\alpha(a) \cdot \alpha(b)) \otimes 1_A - \alpha(b) \otimes \alpha(a)} \\ &= \alpha(a) \cdot \overline{(\alpha(b) \otimes 1_A - 1_A \otimes \alpha(b))} + \overline{(\alpha(a) \otimes 1_A - 1_A \otimes \alpha(a))} \cdot \alpha(b) \\ &= \overline{(\alpha(a) \otimes 1_A - 1_A \otimes \alpha(a))} \cdot \alpha(b) + \alpha(a) \cdot \overline{(\alpha(b) \otimes 1_A - 1_A \otimes \alpha(b))} \\ &= d_{A/K}(a) \cdot \alpha(b) + \alpha(a) \cdot d_{A/K}(b). \end{aligned}$$

As $d_{A/K}$ is clearly $\mathbb{K}$ -linear, we conclude that $d_{A/K}$ is an α -derivation. The map $d_{A/\mathbb{K}}$ is surjective by definition, that is, $d_{A/\mathbb{K}}(A) = \Omega_\alpha(A) = I/I^2$. Since I is generated by the elements of the form $\alpha(a) \otimes 1_A - 1_A \otimes \alpha(a)$, then for all $z \in I$, $z = \sum_{i \in J: \text{finite}} a_i(\alpha(b_i) \otimes 1_A - 1_A \otimes \alpha(b_i))$, with $a_i, b_i \in A$. Thus, $z \in \Omega_\alpha(A) = I/I^2$,

$$\dot{z} = \sum_{i \in J: \text{finite}} a_i \overline{(\alpha(b_i) \otimes 1_A - 1_A \otimes \alpha(b_i))} = \sum_{i \in J: \text{finite}} a_i d_{A/\mathbb{K}}(b_i).$$

□

Lemma 1. *Let M be an A -module, $A \ltimes M$ be the semi-direct product and $d : A \rightarrow M$ be an α -derivation, then the map $\tilde{d} : A \rightarrow A \ltimes M$ given by $\tilde{d}(a) = (a, d(\alpha^{-1}(a)))$, for $a \in A$, is an algebra homomorphism with $\pi_1 \circ \tilde{d} = Id_A$, where $\pi_1(a, m) = a$.*

Conversely, for an algebra homomorphism $h : A \rightarrow A \ltimes M$ satisfying $\pi_1 \circ \tilde{d} = Id_A$, there is a unique α -derivation $d : A \rightarrow M$ with $h = \tilde{d}$.

Proof. Let $a, b \in A$. From the definition of $\tilde{d}$ and π_1 we get

$$\tilde{d}(ab) = (ab, d(\alpha^{-1}(ab))) = (a, d(\alpha^{-1}(a)))(b, d(\alpha^{-1}(b))) = \tilde{d}(a)\tilde{d}(b),$$

and $\pi_1 \circ \tilde{d} = Id_A$.

Conversely, let $h : A \rightarrow A \ltimes M$ be an algebra homomorphism satisfying $\pi_1 \circ \tilde{d} = Id_A$. Then for $a \in A$, we can write $h(a) = (a, h_1(a))$ where $h_1 : A \rightarrow M$ is a $\mathbb{K}$ -linear map. Define $d : A \rightarrow M$ by $d(a) = h_1(\varphi(a))$ for $a \in A$. It follows that d is an α -derivation and the map $\tilde{d} = h$. □

In the following theorem, we give the universal property of the couple $(\Omega_\alpha(A), d_{A/\mathbb{K}})$.

Theorem 1 ([12]). *For any A -module M and for any α -derivation $\varphi : A \rightarrow M$, there exists a unique homomorphism $\tilde{\varphi} : \Omega_\alpha(A) \rightarrow M$ of A -modules such that $\tilde{\varphi} \circ d_{A/\mathbb{K}} = \varphi$.*

Proof. Let M be an A -module and $\varphi : A \rightarrow M$ be an α -derivation. According to Lemma 1, we have an algebra homomorphism $\psi : A \rightarrow A \ltimes M$ given by $\psi(a) = (a, \varphi(\alpha^{-1}(a)))$ for $a \in A$. If we consider the map $i : A \rightarrow A \ltimes M$ defined by $i(a) = (a, 0)$, then we get an R -algebra homomorphism $h : A \otimes A \rightarrow A \ltimes M$ defined by

$$h(a \otimes b) = \psi(a)i(b) = (ab, b\varphi(\alpha^{-1}(a)));$$

for all $a, b \in A$. So, for $x = \alpha(a) \otimes 1 - 1 \otimes \alpha(a) \in I$, we have $h(x) = (0, \varphi(a))$. Thus h vanishes on $I^2 \subset A \otimes A$. So it induces an A -module homomorphism $\tilde{\varphi} : I/I^2 = \Omega_\alpha(A) \longrightarrow M$ such that

$$\tilde{\varphi}(d_{A/\mathbb{K}}(a)) = \varphi(a),$$

for all $a \in A$. □

Therefore, the map

$$\text{Hom}_A(\Omega_\alpha(A), M) \longrightarrow \text{Der}_\alpha(A, M), \psi \longmapsto \psi \circ d_{A/\mathbb{K}}$$

is an isomorphism of A -modules. In particular, $\Omega_\alpha(A)^* \simeq \text{Der}_\alpha(A)$.

Proposition 8. *The map $D_{A/\mathbb{K}} : A \longrightarrow A \oplus \Omega_\alpha(A)$ such that, for all $a \in A$,*

$$D_{A/\mathbb{K}}(a) = \alpha(a) + d_{A/\mathbb{K}}(a)$$

is a first-order α -differential operator.

Proof. For all $a, b \in A$, we have

$$\begin{aligned} & D_{A/\mathbb{K}}(ab) - \alpha(a) D_{A/\mathbb{K}}(b) - \alpha(b) D_{A/\mathbb{K}}(a) + \alpha(a) \alpha(b) D_{A/\mathbb{K}}(1_A) \\ = & \alpha(a) \alpha(b) + d_{A/\mathbb{K}}(ab) - \alpha(a) (\alpha(b) + d_{A/\mathbb{K}}(b)) \\ & - \alpha(b) (\alpha(a) + d_{A/\mathbb{K}}(a)) \\ & + \alpha(a) \alpha(b) (\alpha(1_A) + d_{A/\mathbb{K}}(1_A)) \\ = & d_{A/\mathbb{K}}(ab) - \alpha(a) d_{A/\mathbb{K}}(b) - \alpha(b) d_{A/\mathbb{K}}(a). \end{aligned}$$

Since $d_{A/\mathbb{K}}(a)$ is an α -derivation, we get

$$D_{A/\mathbb{K}}(ab) = \alpha(a) D_{A/\mathbb{K}}(b) + \alpha(b) D_{A/\mathbb{K}}(a) - \alpha(a) \alpha(b) D_{A/\mathbb{K}}(1_A),$$

for all $a, b \in A$. Thus, $D_{A/\mathbb{K}} \in \text{Diff}_\alpha(A, A \oplus \Omega_\alpha(A))$. □

For all $x \in A \oplus \Omega_\alpha(A)$, $x = \sum_{i \in J: \text{finite}} a_i D_{A/\mathbb{K}}(b_i)$, with $a_i, b_i \in A$, that is, the image of $D_{A/\mathbb{K}}$ generates the A -module $A \oplus \Omega_\alpha(A)$.

In the following theorem, we give the universal property of the couple $(A \oplus \Omega_\alpha(A), D_{A/\mathbb{K}})$.

Theorem 2. *For any A -module M and for any first-order α -differential operator $\varphi : A \longrightarrow M$, there exists a unique homomorphism $\tilde{\varphi} : A \oplus \Omega_\alpha(A) \longrightarrow M$ of A -modules such that $\tilde{\varphi} \circ D_{A/\mathbb{K}} = \varphi$.*

Proof. Since $\varphi : A \longrightarrow M$ is first-order α -differential operator, according to Proposition 1, the map $\varphi - L_{\varphi(1_A)} : A \longrightarrow M$ is an α -derivation. By the Theorem 1, there exists a unique homomorphism $\psi : \Omega_\alpha(A) \longrightarrow M$ such that $\psi \circ d_{A/\mathbb{K}} = \varphi - L_{\varphi(1_A)}$. The map $\tilde{\varphi} : A \oplus \Omega_\alpha(A) \longrightarrow M$ such that, for all $a \in A$ and $x \in \Omega_\alpha(A)$,

$$\tilde{\varphi}(a + x) = a\varphi(1_A) + \psi(x)$$

satisfies

$$\begin{aligned}\tilde{\varphi} \circ D_{A/\mathbb{K}}(a) &= \tilde{\varphi}(\alpha(a) + d_{A/\mathbb{K}}(a)) \\ &= \alpha(a)\varphi(1_A) + \psi \circ d_{A/\mathbb{K}}(a) \\ &= \alpha(a)\varphi(1_A) + (\varphi - L_{\varphi(1_A)})(a).\end{aligned}$$

It follows that

$$\tilde{\varphi} \circ D_{A/\mathbb{K}}(a) = \alpha(a)\varphi(1_A) + \varphi(a) - \alpha(a)\varphi(1_A) = \varphi(a),$$

for all $a \in A$. Thus, $\tilde{\varphi} \circ D_{A/\mathbb{K}} = \varphi$. $\square$

Definition 14 ([10]). Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and A be a vector space over $\mathbb{K}$. A Hom-algebra is a triple $(A, \cdot, \alpha)$ consisting of a vector space A , a bilinear map $\cdot : A \times A \rightarrow A$ and a vector space homomorphism $\alpha : A \rightarrow A$.

We introduce the following definition.

Definition 15. Let $(A, \cdot, \alpha)$ be a Hom-algebra and let M be an A -module. A skew-symmetric A -multilinear map $Q : A \times A \times \cdots \times A \rightarrow M$ of degree k is a skew-symmetric α -derivation of degree k if

$$Q(a_1b, a_2, \dots, a_k) = \alpha(a_1)Q(b, a_2, \dots, a_k) + \alpha(b)Q(a_1, a_2, \dots, a_k)$$

for all $b, a_i \in A$, $i \in \{1, \dots, k\}$.

Example 5. If $(A, \cdot, \{-, -\}, \alpha)$ is a purely Hom-Poisson algebra ([7, Definition 2.7]), then the bracket $\{-, -\} : A \times A \rightarrow A$ is a skew-symmetric α -derivation of degree 2, that is,

$$\{b \cdot c, a\} = \alpha(b) \cdot \{a, c\} + \{a, b\} \cdot \alpha(c),$$

for all $a, b, c \in A$. When $\alpha = id_A$, we recover the notion of Poisson algebras given in [6, 8, 9, 13].

Definition 16. Let $(A, \cdot, \alpha)$ be a Hom-algebra and let M be an A -module. A skew-symmetric A -multilinear map $\varphi : A \times A \times \cdots \times A \rightarrow M$ of degree k is a skew-symmetric first-order α -differential operator of degree k if

$$\begin{aligned}\varphi(a_1b, a_2, \dots, a_k) &= \alpha(a_1)\varphi(b, a_2, \dots, a_k) + \alpha(b)\varphi(a_1, a_2, \dots, a_k) \\ &\quad - \alpha(a_1)\alpha(b)\varphi(1_A, a_2, \dots, a_k),\end{aligned}$$

for all $b, a_i \in A$, $i \in \{1, \dots, k\}$.

We denote the A -module of all skew-symmetric first-order α -differential operators of degree k on A with coefficients in M by $Diff_\alpha^k(A, M)$ and we denote by $\mathfrak{L}_{sk}^k(A \oplus \Omega_\alpha(A), M)$ the A -module of all skew-symmetric A -multilinear maps of degree k on A with coefficients in M .

Consider the map $D_{A/\mathbb{K}}^{(k)} = D_{A/\mathbb{K}} \times D_{A/\mathbb{K}} \times \cdots \times D_{A/\mathbb{K}} : A^k \rightarrow (A \oplus \Omega_\alpha(A))^k$ such that

$$D_{A/\mathbb{K}}^{(k)}(a_1, a_2, \dots, a_k) = D_{A/\mathbb{K}}(a_1) \times D_{A/\mathbb{K}}(a_2) \times \cdots \times D_{A/\mathbb{K}}(a_k),$$

for all $a_1, a_2, \dots, a_k \in A$.

Theorem 3. *For any A -module M and for any skew-symmetric first-order α -differential operator $\varphi : A^k \rightarrow M$ of degree k , there exists a unique skew-symmetric A -multilinear map $\tilde{\varphi} : (A \oplus \Omega_\alpha(A))^k \rightarrow M$ of degree k such that*

$$\tilde{\varphi}(D_{A/\mathbb{K}}(a_1), \dots, D_{A/\mathbb{K}}(a_k)) = \varphi(a_1, \dots, a_k),$$

for all $a_1, a_2, \dots, a_k \in A$.

Proof. By definition of skew-symmetric first-order α -differential operator $\varphi : A^k \rightarrow M$ of degree k , the map $\varphi^i : A \rightarrow M$, such that

$$\varphi^i(a_i) = \varphi(a_1, a_2, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_k)$$

is a first-order α -differential operator, for all, $a_i \in A$, $i \in \{1, \dots, k\}$. According to Theorem 2, there exists a unique homomorphism $\tilde{\varphi}^i : A \oplus \Omega_\alpha(A) \rightarrow M$ of A -modules such that $\tilde{\varphi}^i \circ D_{A/\mathbb{K}} = \varphi^i$, that is,

$$\tilde{\varphi}^i[D_{A/\mathbb{K}}(a_i)] = \varphi^i(a_i) = \varphi(a_1, a_2, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_k)$$

We deduce the existence and uniqueness of the skew-symmetric A -multilinear map $\tilde{\varphi} : (A \oplus \Omega_\alpha(A))^k \rightarrow M$ of degree k such that

$$\tilde{\varphi}(D_{A/\mathbb{K}}(a_1), \dots, D_{A/\mathbb{K}}(a_k)) = \varphi(a_1, \dots, a_k)$$

for all $a_1, a_2, \dots, a_k \in A$. □

4.2 Canonical form of purely Hom-Jacobi algebra

If $(A, \cdot, \{-, -\}, \alpha)$ is a purely Hom-Jacobi algebra, then the map $\xi : A \rightarrow A$, $a \mapsto \{1_A, a\}$ is an α -derivation. Indeed, for all $a, b \in A$, $\xi(a \cdot b) = \{1_A, a \cdot b\}$. By the definition of a purely Hom-Jacobi algebra, for all $a, b \in A$, we have

$$\begin{aligned} \xi(a \cdot b) &= \alpha(a) \cdot \{1_A, b\} + \{1_A, a\} \cdot \alpha(b) - \alpha(b) \cdot \alpha(b) \cdot \{1_A, 1_A\} \\ &= \alpha(a) \cdot \xi(b) + \alpha(b) \cdot \xi(a). \end{aligned}$$

Theorem 4. *The following statements are equivalent:*

- (1) $(A, \cdot, \{-, -\}, \alpha)$ is a purely Hom-Jacobi algebra.
- (2) There exists a unique skew-symmetric 2-form

$$\omega_A : [A \oplus \Omega_\alpha(A)] \times [A \oplus \Omega_\alpha(A)] \rightarrow A$$

such that, for all $a, b \in A$,

$$\{a, b\} = \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(b)) \tag{13}$$

defines a Hom-Lie algebra structure on A .

(3) There exists a skew-symmetric 2-form

$$\pi : \Omega_\alpha(A) \times \Omega_\alpha(A) \longrightarrow A$$

and there exists an α -derivation ξ of A such that, for all $a, b \in A$,

$$\{a, b\} = \pi(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)) + \alpha(a)\xi(b) - \alpha(b)\xi(a) \quad (14)$$

defines a Hom-Lie algebra structure on A .

Proof. (1) $\Rightarrow$ (2) If $(A, \cdot, \{-, -\}, \alpha)$ is a purely Hom-Jacobi algebra, the bracket $\{-, -\}$ is a skew-symmetric first-order α -differential operator of degree 2. According to Theorem 3, there exists a unique $\omega_A \in \mathfrak{L}_{sk}^2(A \oplus \Omega_\alpha(A), A)$ such that $\{a, b\} = \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(b))$, for all $a, b \in A$.

(2) $\Rightarrow$ (3) If there exists a skew-symmetric 2-form ω_A such that, for all $a, b \in A$, $\{a, b\} = \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(b))$, then

$$\begin{aligned} \{a, b\} &= \omega_A(\alpha(a) + d_{A/\mathbb{K}}(a), \alpha(b) + d_{A/\mathbb{K}}(b)) \\ &= \alpha(a) \cdot \omega_A(1_A, d_{A/\mathbb{K}}(b)) + \alpha(b) \cdot \omega_A(d_{A/\mathbb{K}}(a), 1_A) \\ &\quad + \omega_A(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)). \end{aligned}$$

Since for all $a \in A$,

$$\omega_A(1_A, \alpha(a)) + \omega_A(1_A, d_{A/\mathbb{K}}(a)) = \omega_A(1_A, D_{A/\mathbb{K}}(a)) = \{1_A, a\} = ad(1_A)(a),$$

then there exists an α -derivation $\xi = ad(1_A)$ such that, for all $a \in A$,

$$\xi(a) = ad(1_A)(a) = \{1_A, a\} = \omega_A(1_A, d_{A/\mathbb{K}}(a)), \quad (15)$$

and there exists a skew-symmetric 2-form $\pi = \omega_A|_{\Omega_\alpha(A) \times \Omega_\alpha(A)} : \Omega_\alpha(A) \times \Omega_\alpha(A) \longrightarrow A$, such that, for all $a, b \in A$,

$$\begin{aligned} \{a, b\} &= \pi(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)) + \alpha(a) \cdot \omega_A(1_A, d_{A/\mathbb{K}}(b)) + \alpha(b) \cdot \omega_A(d_{A/\mathbb{K}}(a), 1_A) \\ &= \pi(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)) + \alpha(a)\xi(b) - \alpha(b)\xi(a). \end{aligned}$$

Here, π is the restriction of $\omega_A : [A \oplus \Omega_\alpha(A)] \times [A \oplus \Omega_\alpha(A)] \longrightarrow A$ to $\Omega_\alpha(A)$, that is $\pi(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)) = \omega_A(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b))$.

(3) $\Rightarrow$ (1) If the bracket $\{-, -\}$, such that, for all $a, b \in A$,

$$\{a, b\} = \pi(d_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)) + \alpha(a)\xi(b) - \alpha(b)\xi(a) = \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(b)),$$

defines a Hom-Lie algebra structure on A , then the bracket satisfies the Hom-Jacobi identity. For all $a, b, c \in A$, we have

$$\{a, b \cdot c\} = \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(b \cdot c)).$$

Since $D_{A/\mathbb{K}}$ is a first-order α -differential operator, for all $a, b, c \in A$,

$$\begin{aligned} \{a, b \cdot c\} &= \alpha(b) \cdot \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(c)) + \alpha(c) \cdot \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(c)) \\ &\quad - \alpha(b) \cdot \alpha(c) \cdot \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(1_A)) \\ &= \alpha(b) \cdot \{a, c\} + \{a, b\} \cdot \alpha(c) - \alpha(b) \cdot \alpha(c) \cdot \{a, 1_A\}, \end{aligned}$$

that is, $ad(a)$ is a first-order α -differential operator. Therefore, $(A, \cdot, \{-, -\}, \alpha)$ is a purely Hom-Jacobi algebra. $\square$

In this case, we say that ω_A or the pair (π, ξ) defines a purely Hom-Jacobi structure on A and the pair (A, ω_A) or the triple (A, π, ξ) is a purely Hom-Jacobi algebra. The pair (π, ξ) is called purely Hom-Jacobi pair. If $\xi = 0$, the pair (A, π) is called purely Hom-Poisson algebra. The skew-symmetric 2-form ω_A on $A \oplus \Omega_\alpha(A)$ is called canonical purely Hom-Jacobi 2-form of purely Hom-Jacobi algebra.

When ω_A is a purely Hom-Jacobi 2-form, the map $i_{1_A}\omega_A : A \oplus \Omega_\alpha(A) \longrightarrow A, x \longmapsto \omega_A(x, 1_A)$ is called 1-form of purely Hom-Jacobi algebra.

Proposition 9. *If $(A, \cdot, \{-, -\}, \alpha)$ is a purely Hom-Jacobi algebra, then for $a \in A$, the map $\Phi_a : A \longrightarrow A, b \longmapsto \{a, b\} - \alpha(b) \{a, 1_A\}$ is an α -derivation.*

Moreover, the map $X : A \longrightarrow \text{Der}_\alpha(A), a \longmapsto \Phi_a$ is a first-order α -differential operator.

Proof. For all $a, b, c \in A$,

$$\begin{aligned} \Phi_a(b \cdot c) - \alpha(b) \cdot \Phi_a(c) - \alpha(c) \cdot \Phi_a(b) &= \{a, b \cdot c\} + \alpha(b \cdot c) \{a, 1_A\} - \alpha(b) \cdot (\{a, c\} - \alpha(c) \{a, 1_A\}) \\ &\quad - \alpha(c) \cdot (\{a, b\} - \alpha(b) \{a, 1_A\}). \end{aligned}$$

By generalized Hom-Leibniz identity, we get

$$\Phi_a(b \cdot c) - \alpha(b) \cdot \Phi_a(c) - \alpha(c) \cdot \Phi_a(b) = 0.$$

For all $a, b \in A$, we have

$$X(a)(b) = \omega_A(D_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(b)). \quad (16)$$

Indeed, for all $a, b \in A$, $X(a)(b) = \Phi_a(b) = \{a, b\} - \alpha(b) \{a, 1_A\}$. By (13), we have

$$\begin{aligned} X(a)(b) &= \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(b)) - \alpha(b) \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(1_A)) \\ &= \omega_A(D_{A/\mathbb{K}}(a), D_{A/\mathbb{K}}(b) - \alpha(b)). \end{aligned}$$

For all $a, b, c \in A$, we have $X(a \cdot b)(c) = \omega_A(D_{A/\mathbb{K}}(a \cdot b), d_{A/\mathbb{K}}(c))$. Since $D_{A/\mathbb{K}}$ is a first-order α -differential operator, then

$$\begin{aligned} X(a \cdot b)(c) &= \alpha(b) \cdot \omega_A(D_{A/\mathbb{K}}(a), d_{A/\mathbb{K}}(c)) + \alpha(a) \cdot \omega_A(D_{A/\mathbb{K}}(b), d_{A/\mathbb{K}}(c)) \\ &\quad - \alpha(a) \cdot \alpha(b) \cdot \omega_A(D_{A/\mathbb{K}}(1_A), d_{A/\mathbb{K}}(c)). \end{aligned}$$

Thus, for all $a, b, c \in A$,

$$X(a \cdot b)(c) = (\alpha(b) \cdot X(a) + \alpha(a) \cdot X(b) - \alpha(a) \cdot \alpha(b) \cdot X(1_A))(c)$$

$\square$

Since the map $X : A \longrightarrow \text{Der}_\alpha(A)$, $a \longmapsto \Phi_a$ is a first-order α -differential operator, then according to Theorem 2, there exists a unique A -linear map $\tilde{X} : A \oplus \Omega_\alpha(A) \longrightarrow \text{Der}_\alpha(A)$ such that

$$\tilde{X} \circ D_{A/\mathbb{K}} = X. \quad (17)$$

Proposition 10. *If (A, ω_A) is a purely Hom-Jacobi algebra, then*

$$[\tilde{X}(x)](a) = \omega_A(x, d_{A/\mathbb{K}}(a)), \quad (18)$$

$$[\widetilde{ad}(x)](a) = \omega_A(x, D_{A/\mathbb{K}}(a)), \quad (19)$$

$$[\widetilde{\widetilde{ad}(x)}](y) = \omega_A(x, y), \quad (20)$$

for all $a, b \in A$ and $x, y \in A \oplus \Omega_\alpha(A)$.

Proof. Since the image of $D_{A/\mathbb{K}}$ generates the A -module $A \oplus \Omega_\alpha(A)$, for $x \in A \oplus \Omega_\alpha(A)$, $x = bD_{A/\mathbb{K}}(c)$ with $b, c \in A$. We have $[\tilde{X}(x)](a) = [b \cdot (\tilde{X} \circ D_{A/\mathbb{K}})(c)](a)$. By (17), $[\tilde{X}(x)](a) = [b \cdot X(c)](a)$, for all $a, b \in A$. By (16),

$$\begin{aligned} [\tilde{X}(x)](a) &= b \cdot \omega_A(D_{A/\mathbb{K}}(c), d_{A/\mathbb{K}}(a)) \\ &= \omega_A(x, d_{A/\mathbb{K}}(a)). \end{aligned}$$

For (19), for all $a \in A$ and for all $x \in A \oplus \Omega_\alpha(A)$, $x = bD_{A/\mathbb{K}}(c)$ with $b, c \in A$, we have

$$[\widetilde{ad}(x)](a) = [b \cdot (\widetilde{ad} \circ D_{A/\mathbb{K}})(c)](a) = b \cdot [ad(c)](a).$$

That is,

$$[\widetilde{ad}(x)](a) = b \cdot \omega_A(D_{A/\mathbb{K}}(c), D_{A/\mathbb{K}}(a)) = \omega_A(x, D_{A/\mathbb{K}}(a)).$$

For (20), by (18) and (19), for all $x, y \in A \oplus \Omega_\alpha(A)$, $x = a_1D_{A/\mathbb{K}}(b_1)$, $y = a_2D_{A/\mathbb{K}}(b_2)$ with $a_1, a_2, b_1, b_2 \in A$, we get $[\widetilde{\widetilde{ad}(x)}](y) = a_1a_2\omega_A(D_{A/\mathbb{K}}(b_1), D_{A/\mathbb{K}}(b_2))$, that is, $[\widetilde{\widetilde{ad}(x)}](y) = \omega_A(x, y)$. $\square$

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