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Invertibility of elements in the path algebra of a quiver

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Abstract. The current study elucidates the nature of right and left inverses of an element in the path algebra of a quiver. A general characterisation of such elements has been established. An explicit formula to calculate the inverse element has been formulated. It is observed that the left and right inverses of an element in the non-commutative path algebraic structure coincides. Furthermore, it is noted that the Jacobson radical of any finite dimensional path algebra can be easily found using this characterisation.

Keywords: Quiver, Path algebra, Unit element, Noncommutative algebra, Jacobson radical.
AMS Subject Classification 2010: 16U60, 16N20.

1 Introduction and preliminaries

Non-commutative algebra is a branch in mathematics that studies the algebraic structures with a non-commutative multiplication operator. The first encounter with such a structure typically occurs with matrices. One such important non-commutative algebra is the path algebra of a quiver. Quivers [1] are directed graphs with no restriction on the number of vertices, arrows and loops. The term quiver was used by Peter Gabriel for his study in representation theory. Quiver and their representations have applications in certain problems of linear algebra such as simultaneous diagonalisation of two matrices of the same size. The concept of invertible bases and invertible algebras is an important area of study among researchers. However, only a few papers have gathered information in this regard to motivate further investigation. Viji and Chakravarti [5] discussed the invertibility condition for an element in the path algebra. Their study was restricted to the path algebra of a unique path quiver. The result was extended to the path algebra of acyclic quivers, keeping it limited to the finite dimensional cases [3].

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Jacobson radical of a ring was first studied by Nathan Jacobson [2]. The Jacobson radical plays an important role in many theoretical results of rings and modules, such as Nakayama's lemma. It could be interpreted as a collective of "bad" elements, bad in the sense that these acts as annihilators on all simple left and right modules. This paper provides a refined and complete characterisation for the invertibility of an element even in infinite dimensional cases, that is, the path algebra of any quiver (including those with infinite vertices and loops). An element in the path algebra of a quiver is right invertible iff the co-efficient of every stationary path is nonzero. If an element is right invertible, then it is left invertible. In that case, right and left inverses are the same. The precise structure of invertible elements provide an alternate method to demonstrate the well-known conclusion that the Jacobson radical of any finite dimensional algebra is its arrow ideal.

2 Main Results

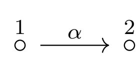
The structure of unit elements in the path algebra of an acyclic quiver influenced the authors to consider the case of a general quiver. With a similar approach, the authors were able to find the structure of right and left invertible elements in the path algebra of any quiver. In this section, an explicit formula for these elements has been developed. The proposed explicit formula elucidates that the right and left inverses of an element coincide even though path algebra has a non-commutative structure. The major objectives of this work include

- To discuss the nature of right and left inverse of an element in the path algebra of a quiver.
- To establish a general characterization for the right and left inverse elements.
- To formulate an explicit formula to calculate the inverse element.

Definition 1. *A quiver [1] $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two sets: Q_0 (whose elements are called points, or vertices) and Q_1 (whose elements are called arrows) and two maps $s,t : Q_1 \to Q_0$ which associates to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$, respectively. Hereafter, the notation $Q = (Q_0, Q_1)$ or simply Q is used to denote a quiver.*

Example 1.

1.



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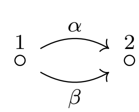

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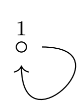


Definition 2. *A path of length l in Q is an expression of the form $(x|\alpha_1, \alpha_2, ..., \alpha_l|y)$ where $x, y \in Q_0$ and $\alpha_i \in Q_1$ such that $s(\alpha_1) = x, t(\alpha_i) = s(\alpha_i + 1)$ for $1 \leq i < l$ and $t(\alpha_l) = y$. If $l = 0$, we impose the condition that $x = y$ (Note that $x = y$ need not imply $l = 0$).*

A stationary path is a path of length 0. It is of the form $(x|x)$ for $x \in Q_0$ and is denoted by ϵ_x. A path is a *cycle* if its source and target coincides. A quiver Q is said to be *acyclic* if it contains no cycles.

Definition 3 ([5])**.** *Let Q be a quiver, and let P be the set of all paths in Q. A path algebra $\overline{KQ}$ of Q is defined as the set of all linear combinations of paths in P (not necessarily finite) with scalars from the field K . That is,*

$$\left\{ \sum_{\alpha \in P} c_\alpha \alpha \mid c_\alpha \in K, \alpha \in P \right\}$$

where c_α corresponds to the coefficient of path α in P. Addition and scalar multiplication is defined component-wise. The product of two paths is defined as the concatenation if the target of the first path is same as the source of the second path and is zero, otherwise. That is

$$(a \mid \alpha_1, \alpha_2, ..., \alpha_l \mid b).(c \mid \beta_1, \beta_2, ..., \beta_m \mid d) = \delta_{bc}(a \mid \alpha_1, ..., \alpha_l, \beta_1, ...\beta_m \mid d)$$

The product of two arbitrary elements of $\overline{KQ}$ can be defined by assuming distributivity of multiplication of paths over arbitrary summation.

$$\left(\sum_{\alpha \in P} c_\alpha \alpha \right) \left(\sum_{\beta \in P} d_\beta \beta \right) = \sum_{\alpha, \beta \in P} c_\alpha d_\beta \alpha \beta \tag{1}$$

This is well defined since $\alpha\beta = 0$ if $t(\alpha) \neq s(\beta)$. For a finite acyclic quiver Q, the set of all its paths will serve as a basis for $\overline{KQ}$.

Theorem 1. *Let $Q = (Q_0, Q_1)$ be any quiver. Then an element $a \in \overline{KQ}$ is right invertible if and only if the co-efficient of stationary path, ϵ_x is nonzero for all $x \in Q_0$.*

Proof. Let $a = \sum_{\alpha \in P} a_\alpha \alpha \in \overline{KQ}$ be right invertible. Then there exists an element $b = \sum_{\beta \in P} b_\beta \beta \in \overline{KQ}$ such that

$$ab = \sum_{x \in Q_0} \epsilon_x \tag{2}$$

$$(\sum_{\alpha \in P} a_\alpha \alpha)(\sum_{\beta \in P} b_\beta \beta) = \sum_{x \in Q_0} \epsilon_x \tag{3}$$

$$\sum_{\alpha, \beta \in P} (a_\alpha b_\beta) \alpha \beta = \sum_{x \in Q_0} \epsilon_x \tag{4}$$

For each $x \in Q_0$, $\alpha\beta = \epsilon_x$ iff $\alpha = \epsilon_x$ and $\beta = \epsilon_x$.
Coefficient of ϵ_x in LHS of (3) is $a_{\epsilon_x} b_{\epsilon_x}$.
Equating coefficients of ϵ_x on both sides of (3),

$$a_{\epsilon_x} b_{\epsilon_x} = 1 \implies a_{\epsilon_x} \neq 0 \tag{5}$$

Thus the coefficient of stationary paths in the element a is non zero.
Conversely, let $a = \sum\limits_{\alpha \in P} a_\alpha \alpha \in \overline{KQ}$ such that for all $x \in Q_0, a_{\epsilon_x} \neq 0$. Define $b = \sum\limits_{\alpha \in P} b_\alpha \alpha \in \overline{KQ}$ by defining b_α inductively on the length of α as follows :
Let $b_{\epsilon_x} = a_{\epsilon_x}^{-1}$ for all $x \in Q_0$. This defines b_α on all paths α of length 0. Assume by induction that b_α has been defined whenever the length of α is less than l. Now take an α of length $l \geq 1$, say $\alpha = (x|\alpha_1, \alpha_2, ..., \alpha_l|y)$. Define

$$b_\alpha = \frac{-1}{a_{\epsilon_x}} \sum_{\substack{\alpha' \beta\prime = \alpha, \\ \alpha\prime \neq \epsilon_x}} a_{\alpha\prime} b_{\beta\prime} \tag{6}$$

where $x = s(\alpha)$.
Coefficient of ϵ_x in ab is

$$a_{\epsilon_x} b_{\epsilon_x} = a_{\epsilon_x} . \frac{1}{a_{\epsilon_x}} = 1 \, \forall x \in Q_0 \tag{7}$$

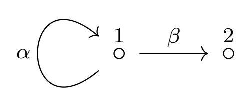
For $l \geq 1$, coefficient of α in ab is

$$a_{\epsilon_x} b_\alpha + \sum_{\substack{\alpha' \beta\prime = \alpha, \\ \alpha\prime \neq \epsilon_x}} a_{\alpha\prime} b_{\beta\prime} = 0. \tag{8}$$

Thus $ab = \sum\limits_{x \in Q_0} \epsilon_x$ which implies b is the right inverse of a.

□

Example 2. Consider the quiver Q with two vertices consisting of one loop α at 1 and arrow β from 1 to 2.



Let $a = 2\epsilon_1 + 3\epsilon_2 + \alpha + \alpha\beta$. By the above theorem, it is known that a is a right invertible element in the path algebra. Following the algorithm in proof, one can get

$$b_{\epsilon_1} = \frac{1}{a_{\epsilon_1}} = \frac{1}{2}, b_{\epsilon_2} = \frac{1}{a_{\epsilon_2}} = \frac{1}{3}$$
$$b_\alpha = \frac{-1}{a_{\epsilon_1}}(a_\alpha b_{\epsilon_1}) = \frac{-1}{2}(1.\frac{1}{2}) = \frac{-1}{4}$$

$$\begin{aligned}
b_\beta &= \frac{1}{a_{\epsilon_1}}(a_\beta b_{\epsilon_2}) = \frac{-1}{2}(0) = 0\\
b_{\alpha^2} &= \frac{-1}{a_{\epsilon_1}}(a_{\alpha^2} b_{\epsilon_1} + a_\alpha b_\alpha) = \frac{-1}{2}(0 + 1.\frac{-1}{4}) = \frac{1}{8}\\
b_{\alpha\beta} &= \frac{-1}{a_{\epsilon_1}}(a_{\alpha\beta} b_{\epsilon_2} + a_\alpha b_\beta) = \frac{-1}{2}(1.\frac{1}{3} + 1.0) = \frac{-1}{6}\\
b_{\alpha^3} &= \frac{-1}{a_{\epsilon_1}}(a_{\alpha^3} b_{\epsilon_1} + a_{\alpha^2} b_\alpha + a_\alpha b_{\alpha^2}) = \frac{-1}{2}(0 + 0 + 1.\frac{1}{8}) = \frac{-1}{16}\\
b_{\alpha^2\beta} &= \frac{-1}{a_{\epsilon_1}}(a_{\alpha^2\beta} b_{\epsilon_2} + a_{\alpha^2} b_\beta + a_\alpha b_{\alpha\beta}) = \frac{-1}{2}(0 + 0 + 1.\frac{-1}{6}) = \frac{1}{12}
\end{aligned}$$

In general,

$$\begin{cases} b_\alpha = \frac{-1}{4} \\ b_{\alpha^n} = \frac{(-1)^n}{2}|b_{\alpha^{n-1}}| \end{cases}$$

$$\begin{cases} b_{\alpha\beta} = \frac{-1}{6} \\ b_{\alpha^n\beta} = \frac{(-1)^n}{2}|b_{\alpha^{n-1}\beta}| \end{cases}$$

which gives

$$b = \frac{\epsilon_1}{2} + \frac{\epsilon_2}{3} - \frac{\alpha}{4} + \frac{\alpha^2}{8} - \frac{\alpha^3}{16} + \cdots - \frac{\alpha\beta}{6} + \frac{\alpha^2\beta}{12} - \frac{\alpha^3\beta}{24} + \cdots$$

It is easy to verify that $ab = \sum_{x \in Q_0} \epsilon_x$, i.e., b is the right inverse of a.

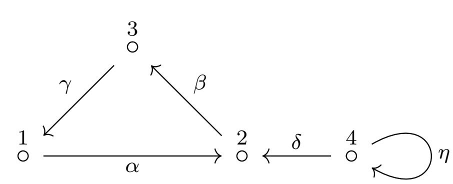
Corollary 1. *If $a \in \overline{KQ}$ is right invertible, then it is left invertible.*

Proof. Let $a \in \overline{KQ}$ be right invertible. Then by theorem 1, coefficient of every ϵ_x is nonzero. So left inverse c of a can be defined inductively in a similar form as follows

$$\begin{aligned}
c_{\epsilon_x} &= \frac{1}{a_{\epsilon_x}}\\
c_\alpha &= \frac{-1}{a_{\epsilon_y}} \sum_{\substack{\alpha'\beta'=\alpha,\\ \beta'\neq\epsilon_y}} c_{\alpha'} a_{\beta'}
\end{aligned}$$

It can be easily verified that $ca = \sum_{x \in Q_0} \epsilon_x$. □

Example 3. Consider the following quiver which has a loop η and a cycle $\alpha\beta\gamma$.



Let us find the right inverse of an element $a = \epsilon_1 + 2\epsilon_2 + \epsilon_3 + \frac{1}{2}\epsilon_4 + 2\alpha + \alpha\beta\gamma - \eta\delta$. Following the algorithm in the proof,

$$\begin{aligned}
b_{\epsilon_1} &= \frac{1}{a_{\epsilon_1}} = 1 \\
b_{\epsilon_2} &= \frac{1}{a_{\epsilon_2}} = \frac{1}{2} \\
b_{\epsilon_3} &= \frac{1}{a_{\epsilon_3}} = 1 \\
b_{\epsilon_4} &= \frac{1}{a_{\epsilon_4}} = 2 \\
b_{\alpha} &= \frac{-1}{a_{\epsilon_1}}(a_\alpha b_{\epsilon_2}) = -1(2.\frac{1}{2}) = -1 \\
b_{\beta} &= \frac{-1}{a_{\epsilon_2}}(a_\beta b_{\epsilon_3}) = 0 \\
b_\gamma &= b_\delta = b_\eta = 0 \\
b_{\eta\delta} &= \frac{-1}{a_{\epsilon_4}}(a_{\eta\delta} b_{\epsilon_2} + a_\eta b_\delta) = -2(-1.\frac{1}{2} + 0) = 1
\end{aligned}$$

The coefficient of all other paths of length 2 is 0. The paths with nonzero coeeficients are calculated below.

$$\begin{aligned}
b_{\alpha\beta\gamma} &= \frac{-1}{a_{\epsilon_1}}(a_{\alpha\beta\gamma} b_{\epsilon_1} + a_{\alpha\beta} b_\gamma + a_\alpha b_{\beta\gamma}) = -1(1.1 + 0 + 2.0) = -1 \\
b_{\alpha\beta\gamma\alpha} &= \frac{-1}{a_{\epsilon_1}}(a_{\alpha\beta\gamma\alpha} b_{\epsilon_2} + a_{\alpha\beta\gamma} b_\alpha + a_{\alpha\beta} b_{\gamma\alpha} + a_\alpha b_{\beta\gamma\alpha}) \\
&= -1(0 + 1. - 1 + 0 + 0) = 1 \\
b_{\alpha\beta\gamma\alpha\beta\gamma} &= \frac{-1}{a_{\epsilon_1}}(a_{\alpha\beta\gamma\alpha\beta\gamma} b_{\epsilon_1} + a_{\alpha\beta\gamma\alpha\beta b_\gamma} + a_{\alpha\beta\gamma\alpha} b_{\beta\gamma} + a_{\alpha\beta\gamma} b_{\alpha\beta\gamma} + a_{\alpha\beta} b_{\gamma\alpha\beta\gamma} + a_\alpha b_{\beta\gamma\alpha\beta\gamma}) \\
&= -1(0 + 0 + 0 + 1. - 1 + 0 + 0) = 1
\end{aligned}$$

By similar calculations, it can be found that the path $(\alpha\beta\gamma)^n$ and $(\alpha\beta\gamma)^n\alpha$ will have nonzero coefficient for all n. It could be summarised as follows:

$$\begin{cases} b_{(\alpha\beta\gamma)^n} = (-1)^n \\ b_{(\alpha\beta\gamma)^n\alpha} = (-1)^{n-1} \end{cases}$$

Thus the right inverse,

$$b = \epsilon_1 + \frac{\epsilon_2}{2} + \epsilon_3 + 2\epsilon_4 - \alpha - \alpha\beta\gamma + (\alpha\beta\gamma)^2 - \cdots + \alpha\beta\gamma\alpha - (\alpha\beta\gamma)^2\alpha + \cdots + \eta\delta$$

and

$$ab = (\epsilon_1 - \alpha - \alpha\beta\gamma + (\alpha\beta\gamma)^2 + \cdots + \alpha\beta\gamma\alpha - (\alpha\beta\gamma)^2\alpha + \cdots)(\epsilon_2) + (\epsilon_3) + (\epsilon_4 + \frac{\eta\delta}{2}) + (\alpha)$$

$$
\begin{aligned}
&+ (\alpha\beta\gamma - (\alpha\beta\gamma)^2 + \ldots - \alpha\beta\gamma\alpha + (\alpha\beta\gamma)^2\alpha - (\alpha\beta\gamma)^3\alpha + \ldots) - \frac{\eta\delta}{2} \\
&= \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4.
\end{aligned}
$$

Now, let's try to find the left inverse of the same element a and verify whether $b = c$. To find the left inverse , the following formula is used

$$
\begin{aligned}
c_{\epsilon_x} &= \frac{1}{a_{\epsilon_x}} \\
c_\alpha &= \frac{-1}{a_{\epsilon_y}} \sum_{\substack{\alpha'\beta\prime=\alpha, \\ \beta\prime\neq\epsilon_y}} c_{\alpha\prime} a_{\beta\prime}
\end{aligned}
$$

$$
\begin{aligned}
c_{\epsilon_1} &= \frac{1}{a_{\epsilon_1}} = 1, \quad c_{\epsilon_2} = \frac{1}{a_{\epsilon_2}} = \frac{1}{2} \\
c_{\epsilon_3} &= \frac{1}{a_{\epsilon_3}} = 1, \quad c_{\epsilon_4} = \frac{1}{a_{\epsilon_4}} = 2 \\
c_\alpha &= \frac{-1}{a_{\epsilon_2}}(c_{\epsilon_1} a_\alpha) = \frac{-1}{2}(1.2) = -1 \\
c_\beta &= \frac{-1}{a_{\epsilon_3}}(c_{\epsilon_2} a_\beta) = 0 \\
c_\gamma &= c_\delta = c_\eta = 0 \\
c_{\eta\delta} &= \frac{-1}{a_{\epsilon_2}}(c_{\epsilon_4} a_{\eta\delta} + c_\eta a_\delta) = \frac{-1}{2}(2. -1 + 0) = 1 \\
c_{\alpha\beta\gamma} &= \frac{-1}{a_{\epsilon_1}}(c_{\epsilon_1} a_{\alpha\beta\gamma} + c_\alpha a_{\beta\gamma} + c_{\alpha\beta} a_\gamma) = -1(1.1 + -1.0 + 0) = -1 \\
c_{\alpha\beta\gamma\alpha} &= \frac{-1}{a_{\epsilon_2}}(c_{\epsilon_1} a_{\alpha\beta\gamma\alpha} + c_{\alpha\beta\gamma} a_\alpha + c_{\alpha\beta} a_{\gamma\alpha} + c_\alpha a_{\beta\gamma\alpha}) \\
&= \frac{-1}{2}(0 + -1.2 + +0 + 0) = 1 \\
c_{\alpha\beta\gamma\alpha\beta\gamma} &= \frac{-1}{a_{\epsilon_1}}(c_{\epsilon_1} a_{\alpha\beta\gamma\alpha\beta\gamma} + c_{\alpha\beta\gamma\alpha\beta} a_\gamma + c_{\alpha\beta\gamma\alpha} a_{\beta\gamma} + c_{\alpha\beta\gamma} a_{\alpha\beta\gamma} + c_{\alpha\beta} a_{\gamma\alpha\beta\gamma} + c_\alpha a_{\beta\gamma\alpha\beta\gamma}) \\
&= -1(0 + 0 + 0 + -1.1 + - + 0) = 1
\end{aligned}
$$

In general,

$$
\begin{cases}
c_{(\alpha\beta\gamma)^n} = (-1)^n \\
c_{(\alpha\beta\gamma)^n\alpha} = (-1)^{n-1}
\end{cases}
$$

Thus the left inverse is given by

$$
\begin{aligned}
c = \epsilon_1 + \frac{\epsilon_2}{2} + \epsilon_3 + 2\epsilon_4 - \alpha - \alpha\beta\gamma + (\alpha\beta\gamma)^2 - \ldots \\
+ \alpha\beta\gamma\alpha - (\alpha\beta\gamma)^2\alpha + \ldots + \eta\delta
\end{aligned}
$$

This means $c = b$. That is, an element is right invertible if and only if it is left invertible. It is verified that two inverses are equal to each other. This result is proved in Theorem 2.

Remark 1. *Introducing an alternate expression of b_α whose credibility is used in proving next theorem. Let $x = s(\alpha)$, $z = t(\alpha) = s(\beta)$ and $y = t(\beta)$. By definition,*

$$b_\alpha = \frac{-1}{a_{\epsilon_{s(\alpha)}}}(a_\alpha b_{\epsilon_{t(\alpha)}}) = \frac{-a_\alpha}{a_{\epsilon_{s(\alpha)}}a_{\epsilon_{t(\alpha)}}} = \frac{-a_\alpha}{a_{\epsilon_x}a_{\epsilon_z}}$$
$$b_\beta = \frac{-1}{a_{\epsilon_{s(\beta)}}}(a_\beta b_{\epsilon_{t(\beta)}}) = \frac{-a_\beta}{a_{\epsilon_{s(\beta)}}a_{\epsilon_{t(\beta)}}} = \frac{-a_\beta}{a_{\epsilon_z}a_{\epsilon_y}}$$
$$b_{\alpha\beta} = \frac{-1}{a_{\epsilon_x}}(-a_{\alpha\beta}b_{\epsilon_y} + a_\alpha b_\beta) = \frac{a_{\alpha\beta}}{a_{\epsilon_x}a_{\epsilon_y}} + \frac{a_\alpha a_\beta}{a_{\epsilon_x}a_{\epsilon_z}a_{\epsilon_y}} = \frac{-a_{\alpha\beta}}{a_{\epsilon_x}a_{\epsilon_y}} + \frac{b_\alpha b_\beta}{b_{\epsilon_z}}$$

Similarly, if there exists an arrow γ such that $\alpha\beta\gamma$ exists, then

$$b_{\alpha\beta\gamma} = \frac{-a_{\alpha\beta\gamma}}{a_{\epsilon_{s(\alpha\beta\gamma)}}a_{\epsilon_{t(\alpha\beta\gamma)}}} + \frac{b_\alpha b_{\beta\gamma}}{b_{\epsilon_{t(\alpha)}}} + \frac{b_{\alpha\beta}b_\gamma}{b_{\epsilon_{t(\alpha\beta)}}} - \frac{b_\alpha b_\beta b_\gamma}{b_{\epsilon_{t(\alpha)}}b_{\epsilon_{t(\beta)}}}$$

It should be noted that, in this case, α, β, γ can be loops and also need not be distinct. Generalizing, if $\alpha = \alpha_1\alpha_2\alpha_3\cdots\alpha_k$ is the decomposition of α into product of edges, that is, $l(\alpha_i) = 1$ for each i where α_i need not be distinct;

$$\begin{aligned} b_\alpha =& \frac{-a_\alpha}{a_{\epsilon_{s(\alpha)}}a_{\epsilon_{t(\alpha)}}} + \frac{b_{\alpha_1}b_{\alpha_2\cdots\alpha_k}}{b_{\epsilon_{t(\alpha_1)}}} + \frac{b_{\alpha_1\cdots\alpha_{k-1}}b_{\alpha_k}}{b_{\epsilon_{t(\alpha_1\cdots\alpha_{k-1})}}} - \frac{b_{\alpha_1}b_{\alpha_2}b_{\alpha_3\cdots\alpha_k}}{b_{\epsilon_{t(\alpha_1)}}b_{\epsilon_{t(\alpha_2)}}} - \frac{b_{\alpha_1}b_{\alpha_2\cdots\alpha_{k-1}}b_{\alpha_k}}{b_{\epsilon_{t(\alpha_1)}}b_{\epsilon_{t(\alpha_2\cdots\alpha_{k-1})}}} \\ &- \frac{b_{\alpha_1\alpha_2\cdots\alpha_{k-2}}b_{\alpha_{k-1}}b_{\alpha_k}}{b_{\epsilon_{t(\alpha_1\alpha_2\cdots\alpha_{k-2})}}b_{\epsilon_{t(\alpha_{k-1})}}} + \cdots + (-1)^k\frac{b_{\alpha_1}b_{\alpha_2}\cdots b_{\alpha_k}}{b_{\epsilon_{t(\alpha_1)}}b_{\epsilon_{t(\alpha_2)}}\cdots b_{\epsilon_{t(\alpha_{k-1})}}}. \end{aligned}$$

Similarly, c_α can be written as,

$$\begin{aligned} c_\alpha =& \frac{-a_\alpha}{a_{\epsilon_{s(\alpha)}}a_{\epsilon_{t(\alpha)}}} + \frac{c_{\alpha_1}c_{\alpha_2\cdots\alpha_k}}{c_{\epsilon_{t(\alpha_1)}}} + \frac{c_{\alpha_1\cdots\alpha_{k-1}}c_{\alpha_k}}{c_{\epsilon_{t(\alpha_1\cdots\alpha_{k-1})}}} - \frac{c_{\alpha_1}c_{\alpha_2}c_{\alpha_3\cdots\alpha_k}}{c_{\epsilon_{t(\alpha_1)}}c_{\epsilon_{t(\alpha_2)}}} - \frac{c_{\alpha_1}c_{\alpha_2\cdots\alpha_{k-1}}c_{\alpha_k}}{c_{\epsilon_{t(\alpha_1)}}c_{\epsilon_{t(\alpha_2\cdots\alpha_{k-1})}}} \\ &- \frac{c_{\alpha_1\alpha_2\cdots\alpha_{k-2}}c_{\alpha_{k-1}}c_{\alpha_k}}{c_{\epsilon_{t(\alpha_1\alpha_2\cdots\alpha_{k-2})}}c_{\epsilon_{t(\alpha_{k-1})}}} + \cdots + (-1)^k\frac{c_{\alpha_1}c_{\alpha_2}\cdots c_{\alpha_k}}{c_{\epsilon_{t(\alpha_1)}}c_{\epsilon_{t(\alpha_2)}}\cdots c_{\epsilon_{t(\alpha_{k-1})}}}. \end{aligned}$$

The following theorem proves that the right and left inverse are identical in the noncommutative path algebraic structure.

Theorem 2. *Let Q be any quiver and $a = \sum_{\alpha\in P} a_\alpha\alpha$ be a unit element in KQ. Then the right inverse and left inverse of a are equal.*

Proof. The proof uses mathematical induction on the length of α to show that $b_\alpha = c_\alpha$ for all α which will imply $b = c$. Let $l(\alpha) = 1$.

$$b_\alpha = \frac{-a_\alpha}{a_{\epsilon_{s(\alpha)}}a_{\epsilon_{t(\alpha)}}} = c_\alpha$$

Assume that the result is true for all α with length less than k. Now, let $l(\alpha) = k$ and $\alpha = \alpha_1\alpha_2\cdots\alpha_k$ be the decomposition of α into product of arrows, that is, $l(\alpha_i) = 1$ for all i. Since each coefficient b_α from the above remark corresponds to a path of length less than k, they are equal to corresponding coefficients of c by induction hypothesis, that is, $b_\alpha = c_\alpha$ for all α. This proves $b = c$. □

Lemma 1. *1. Theorem 2 also establishes the fact that an element in the path algebra of a quiver is left or right invertible if and only if it is invertible.*

2. The sum of two invertible elements in $\overline{KQ}$ may not be invertible.

3. The product of any two invertible elements in $\overline{KQ}$ is always invertible.

3 Units and the Jacobson radical

Jacobson radical is a useful tool in studying the internal structure of a ring. The intersection of all maximal right ideals of a $K - algebra\ A$ is called the Jacobson radical, denoted by $J(A)$. There are a number of characterizations of $J(A)$ involving units. The most basic, and easiest to understand and use, is the following:

Proposition 1 ([4]). *Let A be a finite dimensional algebra. The following statements are equivalent:*

1. $j \in J(A)$.

2. For all $a \in A$, $1 + aj$ has a left inverse (i.e. there is an $s \in A$ such that $s(1 + aj) = 1$).

3. For all $a \in A$, $1 + aj$ is a unit. □

Using the structure of unit elements in the path algebra of finite acyclic quiver, one can easily find the Jacobson radical of the same which apprises about the internal structure as discussed in the introduction of this paper. Using proposition 1, an element j is in $J(KQ)$ iff $1 + aj$ has a left inverse for all $a \in KQ$. Now, $1 + aj$ has a left inverse iff coefficient of each stationary path ϵ_x in $1 + aj$ is non zero. This happens only when $a\epsilon_x \neq -1$ for all a which is the case when $\epsilon_x = 0$. *Hence, Jacobson radical of the path algebra of any finite acyclic quiver consists of those elements for which $\epsilon_x = 0$ for all $x \in Q_0$.*

Acknowledgments

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