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Novel Legendre-Jaiswal functions for solving time-space fractional partial differential equations

Santoshi Tarei[†], **Ankur Kanaujiya**[†*]

[†] *Department of Mathematics, National Institute of Technology Rourkela, Rourkela-769008, India*

Email(s): 522ma1001@nitrkl.ac.in, ankurkanaujiya09@gmail.com

Abstract. This paper examines a new fractional function based on Legendre and Jaiswal polynomials to solve linear and nonlinear time-space fractional partial differential equations of linear and nonlinear class. The Caputo sense is applied while using the fractional derivative. These problems can be solved using the collocation method, operational, and pseudo-operational matrices of integer and fractional-order integration. Using operational matrices, pseudo-operational matrices, and the collocation method, the problem is transformed into a system of algebraic equations. An upper bound on the error of the fractional-order integral operational matrix is computed. Furthermore, a detailed stability and convergence analysis of the collocation scheme presented to validate the robustness of the numerical approach. The applicability and effectiveness of the approach are demonstrated through several benchmark examples, including linear and non-linear fractional convection-diffusion, convection-diffusion-reaction, and nonlinear Fisher's equation. The numerical results confirm that the proposed method is stable, rapidly convergent, and highly accurate, outperforming several existing techniques in both efficiency and precision.

Keywords: Error analysis, fractional partial differential equation, Legendre polynomial, Jaiswal polynomial.
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1 Introduction

Fractional-order differential equations are highly useful tools for modeling a variety of phenomena in physics, engineering, mathematical biology, fluid mechanics, finance, electrochemistry, and other science; see [15, 20, 27] and the references therein. The existence and uniqueness of the solutions to the fractional-order differential equations have been investigated in [15]. The fact that the majority of differential equations with fractional orders do not have closed-form solutions must be noted. Therefore,

*Corresponding author
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suggesting new methods for finding these equation's numerical solutions is crucial. The development of numerical techniques for solving fractional-order differential equations has attracted much interest due to the growing applications. The solution of fractional-order differential equations can be achieved through a variety of numerical techniques, including operational method [18], the variational iteration method [10], finite difference method [26], homotopy perturbation method [30] and Sinc-Legendre collocation method [24].

Different classes of fractional partial differential equations (FPDEs) can be solved analytically and numerically are as follows: In [29], Uddin and Haq introduced radial basis functions approximation method for time fractional advection-diffusion equation. Bu et al. [2] introduced a space-time finite element method for the multi-term time-space fractional diffusion equation. Hamou et al. [6] solved the reaction-diffusion problem by approximating the space derivative by the finite element method and the time-fractional derivative by Euler's method. Gasmi et al. [5] solved the $(1+1)$ telegraph equation with space-and time-fractional derivatives using the generalized Kudryashov method. Hammouch and Mekkaoui [8] implemented a Functional Variable Method to solve the time-space fractional-order generalized Zakharov equation. Uddin et al. [28] derived an analytical solutions for nonlinear directional couplers with metamaterials by incorporating spatial-temporal fractional beta-derivative evolution. Rashid et al. [22] proposed a new fractional operator based on a weight function for Atangana–Baleanu (A-B) fractional operators. Jafari et al. [11] solved the time-fractional diffusion equation using the Hosoya polynomial. Jafari et al. [12] solved multi-order time-fractional differential equations using operational matrices based on the Hosoya polynomial. Hamou et al. [7] solved fractional nonlinear reaction-diffusion problems with periodic conditions. Mekkaoui and Hammouch [19] used the Fractional Iteration Method to approximate analytical solutions to the Bagley-Torvik equation. The behavior of particular dispersive waves in a novel $(3+1)$-dimensional Hirota bilinear (3D-HB) equation was investigated by Hosseini [9]. Zhou and Xu [31] solved the time-fractional convection-diffusion equations with variable coefficient by Chebyshev wavelets collocation method. Dehestani et al. [4] solved time-space FPDEs (TSFPDEs) with variable coefficients by using Legendre-Lagueree functions. Chen et al. [3] utilized Haar wavelet for solving a class of space-time fractional convection-diffusion equations with variable coefficients. Bhawry et al. [1] solved the time-fractional diffusion equation by spectral tau algorithm based on the Jacobi operational matrix. Sarvestani et al. [25] used the Galerkin method based on the second kind of Chebyshev wavelets for solving the multi-term time-fractional diffusion-wave equation. Rahimkhani et al. [21] used Bernoulli wavelet for solving fractional delay differential equations. Researchers widely applied the TSFPDE in the following scientific domains:

- **Anomalous diffusion in porous media:** In geophysics and petroleum engineering, the transport of fluids through heterogeneous or fractured porous media often displays non-Fickian behavior characterized by long-tailed waiting times and spatial jumps. The TSFPDEs effectively capture this behavior by modeling sub-diffusion through time-fractional derivatives and Lévy flight phenomena through space-fractional operators.

- **Heat conduction with memory:** Traditional heat equations fail to accurately model thermal processes in materials with memory, such as polymers, biological tissues. The TSFPDEs incorporate time-fractional derivatives to account for the hereditary thermal response and space-fractional terms to describe spatial heterogeneity. Such models are essential in biomedical applications, including hyperthermia cancer treatment, where precise thermal predictions are critical.

- **Reaction-Diffusion systems in biology and chemistry:** The TSFPDEs are employed to describe complex reaction-diffusion systems that exhibit anomalous diffusion and non-instantaneous reaction kinetics. In chemical systems such as the Belousov-Zhabotinsky (BZ) reaction, and biological systems involving morphogenesis, fractional models help explain the emergence of spatial-temporal patterns such as animal coat markings.

This study introduces novel fractional-order Legendre-Jaiswal functions (FOLJFs) combining fractional order Legendre functions (FOLFs) and fractional-order Jaiswal functions (FOJFs) to solve linear and nonlinear TSFPDEs, with the Caputo derivative for enhanced memory representation and convergence. Utilizing operational and pseudo-operational matrices within a collocation framework, the method simplifies problems into algebraic equations while ensuring high accuracy and reliability with derived error bounds.
The originality of this work lies in the following contributions:

- Fractional Legendre-Jaiswal function development: A new function designed to overcome limitations in existing fractional approaches.

- Rigorous validation: Numerical results and comparisons with existing techniques confirm this approach's efficacy and broad applicability, making it a significant advancement in the field.

Advantages of Legendre-Jaiswal function: The combination of Legendre and Jaiswal polynomials has been deliberately chosen to exploit the strengths of both families. Legendre polynomials provide orthogonality and numerical stability, ensuring well-conditioned system matrices and spectral convergence. Jaiswal polynomials, on the other hand, offer adjustable parameters that enhance flexibility and allow the basis to capture non-smooth or fractional-order behaviors more effectively. The hybrid Legendre–Jaiswal basis thus achieves superior approximation accuracy and faster convergence compared to using either basis alone or other classical functions such as Chebyshev or Laguerre polynomials.

The structure of this article is as follows. Some basic definitions and important properties of Riemann-Liouville fractional integral (RLFI) and the Caputo's fractional derivative (CFD) can be seen in the beginning part of the article in Section 2. The FOLJFs and function approximations are introduced in Section 3. Section 4 discusses the operational matrices (integral pseudo-operational matrices (IP-OM)) with fractional and integer orders for FOLJFs. For solving both linear and non-linear TSFPDEs an algorithm is presented in Section 5. Section 6 provides an overview of stability and convergence analysis. Section 7 presents figures and numerical results that show how accurate the suggested numerical approach is. The conclusion and references are given in Section 8.

2 Preliminaries and basic definitions

In this section, basic definitions regarding RLFI and the CFD of fractional calculus are explained.

Definition 1. *The definition of the RLFI operator for order $\eta \geq 0$ is*

$$I^{\eta}u(z) = \frac{1}{\Gamma(\eta)}\int_0^z (z-s)^{\eta-1}u(s)ds,\ z \geq 0,$$
$$I^0u(z) = u(z).$$

The operator I^{η} has following features for $\Upsilon > -1$:

a) $I^{\eta} I^{\Upsilon} u(z) = I^{\eta+\Upsilon} u(z)$, b) $I^{\eta} I^{\Upsilon} u(z) = I^{\Upsilon} I^{\eta} u(z)$, c) $I^{\eta} z^{\Upsilon} = \frac{\Gamma(\Upsilon+1)}{\Gamma(\eta+\Upsilon+1)} z^{\eta+\Upsilon}$.

Definition 2. *The definition of the CFD of order η is*

$$D^{\eta} u(z) = I^{m-\eta}(D^{m} u(z)) = \frac{1}{\Gamma(m-\eta)} \int_0^z (z-s)^{m-\eta-1} u^{(m)}(s) ds, \; m-1 < \eta \leq m, \; m \in \mathbb{N}, \; z > 0.$$

3 Fractional-order functions

In this section, we introduce FOLJFs to solve fractional partial differential equations.

3.1 Fractional-order Legendre functions

The shifted Legendre functions $L_m^{\beta}(z)$, $\beta > 0$ on the interval $[0,1]$ is defined as [14]

$$L_0^{\beta}(z) = 1, \; L_1^{\beta}(z) = 2z^{\beta} - 1,$$

$$L_{m+1}^{\beta}(z) = \frac{(2m+1)(2z^{\beta}-1)}{m+1} L_m^{\beta}(z) - \frac{m}{m+1} L_{m-1}^{\beta}(z), \; m \geq 1. \tag{1}$$

The FOLFs are specific solution of the normalized eigenfunctions of the singular Sturm-Liouville problem [14]:

$$((z - z^{1+\beta}) L_m'^{\beta}(z))' + \beta^2 m(m+1) z^{\beta-1} L_m^{\beta}(z) = 0, \; z \in (0,1).$$

The explicit formula for $L_m^{\beta}(z)$ of degree $m\beta$ is given by

$$L_m^{\beta}(z) = \sum_{p=0}^{m} \frac{(-1)^{p+m}(m+p)!}{(m-p)!(p!)^2} z^{p\beta}. \tag{2}$$

Let f be a function defined over $L^2[c,d]$ which can be expressed in terms of FOLFs as

$$f(z) = \sum_{m=0}^{\infty} r_m L_m^{\beta}(z).$$

The truncated series of f can be written as

$$f(z) \simeq \sum_{m=0}^{M} r_m L_m^{\beta}(z) = R^T L_{\beta}(z), \tag{3}$$

where $R = [r_0, r_1, \ldots, r_M]^T$ and $L_{\beta}(z) = [L_0^{\beta}(z), L_1^{\beta}(z), \ldots, L_M^{\beta}(z)]^T$.

3.2 Fractional-order Jaiswal functions

The FOJF, denoted by $J_n^{\gamma}(t)$, $\gamma > 0$, is defined as [13]

$$J_1^{\gamma}(t) = 1,\ J_2^{\gamma}(t) = 2t^{\gamma},\ J_3^{\gamma}(t) = 4t^{2\gamma},$$
$$J_{n+3}^{\gamma}(t) = 2t^{\gamma} J_{n+2}^{\gamma}(t) - J_n(t),\ n \geq 1. \tag{4}$$

The explicit formula of J_n^{γ} is given as

$$J_n^{\gamma}(t) = \sum_{q=0}^{\lfloor \frac{n-1}{3} \rfloor} \frac{(-1)^q 2^{n-1-3q}(n-1-2q)!}{q!(n-1-3q)!} t^{(n-1-3q)\gamma},\ \ n \geq 1. \tag{5}$$

Let h be a function defined over $L^2[0,T]$ which can be expressed in terms of FOJFs as

$$h(t) = \sum_{n=1}^{\infty} w_n J_n^{\gamma}(t).$$

The truncated series of h can be written as

$$h(t) \simeq \sum_{n=1}^{N+1} w_n J_n^{\gamma}(t) = W^T J_{\gamma}(t), \tag{6}$$

where $W = [w_1, w_2, \ldots, w_{N+1}]^T$, $J_{\gamma}(t) = [J_1^{\gamma}(t), J_2^{\gamma}(t), \ldots, J_{N+1}^{\gamma}(t)]^T$.

3.3 Fractional-order Legendre-Jaiswal functions

Here, we introduce two variable functions to solve the two-dimensional problems. Let

$$\Psi^{\beta,\gamma}(z,t) = L_m^{\beta}(z) J_n^{\gamma}(t),\ (z,t) \in \Omega = [c,d] \times [0,T],\ m = 0(1)M,\ n = 1(1)N+1. \tag{7}$$

4 Operational and pseudo-operational matrix of integration

In this section, integer and fractional-order different operational matrices are developed.

4.1 Integer order integral pseudo-operational matrix

Integer order IP-OM of FOLFs will be computed using Taylor functions of fractional-order, which are defined as follows [17]:

$$T_i^{\beta}(z) = z^{i\beta},\ i = 0(1)M.$$

Then, FOLFs can be written as

$$L_{\beta}(z) = A_1 T_{\beta}(z), \tag{8}$$

where $T_{\beta}(z) = [1, z^{\beta}, z^{2\beta}, \ldots, z^{M\beta}]^T$ and

$$A_1 = [a_{jk}^1]_{(M+1)\times(M+1)},\ a_{jk}^1 = \begin{cases} \frac{(-1)^{(j+k)}(j+k)!}{(j-k)!(k!)^2}, & j \geq k, \\ 0, & \text{otherwise}, \end{cases} \quad j,\ k = 0(1)M.$$

Integrating FOLFs yields

$$\int_0^z L_\beta(s)ds = \int_0^z A_1 T_\beta(s)ds = A_1 \int_0^z T_\beta(s)ds$$
$$= zA_1B_1T_\beta(z) = zA_1B_1A_1^{-1}L_\beta(z) = zQ_1L_\beta(z),$$

where $Q_1 = A_1B_1A_1^{-1}$ is the IP-OM of FOLFs and

$$B_1 = [b^1_{jk}]_{(M+1)\times(M+1)},\ b^1_{jk} = \begin{cases} \frac{1}{j\beta+1}, & j = k, \\ 0, & \text{otherwise}, \end{cases} \quad j,\ k = 0(1)M.$$

Similarly, we can write FOJFs as

$$J_\gamma(t) = A_2 T_\gamma(t), \tag{9}$$

where $T_\gamma(t) = [1, t^\gamma, t^{2\gamma}, \ldots, t^{N\gamma}]^T$ and $A_2 = [a^2_{jk}]_{(N+1)\times(N+1)}$ with

$$a^2_{jk} = \begin{cases} \binom{j-2\lfloor\frac{j-k}{3}\rfloor}{\lfloor\frac{j-k}{3}\rfloor}(-1)^{\lfloor\frac{j-k}{3}\rfloor}2^{j-3\lfloor\frac{j-k}{3}\rfloor}, & \text{if } j \geq k,\ j \equiv k \pmod 3, \\ 0, & \text{otherwise}, \end{cases}$$

for $j,k = 0(1)N$.
Integrating FOJFs, we get

$$\int_0^t J_\gamma(s)ds = \int_0^t A_2T_\gamma(s)ds = A_2\int_0^t T_\gamma(s)ds = tA_2B_2T_\gamma(t) = tA_2B_2A_2^{-1}J_\gamma(t) = tQ_2J_\gamma(t),$$

where $Q_2 = A_2B_2A_2^{-1}$ is the IP-OM of FOJFs and

$$B_2 = [b^2_{jk}]_{(N+1)\times(N+1)},\ b^2_{jk} = \begin{cases} \frac{1}{j\gamma+1}, & j = k, \\ 0, & \text{otherwise}, \end{cases} \quad j,\ k = 0(1)N.$$

4.2 Fractional-order integral operational and pseudo-operational matrix

Here, using specific properties of the RLFI, we obtain the fractional-order operational matrices of fractional integration of FOLJFs.

Theorem 1. *For $\zeta > 0$, ζ order fractional integration of FOLFs is*

$$I^\zeta(L_\beta(z)) \simeq \mu^{\beta,\zeta}L_\beta(z), \tag{10}$$

where $\mu^{\beta,\zeta}$ is the $(M+1)\times(M+1)$ dimensional operational matrix of fractional integration.

Proof. Using the FOLF's explicit formula, we get

$$I^\zeta(L_\beta(z)) = I^\zeta\left(\sum_{p=0}^{m}\frac{(-1)^{(p+m)}(m+p)!}{(m-p)!(p!)^2}z^{p\beta}\right) = \sum_{p=0}^{m}\frac{(-1)^{(p+m)}(m+p)!}{(m-p)!(p!)^2}I^\zeta(z^{p\beta})$$

$$= \sum_{p=0}^{m} \frac{(-1)^{(p+m)}(m+p)!}{(m-p)!(p!)^2} \frac{\Gamma(p\beta+1)}{\Gamma(p\beta+\zeta+1)} z^{p\beta+\zeta}$$
$$= \sum_{p=0}^{m} d_{mp}^{\beta,\zeta} z^{p\beta+\zeta}, \tag{11}$$

where

$$d_{mp}^{\beta,\zeta} = \frac{(-1)^{(p+m)}(m+p)!}{(m-p)!(p!)^2} \frac{\Gamma(p\beta+1)}{\Gamma(p\beta+\zeta+1)}. \tag{12}$$

Using FOLFs, we can expand $z^{p\beta+\zeta}$ as

$$z^{p\beta+\zeta} \simeq \sum_{i=0}^{M} o_{pi}^{\beta,\zeta} L_i^{\beta}(z). \tag{13}$$

Substituting Eq. (13) in Eq. (11), we get

$$I^{\zeta}(L_{\beta}(z)) = \sum_{p=0}^{m} d_{mp}^{\beta,\zeta} \left(\sum_{i=0}^{M} o_{pi}^{\beta,\zeta} L_i^{\beta}(z) \right) = \sum_{i=0}^{M} \left(\sum_{p=0}^{m} \chi_{mip}^{\beta,\zeta} \right) L_i^{\beta}(z) = \sum_{i=0}^{M} \mu_{mip}^{\beta,\zeta} L_i^{\beta}(z),\ m = 0(1)M,$$

where $\chi_{mip}^{\beta,\zeta} = d_{mp}^{\beta,\zeta} o_{pi}^{\beta,\zeta}$. Then $\mu^{\beta,\zeta}$ can be found as

$$\mu^{\beta,\zeta} = \begin{pmatrix} \chi_{000}^{\beta,\zeta} & \chi_{010}^{\beta,\zeta} & \chi_{020}^{\beta,\zeta} & \cdots & \chi_{0M0}^{\beta,\zeta} \\ \sum_{p=0}^{1} \chi_{10p}^{\beta,\zeta} & \sum_{p=0}^{1} \chi_{11p}^{\beta,\zeta} & \sum_{p=0}^{1} \chi_{12p}^{\beta,\zeta} & \cdots & \sum_{p=0}^{1} \chi_{1Mp}^{\beta,\zeta} \\ \sum_{p=0}^{2} \chi_{20p}^{\beta,\zeta} & \sum_{p=0}^{2} \chi_{21p}^{\beta,\zeta} & \sum_{p=0}^{2} \chi_{22p}^{\beta,\zeta} & \cdots & \sum_{p=0}^{2} \chi_{2Mp}^{\beta,\zeta} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{p=0}^{M} \chi_{M0p}^{\beta,\zeta} & \sum_{p=0}^{M} \chi_{M1p}^{\beta,\zeta} & \sum_{p=0}^{M} \chi_{M2p}^{\beta,\zeta} & \cdots & \sum_{p=0}^{M} \chi_{MMp}^{\beta,\zeta} \end{pmatrix}.$$

□

Using the transfer matrix, we can derive fractional pseudo-operational matrix of FOLFs as follows:

$$I^{\zeta}(L_{\beta}(z)) = I^{\zeta}(A_1 T_{\beta}(z)) = A_1 I^{\zeta}(T_{\beta}(z)) = z^{\zeta} A_1 \rho^{\beta,\zeta} T_{\beta}(z) = z^{\zeta} A_1 \rho^{\beta,\zeta} A_1^{-1} L_{\beta}(z) = z^{\zeta} Q_1^{\beta,\zeta} L_{\beta}(z), \tag{14}$$

where $Q_1^{\beta,\zeta}$ is the IP-OM of fractional-order for FOLFs, $Q_1^{\beta,\zeta} = A_1 \rho^{\beta,\zeta} A_1^{-1}$, and

$$\rho^{\beta,\zeta} = [\rho_{jk}^{\beta,\ \zeta}]_{(M+1)\times(M+1)},\ \rho_{jk}^{\beta,\zeta} = \begin{cases} \frac{\Gamma(j\beta+1)}{\Gamma(j\beta+\zeta+1)}, & j = k \\ 0, & \text{otherwise,} \end{cases} \quad j,\ k = 0(1)M.$$

Hence, we get

$$I^{\zeta}(L_{\beta}(z)) = z^{\zeta} Q_1^{\beta,\zeta} L_{\beta}(z) \simeq \mu^{\beta,\zeta} L_{\beta}(z).$$

Theorem 2. *For $\alpha > 0$, α order fractional integration of FOJFs is*

$$I^{\alpha}(J_{\gamma}(t)) \simeq \theta^{\gamma,\alpha} J_{\gamma}(t), \tag{15}$$

where $\theta^{\gamma,\alpha}$ is operational matrix of dimension $(N+1)\times(N+1)$ of α order fractional integration.

Proof. Using the FOJFs explicit formula, we get

$$\begin{aligned} I^{\alpha}(J_{\gamma}(t)) &= I^{\alpha}\left(\sum_{q=0}^{\lfloor\frac{n-1}{3}\rfloor} \frac{(-1)^q 2^{n-1-3q}(n-1-2q)!}{q!(n-1-3q)!} t^{(n-1-3q)\gamma}\right) \\ &= \sum_{q=0}^{\lfloor\frac{n-1}{3}\rfloor} \frac{(-1)^q 2^{n-1-3q}(n-1-2q)!}{q!(n-1-3q)!} I^{\alpha}(t^{(n-1-3q)\gamma}) \\ &= \sum_{q=0}^{\lfloor\frac{n-1}{3}\rfloor} \frac{(-1)^q 2^{n-1-3q}(n-1-2q)!}{q!(n-1-3q)!} \frac{\Gamma((n-1-3q)\gamma+1)}{\Gamma((n-1-3q)\gamma+\alpha+1)} t^{(n-1-3q)\gamma+\alpha} \\ &= \sum_{q=0}^{\lfloor\frac{n-1}{3}\rfloor} \hat{d}_{nq}^{\gamma,\alpha} t^{(n-1-3q)\gamma+\alpha}, \end{aligned} \tag{16}$$

where

$$\hat{d}_{nq}^{\gamma,\alpha} = \frac{(-1)^q 2^{n-1-3q}(n-1-2q)!}{q!(n-1-3q)!} \frac{\Gamma((n-1-3q)\gamma+1)}{\Gamma((n-1-3q)\gamma+\alpha+1)}. \tag{17}$$

Using FOJFs, we can expand $t^{(n-1-3q)\gamma+\alpha}$ as

$$t^{(n-1-3q)\gamma+\alpha} \simeq \sum_{i=1}^{N+1} \tilde{o}_{qi}^{\gamma,\alpha} J_i^{\gamma}(t). \tag{18}$$

Substituting Eq. (18) in Eq. (16), we get

$$I^{\alpha}(J_{\gamma}(t)) = \sum_{q=0}^{\lfloor\frac{n-1}{3}\rfloor} \hat{d}_{nq}^{\gamma,\alpha} \left(\sum_{i=1}^{N+1} \tilde{o}_{qi}^{\gamma,\alpha} J_i^{\gamma}(t)\right) = \sum_{i=1}^{N+1} \left(\sum_{q=0}^{\lfloor\frac{n-1}{3}\rfloor} \xi_{niq}^{\gamma,\alpha}\right) J_i^{\gamma}(t) = \sum_{i=1}^{N+1} \theta_{niq}^{\gamma,\alpha} J_i^{\gamma}(t),\ n = 1(1)N+1,$$

where $\theta_{niq}^{\gamma,\alpha} = \hat{d}_{nq}^{\gamma,\alpha} \tilde{o}_{qi}^{\gamma,\alpha}$. Then $\theta_{niq}^{\gamma,\alpha}$ can be found as

$$\theta^{\gamma,\alpha} = \begin{pmatrix} \xi_{110}^{\gamma,\alpha} & \xi_{120}^{\gamma,\alpha} & \xi_{130}^{\gamma,\alpha} & \cdots & \xi_{1N+10}^{\gamma,\alpha} \\ \sum_{q=0}^{1} \xi_{21q}^{\gamma,\alpha} & \sum_{q=0}^{1} \xi_{22q}^{\gamma,\alpha} & \sum_{q=0}^{1} \xi_{23q}^{\gamma,\alpha} & \cdots & \sum_{q=0}^{1} \xi_{2N+1q}^{\gamma,\alpha} \\ \sum_{q=0}^{2} \xi_{31q}^{\gamma,\alpha} & \sum_{q=0}^{2} \xi_{32q}^{\gamma,\alpha} & \sum_{q=0}^{2} \xi_{33q}^{\gamma,\alpha} & \cdots & \sum_{q=0}^{2} \xi_{3N+1q}^{\gamma,\alpha} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{q=0}^{N} \xi_{N+11q}^{\gamma,\alpha} & \sum_{q=0}^{N} \xi_{N+12q}^{\gamma,\alpha} & \sum_{q=0}^{N} \xi_{N+13q}^{\gamma,\alpha} & \cdots & \sum_{q=0}^{N} \xi_{N+1N+1q}^{\gamma,\alpha} \end{pmatrix}.$$

□

We can derive the pseudo-operational matrix of fractional integration for FOJFs based on the transfer matrix as follows:

$$I^{\alpha}(J_{\gamma}(t)) = I^{\alpha}(A_2 T_{\gamma}(t)) = A_2 I^{\alpha}(T_{\gamma}(t)) = t^{\alpha} A_2 \rho^{\gamma,\alpha} T_{\gamma}(t) = t^{\alpha} A_2 \rho^{\gamma,\alpha} A_2^{-1} J_{\gamma}(t) = t^{\gamma} Q_2^{\gamma,\alpha} J_{\gamma}(t). \tag{19}$$

Here $Q_2^{\gamma,\alpha}$ is the IP-OM of fractional-order for FOJFs and $Q_2^{\gamma,\alpha} = A_2 \rho^{\gamma,\alpha} A_2^{-1}$, where

$$\rho^{\gamma,\alpha} = [\rho_{jk}^{\gamma,\alpha}]_{(N+1)\times(N+1)}, \ \rho_{jk}^{\gamma,\alpha} = \begin{cases} \frac{\Gamma(j\gamma+1)}{\Gamma(j\gamma+\alpha+1)}, & j = k, \\ 0, & \text{otherwise}, \end{cases} \quad j,\ k = 0(1)N.$$

Hence, we get

$$I^{\alpha}(J_{\gamma}(t)) = t^{\gamma} Q_2^{\gamma,\alpha} J_{\gamma}(t) \simeq \theta^{\gamma,\alpha} J_{\gamma}(t).$$

4.3 Error bound for the operational matrix of fractional integration

Lemma 1 ([4, 16]). *Assume H is a Hilbert space and V is a closed subspace of H such that $\dim V < \infty$. Let $v_1, v_2, \ldots, v_m$ be a basis for V. Let x be an arbitrary element in H and x^* be the unique best approximation to x out of V. Then*

$$\|x - x^*\|_2^2 = \frac{G(x, v_1, v_2, \ldots, v_m)}{G(v_1, v_2, \ldots, v_m)},$$

where

$$G(x, v_1, v_2, \ldots, v_m) = \begin{vmatrix} \langle x, x\rangle & \langle x, v_1\rangle & \cdots & \langle x, v_m\rangle \\ \langle v_1, x\rangle & \langle v_1, v_1\rangle & \cdots & \langle v_1, v_m\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_m, x\rangle & \langle v_m, v_1\rangle & \cdots & \langle v_m, v_m\rangle \end{vmatrix}.$$

Lemma 2 ([21]). *Suppose $\tilde{g} \in L^2[c,d]$ is approximated by $\tilde{g}_M$*

$$\tilde{g}(z) \simeq \tilde{g}_M(z) = \sum_{m=0}^{M} \tau_m L_m^{\beta}(z).$$

Consider $S_M(\tilde{g}) = \int_c^d [\tilde{g}(z) - \tilde{g}_M(z)]^2 dz$, then $\lim_{M\to\infty} S_M(\tilde{g}) = 0$.

Define the error vector for FOLFs as

$$e_z^{\zeta} = I_z^{\zeta} L_{\beta} - \mu^{\beta,\zeta} L_{\beta}, \tag{20}$$

where $e_z^{\zeta} = [E_{z,m}^{\beta,\zeta}]$, $m = 0(1)M$. Now from Eq. (13) and Lemma 1, we get

$$\|z^{p\beta+\zeta} - \sum_{i=0}^{M} o_{pi}^{\beta,\zeta} L_i^{\beta}(z)\|_{L^2} = \left(\frac{G(z^{p\beta+\zeta}, L_0^{\beta}(z), \ldots, L_M^{\beta}(z))}{G(L_0^{\beta}(z), L_1^{\beta}(z), \ldots, L_M^{\beta}(z))} \right)^{\frac{1}{2}}. \tag{21}$$

Hence, from Eqs. (11), (20) and (21), we get

$$\begin{aligned}
\|E_{z,m}^{\beta,\zeta}\|_{L^2} &= \|I_z^{\zeta}(L_m^{\beta}(z)) - \sum_{p=0}^{m} d_{mp}^{\beta,\zeta}\left(\sum_{i=0}^{M} o_{pi}^{\beta,\zeta} L_i^{\beta}(z)\right)\|_{L^2} \\
&\leq |\sum_{p=0}^{m} \frac{(-1)^{(p+m)}(m+p)!}{(m-p)!(p!)^2}\frac{\Gamma(p\beta+1)}{\Gamma(p\beta+\zeta+1)}|\ \|z^{p\beta+\zeta} - \sum_{i=0}^{M} o_{pi}^{\beta,\zeta} L_i^{\beta}(z)\|_{L^2} \\
&\leq \sum_{p=0}^{m} \frac{(m+p)!}{(m-p)!(p!)^2}\frac{\Gamma(p\beta+1)}{\Gamma(p\beta+\zeta+1)}\left(\frac{G(z^{p\beta+\zeta}, L_0^{\beta}(z), \ldots, L_M^{\beta}(z))}{G(L_0^{\beta}(z), L_1^{\beta}(z), \ldots, L_M^{\beta}(z))}\right)^{\frac{1}{2}}.
\end{aligned} \tag{22}$$

Let $\hat{\Psi}^{\beta,\gamma}$ be a rearrangement of $\Psi^{\beta,\gamma}$. Then the error vector of $\hat{\Psi}^{\beta,\gamma}$ with respect to z is defined as follows:

$$R_z^{\zeta} = I_z^{\zeta}\hat{\Psi}^{\beta,\gamma} - \chi_{\mu}^{\beta,\zeta}\hat{\Psi}^{\beta,\gamma}, \tag{23}$$

with $\hat{\Psi}^{\beta,\gamma} = [\Psi_{01}^{\beta,\gamma}, \ldots, \Psi_{M1}^{\beta,\gamma}, \ldots, \Psi_{0N+1}^{\beta,\gamma}, \ldots, \Psi_{MN+1}^{\beta,\gamma}]^T$ and

$$\chi_{\mu}^{\beta,\zeta} = I_{(N+1)\times(N+1)} \otimes \mu^{\beta,\zeta} = \begin{pmatrix} \mu^{\beta,\zeta} & \hat{O} & \ldots & \hat{O} \\ \hat{O} & \mu^{\beta,\zeta} & \ldots & \hat{O} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{O} & \hat{O} & \ldots & \mu^{\beta,\zeta} \end{pmatrix},$$

where $\hat{O}$ is $(M+1)\times(M+1)$ dimensional zero matrix. Alternatively,

$$R_z^{\zeta} = \begin{pmatrix} W_1^{\beta,\gamma,\zeta} \\ W_2^{\beta,\gamma,\zeta} \\ \vdots \\ W_{N+1}^{\beta,\gamma,\zeta} \end{pmatrix},\ W_n^{\beta,\gamma,\zeta} = \begin{pmatrix} r_{0n}^{\beta,\gamma,\zeta} \\ r_{1n}^{\beta,\gamma,\zeta} \\ \vdots \\ r_{Mn}^{\beta,\gamma,\zeta} \end{pmatrix},\ n = 1(1)N+1,$$

where

$$W_n^{\beta,\gamma,\zeta} = [r_{mn}^{\beta,\gamma,\zeta}] = I_z^{\zeta} L_{\beta}(z) J_n^{\gamma}(t) - \mu^{\beta,\zeta} L_{\beta}(z) J_n^{\gamma}(t),\ m = 0(1)M,\ n = 1(1)N+1. \tag{24}$$

From Eqs. (22)–(24), we get

$$\begin{aligned}
\|W_n^{\beta,\gamma,\zeta}\|_{L^2} &= \|I_z^{\zeta} L_{\beta}(z) J_n^{\gamma}(t) - \mu^{\beta,\zeta} L_{\beta}(z) J_n^{\gamma}(t)\|_{L^2} \\
&= \|[I_z^{\zeta} L_{\beta}(z) - \mu^{\beta,\zeta} L_{\beta}(z)] J_n^{\gamma}(t)\|_{L^2} \\
&= \left(\int_0^T \int_c^d |[I_z^{\zeta} L_{\beta}(z) - \mu^{\beta,\zeta} L_{\beta}(z)] J_n^{\gamma}(t)|^2 dz dt\right)^{\frac{1}{2}} \\
&\leq \left(\int_c^d |I_z^{\zeta} L_{\beta}(z) - \mu^{\beta,\zeta} L_{\beta}(z)|^2 dz\right)^{\frac{1}{2}} \left(\int_0^T |J_n^{\gamma}(t)|^2 dt\right)^{\frac{1}{2}} \\
&= \|I_z^{\zeta} L_{\beta}(z) - \mu^{\beta,\zeta} L_{\beta}(z)\|_{L^2} \|J_n^{\gamma}(t)\|_{L^2} \\
&\leq C_1 T \sum_{q=0}^{\lfloor \frac{n-1}{3} \rfloor} \frac{2^{n-1-3q}(n-1-2q)!}{q!(n-1-3q)!}
\end{aligned}$$

$$\times \sum_{p=0}^{m} \frac{(m+p)!}{(m-p)!(p!)^2} \frac{\Gamma(p\beta+1)}{\Gamma(p\beta+\zeta+1)} \left(\frac{G(z^{p\beta+\zeta}, L_0^{\beta}(z), \ldots, L_M^{\beta}(z))}{G(L_0^{\beta}(z), L_1^{\beta}(z), \ldots, L_M^{\beta}(z))} \right)^{\frac{1}{2}}. \tag{25}$$

Lemma 3 ([21]). *Suppose $\tilde{h} \in L^2[0,T]$ is approximated by $\tilde{h}_N$*

$$\tilde{h}(t) \simeq \tilde{h}_N(t) = \sum_{n=1}^{N+1} \kappa_n J_n^{\gamma}(t).$$

Consider $\tilde{S}_N(\tilde{h}) = \int_0^T [\tilde{h}(t) - \tilde{h}_N(t)]^2 dt$, then $\lim_{N\to\infty} \tilde{S}_N(\tilde{h}) = 0$.

For FOJFs, the error vector is

$$\tilde{e}_t^{\alpha} = I_t^{\alpha} J_{\gamma} - \theta^{\gamma,\alpha} J_{\gamma}, \tag{26}$$

where $\tilde{e}_t^{\alpha} = [\tilde{E}_{t,n}^{\gamma,\alpha}]$, $n = 1(1)N+1$. Using Eq. (18) and Lemma 1, we get

$$\|t^{(n-1-3q)\gamma+\alpha} - \sum_{i=1}^{N+1} \tilde{o}_{qi}^{\gamma,\alpha} J_i^{\gamma}(t)\|_{L^2} = \left(\frac{G(t^{(n-1-3q)\gamma+\alpha}, J_1^{\gamma}(t), \ldots, J_{N+1}^{\gamma}(t))}{G(J_1^{\gamma}(t), J_2^{\gamma}(t), \ldots, J_{N+1}^{\gamma}(t))} \right)^{\frac{1}{2}}. \tag{27}$$

Hence from Eqs. (16), (26) and (27), we get

$$\begin{aligned}
\|\tilde{E}_{t,n}^{\gamma,\alpha}\|_{L^2} &= \|I_t^{\alpha}(J_n^{\gamma}(t)) - \sum_{q=0}^{\lfloor \frac{n-1}{3} \rfloor} \hat{d}_{nq}^{\gamma,\alpha} \left(\sum_{i=1}^{N+1} \tilde{o}_{qi}^{\gamma,\alpha} J_i^{\gamma}(t) \right) \|_{L^2} \\
&\le |\sum_{q=0}^{\lfloor \frac{n-1}{3} \rfloor} \frac{(-1)^q 2^{n-1-3q}(n-1-2q)!}{q!(n-1-3q)!} \frac{\Gamma(n-1-3q)\gamma+1)}{\Gamma((n-1-3q)\gamma+\alpha+1)}| \\
&\quad \times \|t^{(n-1-3q)\gamma+\alpha} - \sum_{i=1}^{N+1} \tilde{o}_{qi}^{\gamma,\alpha} J_i^{\gamma}(t)\|_{L^2} \\
&\le \sum_{q=0}^{\lfloor \frac{n-1}{3} \rfloor} \frac{2^{n-1-3q}(n-1-2q)!}{q!(n-1-3q)!} \frac{\Gamma(n-1-3q)\gamma+1)}{\Gamma((n-1-3q)\gamma+\alpha+1)} \\
&\quad \times \left(\frac{G(t^{(n-1-3q)\gamma+\alpha}, J_0^{\gamma}(t), \ldots, J_N^{\gamma}(t))}{G(J_0^{\gamma}(t), J_1^{\gamma}(t), \ldots, J_N^{\gamma}(t))} \right)^{\frac{1}{2}}, \; n = 1(1)N+1.
\end{aligned} \tag{28}$$

Let $\tilde{\Psi}^{\beta,\gamma}$ be a rearrangement of $\Psi^{\beta,\gamma}$. Then the error vector of $\tilde{\Psi}^{\beta,\gamma}$ with respect to t is

$$\tilde{R}_t^{\alpha} = I_t^{\alpha} \tilde{\Psi}^{\beta,\gamma} - \xi_{\theta}^{\gamma,\alpha} \tilde{\Psi}^{\beta,\gamma}, \tag{29}$$

where $\tilde{\Psi}^{\beta,\gamma} = [\Psi_{01}^{\beta,\gamma}, \ldots, \Psi_{0N+1}^{\beta,\gamma}, \ldots, \Psi_{M1}^{\beta,\gamma}, \ldots, \Psi_{MN+1}^{\beta,\gamma}]^T$,

$$\xi_{\theta}^{\gamma,\alpha} = I_{(M+1)\times(M+1)} \otimes \theta^{\gamma,\alpha} = \begin{pmatrix} \theta^{\gamma,\alpha} & \hat{O} & \ldots & \hat{O} \\ \hat{O} & \theta^{\gamma,\alpha} & \ldots & \hat{O} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{O} & \hat{O} & \ldots & \theta^{\gamma,\alpha} \end{pmatrix},$$

and $\hat{O}$ is $(N+1)\times(N+1)$ dimensional zero matrix. Alternatively

$$\tilde{R}_t^{\alpha}=\begin{pmatrix}\tilde{W}_0^{\beta,\gamma,\alpha}\\ \tilde{W}_1^{\beta,\gamma,\alpha}\\ \vdots\\ \tilde{W}_M^{\beta,\gamma,\alpha}\end{pmatrix},\ \tilde{W}_m^{\beta,\gamma,\alpha}=\begin{pmatrix}\tilde{r}_{m1}^{\beta,\gamma,\alpha}\\ \tilde{r}_{m2}^{\beta,\gamma,\alpha}\\ \vdots\\ \tilde{r}_{mN+1}^{\beta,\gamma,\alpha}\end{pmatrix}\ m=0(1)M,$$

where

$$\tilde{W}_m^{\beta,\gamma,\alpha}=[\tilde{r}_{mn}^{\beta,\gamma,\alpha}]=I_t^{\alpha}L_m^{\beta}(z)J_{\gamma}(t)-\theta^{\gamma,\alpha}L_m^{\beta}(z)J_{\gamma}(t),\ m=0(1)M. \tag{30}$$

According to Eqs. (28)–(30), we get

$$\begin{aligned}\|\tilde{W}_m^{\beta,\gamma,\alpha}\|_{L^2}&=\|I_t^{\alpha}L_m^{\beta}(z)J_{\gamma}(t)-\theta^{\gamma,\alpha}L_m^{\beta}(z)J_{\gamma}(t)\|_{L^2}\\&=\|[I_t^{\alpha}J_{\gamma}(t)-\theta^{\gamma,\alpha}J_{\gamma}(t)]L_m^{\beta}(z)\|_{L^2}\\&=\left(\int_0^T\int_c^d|[I_t^{\alpha}J_{\gamma}(t)-\theta^{\gamma,\alpha}J_{\gamma}(t)]L_m^{\beta}(z)|^2dzdt\right)^{\frac{1}{2}}\\&\le\left(\int_0^T[I_t^{\alpha}J_{\gamma}(t)-\theta^{\gamma,\alpha}J_{\gamma}(t)]^2dt\right)^{\frac{1}{2}}\left(\int_c^d|L_m^{\beta}(z)|^2dz\right)^{\frac{1}{2}}\\&=\|I_t^{\alpha}J_{\gamma}(t)-\theta^{\gamma,\alpha}J_{\gamma}(t)\|_{L^2}\|L_m^{\beta}(z)\|_{L^2}\\&\le C_2(d-c)\sum_{p=0}^{m}\frac{(m+p)!}{(m-p)!(p!)^2}\times\sum_{q=0}^{\lfloor\frac{n-1}{3}\rfloor}\left[\frac{2^{n-1-3q}(n-1-2q)!}{q!(n-1-3q)!}\frac{\Gamma(n-1-3q)\gamma+1)}{\Gamma((n-1-3q)\gamma+\alpha+1)}\right.\\&\quad\left.\times\left(\frac{G(t^{(n-1-3q)\gamma+\alpha},J_1^{\gamma}(t),\ldots,J_{N+1}^{\gamma}(t))}{G(J_1^{\gamma}(t),J_2^{\gamma}(t),\ldots,J_{N+1}^{\gamma}(t))}\right)^{\frac{1}{2}}\right].\end{aligned} \tag{31}$$

5 The numerical method

In this section, we study linear and nonlinear TSFPDEs as

$$\begin{cases}F\left(\frac{\partial^{\zeta}V}{\partial z^{\zeta}},\frac{\partial^{\alpha}V}{\partial t^{\alpha}},\frac{\partial^{\alpha-1}V}{\partial t^{\alpha-1}},\frac{\partial^2V}{\partial z^2},\frac{\partial^2V}{\partial t^2},\frac{\partial V}{\partial z},\frac{\partial V}{\partial t},V\right)=g(z,t),\ \zeta\in(0,1],\ \alpha\in(1,2],\\ V(z,0)=\tilde{\rho}_0(z),\ \frac{\partial v(z,0)}{\partial t}=\tilde{\rho}_1(z),\ z\in(0,1],\\ V(0,t)=\aleph_0(t),\ V(1,t)=\aleph_1(t),\ t\in(0,\infty).\end{cases} \tag{32}$$

Let us assume that

$$\frac{\partial^4V(z,t)}{\partial z^2\partial t^2}\simeq L_{\beta}^T(z)UJ_{\gamma}(t), \tag{33}$$

since the largest order of derivative with regard to z and t is 2. Here the unknown matrix is U of dimension $(M+1)\times(N+1)$. Integrating Eq. (33) with respect to t and using initial conditions, we get

$$\frac{\partial^3V(z,t)}{\partial z^2\partial t}\simeq tL_{\beta}^T(z)UQ_2J_{\gamma}(t)+\tilde{\rho}_1''(z). \tag{34}$$

Again, integrating Eq. (34) with respect to t, we get

$$\frac{\partial^2 V(z,t)}{\partial z^2} \simeq t^2 L_\beta^T(z) U Q_2 \hat{Q}_2 J_\gamma(t) + t\tilde{\rho}_1^{''}(z) + \tilde{\rho}_0^{''}(z). \tag{35}$$

Consider

$$\int_0^t sJ_\gamma(s)ds = \int_0^t sA_2 T_\gamma(s)ds = A_2 \int_0^t sT_\gamma(s)ds = t^2 A_2 \hat{B}_2 T_\gamma(t) = t^2 A_2 \hat{B}_2 A_2^{-1} J_\gamma(t) = t^2 \hat{Q}_2 J_\gamma(t), \tag{36}$$

where

$$\hat{B}_2 = [\hat{b}^2_{jk}]_{(N+1)\times(N+1)},\ b^{\hat{2}}_{jk} = \begin{cases} \frac{1}{j\gamma+2}, & j=k, \\ 0, & \text{otherwise}, \end{cases} \ j,\ k = 0(1)N.$$

Now, integrating Eq. (35) with respect to z, we get

$$\frac{\partial V(z,t)}{\partial z} \simeq zt^2 L_\beta^T(z) Q_1^T U Q_2 \hat{Q}_2 J_\gamma(t) + t(\tilde{\rho}_1^{'}(z) - \tilde{\rho}_1^{'}(0)) + (\tilde{\rho}_0^{'}(z) - \tilde{\rho}_0^{'}(0)) + \frac{\partial V(0,t)}{\partial z}, \tag{37}$$

then

$$\begin{aligned} V(z,t) =& z^2 t^2 L_\beta^T(z) \hat{Q}_1^T Q_1^T U Q_2 \hat{Q}_2 J_\gamma(t) + t(\tilde{\rho}_1(z) - \tilde{\rho}_1(0) - z\tilde{\rho}_1^{'}(0)) + (\tilde{\rho}_0(z) - \tilde{\rho}_0(0) - z\tilde{\rho}_0^{'}(0)) \\ &+ z\frac{\partial V(0,t)}{\partial z} + \aleph_0(t), \end{aligned} \tag{38}$$

where

$$\int_0^z sL_\beta(s)ds = \int_0^z sA_1 T_\beta(s)ds = A_1 \int_0^z sT_\beta(s)ds = z^2 A_1 \hat{B}_1 T_\beta(z) = t^2 A_1 \hat{B}_1 A_1^{-1} L_\beta(z) = z^2 \hat{Q}_1 L_\beta(z), \tag{39}$$

and

$$\hat{B}_1 = [\hat{b}^1_{jk}]_{(M+1)\times(M+1)},\ b^{\hat{1}}_{jk} = \begin{cases} \frac{1}{j\beta+2}, & j=k, \\ 0, & \text{otherwise}, \end{cases} \ j,\ k = 0(1)M.$$

Here, in Eq. (38), $\frac{\partial V(0,t)}{\partial z}$ is unknown. Integrating Eq. (38) with respect to z from 0 to 1 gives

$$\begin{aligned} V(1,t) - V(0,t) =& t^2 \left(\int_0^1 zL_\beta^T(z)dz \right) Q_1^T U Q^2 \hat{Q}^2 J_\gamma(t) + t(\tilde{\rho}_1(1) - \tilde{\rho}_1(0) - \tilde{\rho}_1^{'}(0)) \\ &+ (\tilde{\rho}_0(1) - \tilde{\rho}_0(0) - \tilde{\rho}_0^{'}(0)) + \frac{\partial V(0,t)}{\partial z}, \end{aligned}$$

where $\int_0^1 zL_\beta(z)dz = \int_0^1 zA_1 T_\beta^T(z)dz = S^T A_1^T$, and $S = [s_{j1}],\ s_{j1} = \frac{1}{j\beta+2},\ j = 0(1)M$. Then, we get

$$\begin{aligned} \frac{\partial V(0,t)}{\partial z} =& \aleph_1(t) - \aleph_0(t) - t^2 S^T A_1^T Q_1^T U Q_2 \hat{Q}_2 J_\gamma(t) - t(\tilde{\rho}_1(1) - \tilde{\rho}_1(0) - \tilde{\rho}_1^{'}(0)) \\ &- (\tilde{\rho}_0(1) - \tilde{\rho}_0(0) - \tilde{\rho}_0^{'}(0)). \end{aligned} \tag{40}$$

By using the FOLJFs, the approximate values of $\frac{\partial V(z,t)}{\partial z}$ and $V(z,t)$ are obtained by substituting Eq. (40) into Eqs. (37) and (38). Furthermore, we compute fractional derivatives using the Riemann-Liouville fractional integral properties and fractional-order operational matrices as

$$\begin{aligned}\frac{\partial^\zeta V(z,t)}{\partial z^\zeta} &= I_z^{1-\zeta}(\frac{\partial V(z,t)}{\partial z})\\ &=t^2 L_\beta^T(z)(\mu^{\beta,1-\zeta})^T Q_1^T U Q_2 \hat{Q}_2 J_\gamma(t) + t(I_z^{1-\zeta}\tilde{\rho}_1'(z) - \frac{z^{1-\zeta}}{\Gamma(2-\zeta)}\tilde{\rho}_1'(0)) + (I_z^{1-\zeta}\tilde{\rho}_0'(z)\\ &- \frac{z^{1-\zeta}}{\Gamma(2-\zeta)}\tilde{\rho}_0'(0)) + \frac{z^{1-\zeta}}{\Gamma(2-\zeta)}(\aleph_1(t) - \aleph_0(t) - t^2 S^T A_1^T Q_1^T U Q_2 \hat{Q}_2 J_\gamma(t)\\ &- t(\tilde{\rho}_1(1) - \tilde{\rho}_1(0) - \tilde{\rho}_1'(0)) - (\tilde{\rho}_0(1) - \tilde{\rho}_0(0) - \tilde{\rho}_0'(0))). \end{aligned} \quad (41)$$

We can calculate $\mu^{\beta,1-\zeta}$ as $\mu^{\beta,\zeta}$. Integrating Eq. (33) with respect to z, we have

$$\frac{\partial^3 V(z,t)}{\partial z \partial t^2} \simeq z L_\beta^T(z) Q_1^T U J_\gamma(t) + \frac{\partial^3 V(0,t)}{\partial z \partial t^2} \quad (42)$$

and

$$\frac{\partial^2 V(z,t)}{\partial t^2} \simeq z^2 L_\beta^T(z)\hat{Q}_1^T Q_1^T U J_\gamma(t) + z\frac{\partial^3 V(0,t)}{\partial z \partial t^2} + \aleph_0''(t). \quad (43)$$

Integrating Eq. (42) from 0 to 1 with respect to z, we get

$$\frac{\partial^3 V(0,t)}{\partial z \partial t^2} = \aleph_1''(t) - \aleph_0''(t) - S^T A_1^T Q_1^T U J_\gamma(t).$$

Then

$$\frac{\partial^2 V(z,t)}{\partial t^2} \simeq z^2 L_\beta^T(z)\hat{Q}_1 Q_1^T U J_\gamma(t) + z(\aleph_1''(t) - \aleph_0''(t) - S^T A_1^T Q_1^T U J_\gamma(t)) + \aleph_0''(t). \quad (44)$$

Integrating the above equation with respect to t, we obtain

$$\frac{\partial V(z,t)}{\partial t} \simeq z^2 t L_\beta^T(z)\hat{Q}_1 Q_1^T U Q_2 J_\gamma(t) + z(\aleph_1'(t) - \aleph_0'(t) - t S^T A_1^T Q_1^T U Q_2 J_\gamma(t)) + \aleph_0'(t) + \tilde{\rho}_1(z). \quad (45)$$

Then, we calculate $\frac{\partial^\alpha V(z,t)}{\partial t^\alpha}$ as

$$\begin{aligned}\frac{\partial^\alpha V(z,t)}{\partial t^\alpha} &= I_t^{2-\alpha}\left(\frac{\partial^2 V(z,t)}{\partial^2 t}\right)\\ &= z^2 L_\beta^T(z)\hat{Q}_1^T Q_1^T U \theta^{\gamma,2-\alpha} J_\gamma(t) + z(I_t^{2-\alpha}\aleph_1''(t) - I_t^{2-\alpha}\aleph_0''(t)\\ &- S^T A_1^T Q_1^T U \theta^{\gamma,2-\alpha} J_\gamma(t)) + I_t^{2-\alpha}\aleph_0''(t). \end{aligned} \quad (46)$$

We can calculate $\theta^{\gamma,1-\alpha}$ as $\theta^{\gamma,\alpha}$. We derive an algebraic equation by placing the previously mentioned approximation functions in Eq. (32). Then, we take collocation points as

$$z_j = \frac{2j-1}{2(M+1)},\ j = 1(1)M+1,\ t_k = \frac{2k-1}{2(N+1)},\ k = 1(1)N+1. \quad (47)$$

Then, the unknown matrix U is obtained using Newton's iterative method. Putting U in Eq. (38) we get an approximate solution.

Algorithm 1: Solution of TSFPDE using FOLF and FOJF spectral method

Input: TSFPDE (32) with ICs/BCs; parameters M,N,β,γ; tolerance ε_{tol}.

Step 1:
- Construct FOLFs $L_\beta(z)$ and FOJFs $J_\gamma(t)$ using recurrences (2), (5).
- Compute operational matrices $Q_1, \hat{Q}_1, Q_2, \hat{Q}_2, \mu^{\beta,\zeta}$, and $\theta^{\gamma,\alpha}$.
- Generate collocation points: $z_i = \frac{2i-1}{2(M+1)}$, $t_j = \frac{2j-1}{2(N+1)}$.

Step 2:
- Assume $\frac{\partial^4 V}{\partial z^2 \partial t^2} \approx L_\beta^T(z)\, U\, J_\gamma(t)$.
- Derive approximations of all terms in $F(\cdot)$ using Equations (35), (37), (38), (41), (44), (45)–(46).

Step 3:
- Substitute approximations into TSFPDE (32) at points (z_i, t_j).
- Form nonlinear algebraic system $F(\mathbf{U}) = \mathbf{0}$.

Step 4:
- Solve $F(\mathbf{U}) = \mathbf{0}$ (e.g., via the Newton–Raphson method) until $\|F\| < \varepsilon_{\text{tol}}$.

Step 5:
- Construct approximate solution using Eq. (38).

Output: Approximate solution.

6 Theoretical analysis: stability and convergence

6.1 Stability analysis

Theorem 3 (Conditional Stability of the Discrete System)**.** *Consider the nonlinear algebraic system $F(\mathbf{U}) = \mathbf{0}$ derived from the FOLJF collocation method for the TSFPDE (32), where $\mathbf{U}$ is the vector of unknown coefficients. Let $\mathbf{U}^*$ be the solution corresponding to the original data $\mathbf{D} = (g, \tilde{\rho}_0, \tilde{\rho}_1, \aleph_0, \aleph_1)$, and let $\tilde{\mathbf{U}}^*$ be the solution corresponding to a perturbed data $\tilde{\mathbf{D}} = \mathbf{D} + \delta\mathbf{D}$. Assume that*

1. *The Jacobian matrix $J_F(\mathbf{U})$ of the system is Lipschitz continuous in a neighborhood of $\mathbf{U}^*$, i.e., there exists a constant $L > 0$ such that*

$$\|J_F(\mathbf{V}) - J_F(\mathbf{W})\| \leq L\|\mathbf{V} - \mathbf{W}\|, \ \forall\, \mathbf{V}, \mathbf{W} \in B(\mathbf{U}^*, R).$$

2. *The Jacobian $J_F(\mathbf{U}^*)$ is non-singular, with a condition number $\kappa(J_F(\mathbf{U}^*))$.*

Then, for a sufficiently small perturbation $\|\delta\mathbf{D}\|$, the perturbation in the numerical solution is bounded by

$$\|\tilde{\mathbf{U}}^* - \mathbf{U}^*\| \leq C\,\kappa(J_F(\mathbf{U}^*))\,\|\delta\mathbf{D}\|,$$

where the constant $C > 0$ depends on the Lipschitz constant L and the specific norms used.

Proof. The perturbed system satisfies

$$F(\tilde{\mathbf{U}}^*) + \mathbf{G}(\delta\mathbf{D}) = \mathbf{0},$$

where $\mathbf{G}$ encapsulates the effect of the data perturbation on the discrete system. Using a first-order Taylor expansion of $F(\tilde{\mathbf{U}}^*)$ around $\mathbf{U}^*$, we have

$$F(\tilde{\mathbf{U}}^*) = F(\mathbf{U}^*) + J_F(\mathbf{U}^*)(\tilde{\mathbf{U}}^* - \mathbf{U}^*) + \mathbf{R},$$

where the remainder term $\mathbf{R}$ satisfies $\|\mathbf{R}\| \leq \frac{L}{2}\|\tilde{\mathbf{U}}^* - \mathbf{U}^*\|^2$ by the Lipschitz continuity assumption. Since $F(\mathbf{U}^*) = \mathbf{0}$ and $F(\tilde{\mathbf{U}}^*) = -\mathbf{G}(\delta\mathbf{D})$, we obtain

$$J_F(\mathbf{U}^*)(\tilde{\mathbf{U}}^* - \mathbf{U}^*) + \mathbf{R} = -\mathbf{G}(\delta\mathbf{D}).$$

Solving for the solution perturbation gives

$$\tilde{\mathbf{U}}^* - \mathbf{U}^* = -J_F(\mathbf{U}^*)^{-1}[\mathbf{G}(\delta\mathbf{D}) + \mathbf{R}].$$

Taking norms and using the submultiplicative property we have

$$\begin{aligned}\|\tilde{\mathbf{U}}^* - \mathbf{U}^*\| &\leq \|J_F(\mathbf{U}^*)^{-1}\| \left(\|\mathbf{G}(\delta\mathbf{D})\| + \|\mathbf{R}\|\right) \\ &\leq \|J_F(\mathbf{U}^*)^{-1}\| \left(K\|\delta\mathbf{D}\| + \frac{L}{2}\|\tilde{\mathbf{U}}^* - \mathbf{U}^*\|^2\right),\end{aligned}$$

where we assume $\|\mathbf{G}(\delta\mathbf{D})\| \leq K\|\delta\mathbf{D}\|$ for small perturbations. For sufficiently small $\|\delta\mathbf{D}\|$, the quadratic term becomes negligible compared to the linear term, yielding

$$\|\tilde{\mathbf{U}}^* - \mathbf{U}^*\| \leq \|J_F(\mathbf{U}^*)^{-1}\| K\|\delta\mathbf{D}\| + \mathscr{O}(\|\delta\mathbf{D}\|^2).$$

Using the relationship between the matrix norm and condition number, $\|J_F(\mathbf{U}^*)^{-1}\| \leq \frac{\kappa(J_F(\mathbf{U}^*))}{\|J_F(\mathbf{U}^*)\|}$, we obtain the final bound

$$\|\tilde{\mathbf{U}}^* - \mathbf{U}^*\| \leq C\,\kappa(J_F(\mathbf{U}^*))\,\|\delta\mathbf{D}\|,$$

where $C = K/\|J_F(\mathbf{U}^*)\|$. This completes the proof. □

6.2 Convergence analysis

Theorem 4 (Spectral Convergence for Smooth Solutions)**.** *Let $V(z,t)$ be the exact solution of the TSF-PDE (32). Let $V_{M,N}(z,t)$ be the numerical solution obtained by the FOLJF spectral method with truncation parameters M and N, and basis fractional orders β and γ. Assume that*

1. *The function $V(z^\beta, t^\gamma)$ is analytic on the domain Ω.*
2. *The initial and boundary conditions $\tilde{\rho}_0, \tilde{\rho}_1, \aleph_0, \aleph_1$ are compatible and can be approximated with spectral accuracy by the chosen bases.*
3. *The Jacobian matrix in the Newton-Raphson iteration remains non-singular for all M,N greater than some M_0, N_0.*

Then, the numerical solution converges to the exact solution exponentially fast (spectral convergence). Specifically, there exist constants $C, \rho_M, \rho_N > 1$, independent of M and N, such that

$$\|V - V_{M,N}\|_{L^2(\Omega)} \leq C(\rho_M^{-M} + \rho_N^{-N}).$$

Proof. We decompose the total error into three components:

$$\|V - V_{M,N}\|_{L^2(\Omega)} \leq \underbrace{\|V - \mathscr{P}_{M,N}V\|_{L^2(\Omega)}}_{E_{\text{Proj}}} + \underbrace{\|\mathscr{P}_{M,N}V - \hat{V}_{M,N}\|_{L^2(\Omega)}}_{E_{\text{Disc}}} + \underbrace{\|\hat{V}_{M,N} - V_{M,N}\|_{L^2(\Omega)}}_{E_{\text{Alg}}},$$

where $\mathscr{P}_{M,N}V$ is the best approximation in the FOLJF space, and $\hat{V}_{M,N}$ is the exact solution of the discretized system. The three components are defined as follows:

- $E_{\text{Proj}} = \|V - \mathscr{P}_{M,N}V\|_{L^2(\Omega)}$ denotes the **projection error**, which arises from approximating the exact solution V by its projection onto the finite-dimensional FOLJF space.
- $E_{\text{Disc}} = \|\mathscr{P}_{M,N}V - \hat{V}_{M,N}\|_{L^2(\Omega)}$ denotes the **discretization error**, which measures the difference between the projected exact solution and the exact discrete solution of the numerical scheme.
- $E_{\text{Alg}} = \|\hat{V}_{M,N} - V_{M,N}\|_{L^2(\Omega)}$ denotes the **algorithmic error**, which reflects numerical inaccuracies (e.g., iterative solver tolerance or round-off errors).

By Assumption 1, $V(z^\beta, t^\gamma)$ is analytic. Since the FOLFs $\{L_m^\beta(z)\}_{m=0}^\infty$ and FOJFs $\{J_n^\gamma(t)\}_{n=1}^\infty$ form complete orthogonal systems in $L^2[0,1]$ and $L^2[0,T]$ respectively, their tensor product is complete in $L^2(\Omega)$. For analytic functions, the expansion coefficients decay exponentially. Therefore,

$$E_{\text{Proj}} = \|V - \mathscr{P}_{M,N}V\|_{L^2(\Omega)} \leq K(e^{-\alpha_1 M} + e^{-\alpha_2 N}) \leq C_1(\rho_M^{-M} + \rho_N^{-N}),$$

where $\rho_M = e^{\alpha_1}$, $\rho_N = e^{\alpha_2}$.

The numerical method constructs the solution by integrating the assumption $\frac{\partial^4 V}{\partial z^2 \partial t^2} = L_\beta^T U J_\gamma$. This operational matrix approach is exact within the space $\mathbb{V}_{M,N}$. The collocation procedure is consistent, and for spectral methods applied to smooth problems, the discrete solution is quasi-optimal:

$$\|\mathscr{P}_{M,N}V - \hat{V}_{M,N}\|_{L^2(\Omega)} \leq C_2 \|V - \mathscr{P}_{M,N}V\|_{L^2(\Omega)}.$$

Thus, $E_{\text{Disc}} \leq C_2(\rho_M^{-M} + \rho_N^{-N})$.

Under Assumption 3, the Newton-Raphson method with a sufficiently accurate initial guess converges quadratically. After k iterations we have

$$\|\hat{V}_{M,N} - V_{M,N}^{(k)}\|_{L^2(\Omega)} \leq \mu \|\hat{V}_{M,N} - V_{M,N}^{(k-1)}\|_{L^2(\Omega)}^2.$$

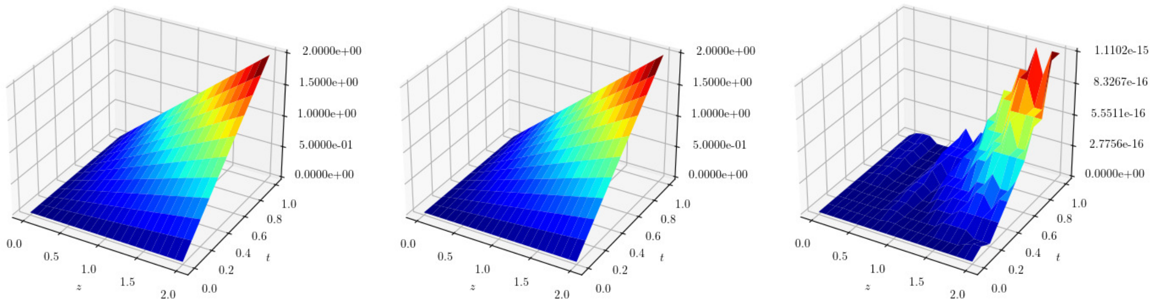
By iterating until the error is below a tolerance ε_{tol}, we can achieve $E_{\text{Alg}} \leq \varepsilon_{\text{tol}}$, which is negligible compared to the exponential terms for large M, N.

Combining all three error components gives

$$\begin{aligned}\|V - V_{M,N}\|_{L^2(\Omega)} &\leq C_1(\rho_M^{-M} + \rho_N^{-N}) + C_2(\rho_M^{-M} + \rho_N^{-N}) + \varepsilon_{\text{tol}} \\ &\leq C(\rho_M^{-M} + \rho_N^{-N}) + \varepsilon_{\text{tol}},\end{aligned}$$

where $C = C_1 + C_2$. For sufficiently large M, N, the exponential decay dominates, proving spectral convergence. □

Remark 1. *The convergence rates ρ_M and ρ_N are critically dependent on the analyticity properties of $V(z^\beta, t^\gamma)$. The optimal choice of basis parameters is often $\beta = \zeta$ and $\gamma = \alpha$, which aligns the structure of the basis functions with the differential operators in the governing equation. This alignment typically maximizes the region of analyticity of $V(z^\beta, t^\gamma)$ and consequently optimizes the convergence rates ρ_M and ρ_N.*



(a) **(b)** **(c)**

Figure 1: (a) Numerical solution, (b) analytical solution and (c) absolute point-wise error for $M = 1,\ N = 1,\ \beta = 0.5$, and $\gamma = 0.5$ for Example 1

7 Numerical experiments

In this section, some numerical experiments are conducted to verify the precision and efficacy of the presented method. Furthermore, L_∞ error of the illustrated examples are presented, which is stated as

$$\varepsilon^{M,N} = \max_{\substack{m=0,1,\dots,M,\\ n=0,1,\dots,N}} |V_m^n - V(z_m,t_n)|,$$

where the exact solution at (z_m,t_n) is $V(z_m,t_n)$, and the approximate solution is V_m^n. The computations for the examples are carried out using Python 3. Newton's method converges within 10 iterations based on the selected tolerance of $\varepsilon = 10^{-10}$.

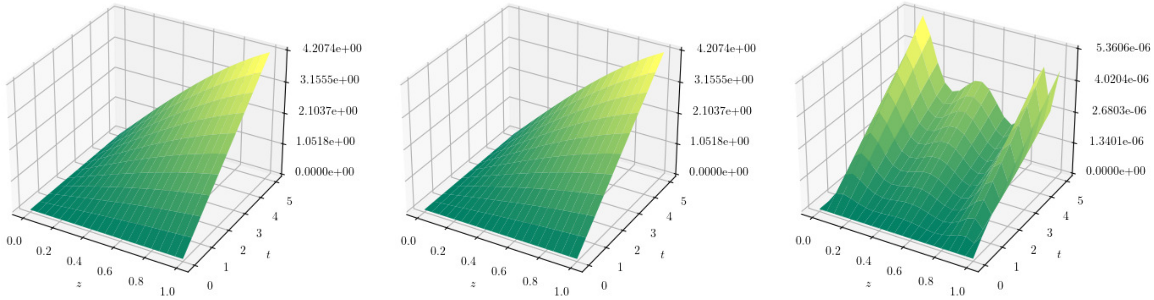
Example 1. Consider the following problem [4]:

$$\begin{cases} z\dfrac{\partial^{\frac{1}{2}} V(z,t)}{\partial z^{\frac{1}{2}}} + \dfrac{\partial^{\frac{1}{2}} V(z,t)}{\partial t^{\frac{1}{2}}} = \dfrac{2zt^{\frac{1}{2}}}{\sqrt{\pi}} + \dfrac{2tz^{\frac{3}{2}}}{\sqrt{\pi}}, \\ V(z,0) = 0,\ z \in [0,2], \\ V(0,t) = 0,\ t \in [0,1]. \end{cases}$$

The exact solution of this example is $V(z,t) = zt$. We compared our results with the results in [4] for this example. Set $M = N = 1$, and now we have varied the value of $z,\ t$ to compute the L_∞ errors. Table 1 contains the comparison of L_∞ errors for different values of z and t. It show that, we have achieved more precise results than the reference [4]. In Table 2, we have presented the L_∞ errors for different values of β,γ,t. Figures 1a, 1b, and 1c present the 3D graphical representations of the approximate solution, the exact solution, and the absolute error of Example 1, respectively.

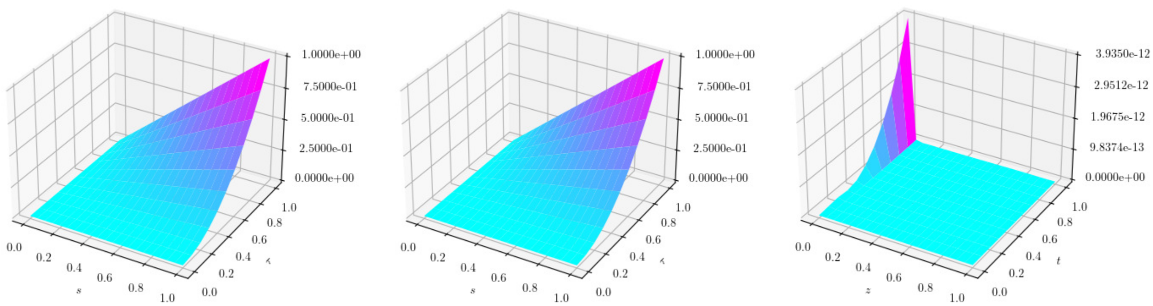
Example 2. Consider the following problem [4]:

$$\begin{cases} \dfrac{\partial^\alpha V(z,t)}{\partial t^\alpha} + \dfrac{\partial V(z,t)}{\partial z} = \dfrac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin(z) + t\cos(z),\ \alpha \in (0,1], \\ V(z,0) = 0,\ z \in [0,1], \\ V(0,t) = 0,\ t \in (0,\infty), \end{cases}$$



(a) **(b)** **(c)**

Figure 2: (a) Numerical solution, (b) analytical solution and (c) absolute point-wise error for $N=4,\ M=6,\ \alpha=0.25,\ \beta=1$, and $\gamma=0.25$ for Example 2



(a) **(b)** **(c)**

Figure 3: (a) Numerical solution, (b) analytical solution and (c) absolute point-wise error for $N=1,\ M=4,\ \beta=\gamma=\alpha=0.25$ for Example 3

where the exact solution is $V(z,t)=t\sin(z)$. Figures 2a, 2b, and 2c present the 3D graphical representations of the approximate solution, the exact solution, and the absolute error of Example 2, respectively. Table 3 displays the L_∞ error for $z\in[0,1]$ and $t\in[0,5]$ with different values of β,γ,N,M. For different N,M and for $\beta=\gamma=\alpha=0.5$ with $t=25,50,100$, and $z\in[0,1]$, L_∞ errors are shown in Table 4.

Example 3. Consider the following problem [4]:

$$\begin{cases}\dfrac{\partial^\alpha V(z,t)}{\partial t^\alpha}+V^2(z,t)=\dfrac{2zt^{2-\alpha}}{\Gamma(3-\alpha)}+z^2t^4,\ \alpha\in(0,1],\\ V(z,0)=0,\ \dfrac{\partial V(z,0)}{\partial t}=0,\quad z\in[0,1],\\ V(0,t)=0,\ V(1,t)=t^2,\quad t\in(0,\infty).\end{cases}$$

This problem has an exact solution $V(z,t)=zt^2$. The comparison of absolute errors for $\beta=\gamma=\alpha=0.25$ and $M=4,\ N=1$ with [4] is displayed in Table 5. Tables 5 and 6 demonstrate that, with the help of a few terms of fractional-order Legendre-Jaiswal functions, we were able to obtain a good approximation of the exact solution. Figures 3a, 3b, and 3c present the 3D graphical representations of the approximate solution, the exact solution, and the absolute error of Example 3, respectively.

Example 4. Consider the following problem [31]:

$$\begin{cases} \dfrac{\partial^\alpha V(z,t)}{\partial t^\alpha} + z\dfrac{\partial V(z,t)}{\partial z} - \dfrac{\partial^2 V(z,t)}{\partial z^2} = g(z,t),\ \alpha \in (0,1], \\ V(z,0) = z^2 - z^3,\ z \in (0,1), \\ V(0,t) = V(1,t) = 0,\ t \in [0,1], \\ g(z,t) = \dfrac{2t^{2-\alpha}}{\Gamma(3-\alpha)}(z^2 - z^3) + (t^2+1)(2z^2 - 3z^3 + 6z - 2). \end{cases}$$

The exact solution of this problem is $V(z,t) = (t^2+1)(z^2-z^3)$. Table 7 presents the comparison of absolute errors between the proposed Legendre-Jaiswal collocation method and existing numerical techniques for various values of α at $\beta = 0.5,\ \gamma = 0.5$ and $N = 2,\ M = 6$, demonstrating that the present method yields significantly smaller errors and superior accuracy. Figures 4a, 4b, and 4c present the 3D graphical representations of the approximate solution, the exact solution, and the absolute error of Example 4, respectively.

Example 5. Consider the following time-fractional Fisher's equation [23]:

$$\begin{cases} \dfrac{\partial^\alpha V(z,t)}{\partial t^\alpha} - \dfrac{\partial^2 V(z,t)}{\partial z^2} - V(z,t)(1 - V^3(z,t)) = g(z,t),\ \alpha \in (0,1], \\ V(z,0) = 0,\ z \in (0,1), \\ V(0,t) = t^{2\alpha},\ V(1,t) = 0,\ t \in [0,1], \\ g(z,t) = e^{2z}(1-z^2)t^\alpha \dfrac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} - 2zt^{2\alpha}(1-4z-2z^2)e^{2z} - [t^{2\alpha}(1-z^2)e^{2z}][1 - (t^{2\alpha}(1-z^2)e^{2z})^3]. \end{cases}$$

The exact solution of this problem is $V(z,t) = (1-z^2)e^{2z}t^{2\alpha}$. The comparison of absolute errors between the suggested Legendre-Jaiswal collocation method and numerical techniques [23] for different values of α, N and $M = 9$, $\beta = \gamma = 0.9$ is shown in Table 8, which shows that the current method produces significantly smaller errors and superior accuracy. Figures 5a, 5b, 5c present the 3D graphical representations of the absolute error for $N = 4,\ M = 9,\ \beta = \gamma = 0.9$, corresponding to $\alpha = 0.5,\ \alpha = 0.8,\ \alpha = 0.95$ respectively.
Tables 1–8 demonstrate the effectiveness and accuracy of the proposed method in solving the time–space fractional convection–diffusion, convection–diffusion–reaction, and nonlinear Fisher equations. They also show that the approximate solutions obtained for each selected α, β, and γ closely match the corresponding exact solutions.

8 Conclusion

This paper presents fractional Legendre Jaiswal functions and its application to the FPDE's numerical solution. We employed the collocation approach using integral operational operational matrices with fractional and integer orders to solve this problem. We have used a limited number of Legendre-Jaiswal

Table 1: Comparison of the L_∞ errors of Example 1 for $N = M = 1$ and $\beta = \gamma = 0.5$

z/t	Present method 0.2	Present method 0.5	Present method 0.8	In Ref. [4] 0.2	In Ref. [4] 0.5	In Ref. [4] 0.8
0.1	0	1.3878e-17	4.1633e-17	3.83e-17	7.34e-17	9.15e-17
0.3	0	2.7756e-17	5.5511e-17	8.02e-17	1.52e-16	1.89e-16
0.5	0	0	0	9.37e-17	1.77e-16	2.17e-16
0.7	2.7756e-17	5.5511e-17	1.1102e-16	8.57e-17	1.59e-16	1.92e-16
0.9	0	0	1.1102e-16	8.57e-17	1.59e-16	1.92e-16
1.1	0	1.1102e-16	2.22045e-16	1.75e-17	2.40e-17	1.56e-17
1.3	0	1.1102e-16	4.4409e-16	3.89e-17	8.71e-17	1.27e-16
1.5	5.5511e-17	2.2204e-16	4.4409e-16	1.08e-16	2.23e-16	3.03e-16
1.7	5.5511e-17	3.3307e-16	6.6613e-16	1.90e-16	3.84e-16	5.09e-16
1.9	0	2.2204e-16	6.6613e-16	2.84e-16	5.67e-16	7.43e-16
CPU Time (Second)	0.025	0.025	0.030	-	-	-

Table 2: L_∞ errors of Example 1 for $z \in [0,2]$

M	N	β	γ	$t = 50$	CPU Time (Second)	$t = 100$	CPU Time (Second)	$t = 200$	CPU Time (Second)
1	1	0.5	0.5	2.3093e-14	0.019	6.7502e-14	0.020	1.9185e-13	0.021
2	1	0.25	0.5	1.2257e-13	0.039	3.1974e-13	0.041	8.7041e-13	0.042
3	2	0.75	0.25	1.0232e-12	0.049	2.9985e-12	0.049	9.0594e-12	0.049

Table 3: L_∞ errors of Example 2 for $z \in [0,1]$ and $t \in [0,5]$ with different values of β, γ, N, M

β	γ	α	M	N	L_∞ error	CPU Time (Second)
1	1/4	1/4	4	4	1.7027e-09	1.299
	1/4	1/4	6	4	2.2401e-09	3.245
1/2	1/2	1/2	3	2	3.1419e-04	0.432
	1/2	1/2	8	2	1.6509e-08	1.932
	1/4	1/2	6	4	2.2114e-05	3.107
	1/4	1/4	8	4	5.5556e-08	5.859
1/3	1/2	1/2	12	2	1.6757e-08	4.444
	1/3	1/3	3	3	4.1361e-03	0.630
	1/3	1/3	9	3	8.1951e-05	4.457
1/4	1/4	1/4	4	4	2.9786e-05	1.575
	1/4	1/4	12	4	9.1010e-06	13.144

Table 4: L_∞ errors of Example 2 for $\beta = \gamma = \alpha = 0.5$ with $z \in [0,1]$

N	M	$t = 25$	CPU Time (Second)	$t = 50$	CPU Time (Second)	$t = 100$	CPU Time (Second)
2	8	6.7282e-04	1.503	3.3788e-03	1.503	1.5451e-02	1.503
2	12	2.0298e-06	4.220	1.0163e-05	4.220	4.6399e-05	4.220
3	9	7.2517e-04	4.330	4.9267e-03	4.330	3.1764e-02	4.330
3	12	1.2758e-05	7.840	8.5885e-05	7.840	5.4999e-04	7.840

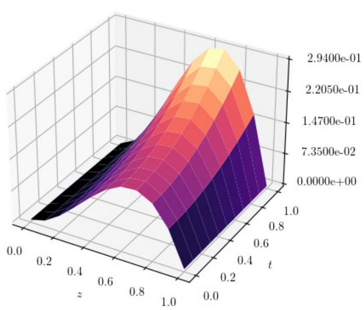
functions and obtained good results. We achieve great accuracy because of the pseudo-operational matrix technique. In conclusion, the numerical results validate the high accuracy and efficacy of the proposed

Table 5: L_∞ errors for $z \in [0,1]$ with various $\beta,\ \gamma, \alpha,\ t$ of Example 3

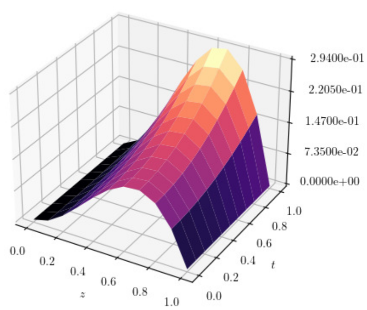
	Present method, $N=1,\ M=4$				In Ref. [4], $N=1,\ M=4$			
	$\alpha=\beta=\gamma=0.25$				$\alpha=\beta=\gamma=0.25$			
z/t	0.2	0.8	1.5	5	0.2	0.8	1.5	5
0.1	8.6736e-19	0	2.7756e-17	1.3323e-15	4.26e-09	2.66e-08	2.40e-07	2.66e-06
0.2	6.2450e-17	1.9151e-15	1.2712e-14	3.0109e-13	2.82e-09	1.76e-08	1.58e-07	1.76e-06
0.3	6.9389e-18	5.5511e-17	1.1102e-16	2.6645e-15	1.17e-09	7.34e-09	6.60e-08	7.34e-07
0.4	6.9389e-18	5.5511e-17	4.4409e-16	1.2434e-14	4.27e-10	2.66e-09	2.40e-08	2.66e-07
0.5	1.3878e-17	1.0547e-15	6.4393e-15	1.4744e-13	1.93e-09	1.20e-08	1.08e-07	1.20e-06
0.6	4.8572e-17	2.3315e-15	1.4655e-14	3.3040e-13	3.33e-09	2.08e-08	1.87e-07	2.08e-06
0.7	6.2450e-17	3.1086e-15	1.9540e-14	4.4409e-13	4.63e-09	2.89e-08	2.60e-07	2.89e-06
0.8	4.1633e-17	3.1086e-15	1.8208e-14	4.0501e-13	5.84e-09	3.65e-08	3.28e-07	3.65e-06
0.9	2.7756e-17	8.8818e-16	6.2172e-15	1.3856e-13	6.96e-09	4.35e-08	3.91e-07	4.35e-06
CPU Time (Second)	2.176	2.176	2.061	2.037	-	-	-	-

Table 6: L_∞ errors of Example 3 for $z \in [0,1]$ and $t \in [0,1]$

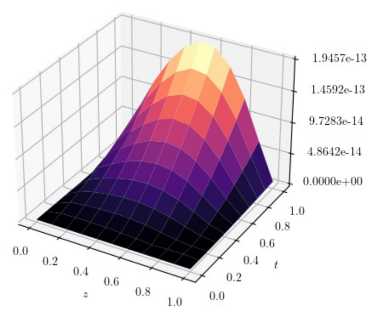
M	N	β	γ	α	Error	CPU Time (Second)
2	2	1	0.5	0.5	0	1.654
		0.5	0.5	1	5.5511e-17	1.678
		1	0.25	0.5	3.4694e-18	0.835
4	4	0.25	0.25	0.5	1.1102e-16	26.157
		1	0.25	0.5	5.5511e-17	25.388
		1	0.5	0.25	5.5511e-17	22.673



(a)



(b)



(c)

Figure 4: (a) Numerical solution, (b) analytical solution and (c) absolute point-wise error for $N=2,\ M=6,\ \beta=\alpha=\gamma=0.5$ for Example 4

method.

Acknowledgements

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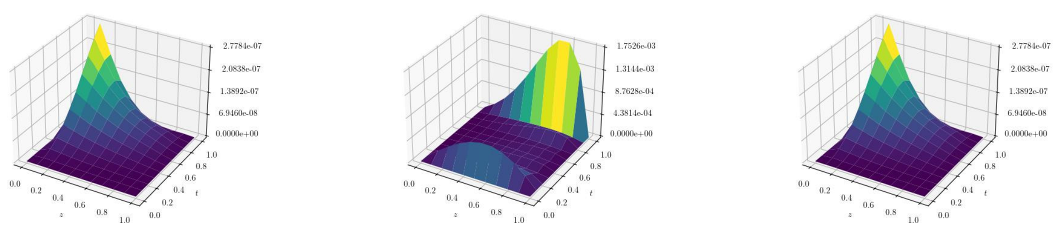
Table 7: Comparison of L_∞ errors of Example 4 for $N=2,\ M=6,\ \beta=0.5,\ \gamma=0.5,\ z\in[0,1]$ and $t\in[0,1]$

	Present method					In Ref. [31]				
(z,t)	$\alpha=0.1$	$\alpha=0.3$	$\alpha=0.5$	$\alpha=0.7$	$\alpha=0.9$	$\alpha=0.1$	$\alpha=0.3$	$\alpha=0.5$	$\alpha=0.7$	$\alpha=0.9$
$(0.1,0.1)$	3.5007e-15	1.0755e-15	4.7757e-15	5.4297e-15	3.8095e-15	1.1507e-06	3.8344e-06	6.3167e-06	1.0481e-05	2.1943e-05
$(0.2,0.2)$	1.2379e-14	3.8719e-15	1.7028e-14	1.2781e-14	1.1359e-14	2.5205e-06	6.6571e-06	9.0144e-06	9.4357e-06	1.1018e-05
$(0.3,0.3)$	2.4786e-14	7.5218e-15	3.5666e-14	2.4244e-14	1.8152e-14	3.8111e-06	8.9655e-06	1.0906e-05	1.0861e-05	9.8651e-06
$(0.4,0.4)$	3.7470e-14	1.3434e-14	5.9577e-14	4.0079e-14	2.3065e-14	4.8742e-06	1.0557e-05	1.2217e-05	1.0863e-05	6.2094e-06
$(0.5,0.5)$	4.7268e-14	2.3509e-14	8.5765e-13	5.5456e-14	2.7506e-14	5.6187e-06	1.1426e-05	1.2120e-05	9.8330e-06	5.2172e-06
$(0.6,0.6)$	5.1875e-14	3.7803e-14	1.0875e-13	6.3838e-14	3.3334e-14	5.8283e-06	1.1257e-05	1.1359e-05	8.6775e-06	4.1069e-06
$(0.7,0.7)$	5.0127e-14	5.3069e-14	1.2076e-13	6.0590e-14	4.0606e-14	5.3973e-06	9.9820e-06	9.6580e-06	7.0242e-06	2.9580e-06
$(0.8,0.8)$	4.1356e-14	6.1701e-14	1.1288e-13	4.4881e-14	4.5186e-14	4.2535e-06	7.5767e-06	7.0716e-06	4.9338e-06	1.9380e-06
$(0.9,0.9)$	2.4064e-14	5.1514e-14	7.6689e-14	2.0345e-14	3.6221e-14	2.4093e-06	4.1526e-06	3.7565e-06	2.5354e-06	1.0246e-06
CPU Time (Second)	4.886	4.695	4.655	4.736	4.729		-	-	-	-

Table 8: Comparison of the absolute errors with the Ref. [23] for $\beta = \gamma = 0.9$, of Example 5

α	Present method ($M = 9$)			Roul ($M = 1000$) [23]	
	N	L_∞ error	CPU Time (Second)	N	L_∞ error
0.5	3	7.7462e-08	209.731	20	3.4000e-03
	4	7.7398e-08	354.729	40	8.6499e-04
	5	7.7514e-08	506.462	80	2.1920e-04
0.8	3	6.5644e-04	212.772	20	4.7000e-03
	4	1.9966e-04	351.899	40	2.2000e-03
	5	8.7640e-05	520.444	80	3.1241e-04
0.95	3	3.9574e-08	219.457	20	4.6000e-03
	4	3.9087e-08	353.541	40	1.2000e-03
	5	3.9476e-08	506.094	80	3.0555e-04



(a) (b) (c)

Figure 5: Absolute point-wise error for Example 5 with $M = 9$, $N = 4$, $\beta = 0.9$, and $\gamma = 0.9$: (a) $\alpha = 0.5$ (b) $\alpha = 0.8$ and (c) $\alpha = 0.95$

Conflict of interest

The authors declare that they have no conflict of interest.

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