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Classification of flow behavior near generalized equilibrium points in piecewise smooth systems

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Abstract. The aim of this paper is to classify various states of flow behavior for piecewise smooth systems near generalized equilibrium points. Seven categories are introduced based on the sign of the vector field across the discontinuity boundary, each encompassing distinct dynamical configurations. We investigate how a small perturbation parameter influences the existence, type, and stability of generalized singular points in planar piecewise linear systems. Starting with a one-dimensional example to illustrate core mechanisms, we extend the analysis to two dimensions, providing a detailed classification grounded in the signs of the system's components. Our results yield a comprehensive framework for understanding how generalized singular points govern local dynamics, including bifurcations induced by parameter variation. This work contributes to the theoretical foundation for analyzing discontinuity-induced phenomena such as sliding modes and nonsmooth bifurcations.

Keywords: Generalized equilibrium point, piecewise smooth systems, bifurcation analysis, perturbation parameter, stability.
AMS Subject Classification 2010: 34C23, 37G10, 34C99.

1 Introduction

The study of dynamical systems has a rich and extensive history, with mathematicians and physicists alike seeking to understand the long-term behavior of complex systems. A fundamental aspect of this endeavor lies in the identification and classification of equilibrium points, also known as singular points, which represent states of stasis or balance within the system. For smooth dynamical systems, the theory surrounding the classification of singular points is well-established, with techniques such as linearization and the analysis of eigenvalues providing powerful tools for understanding local stability and bifurcations [6, 10, 11].

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However, many real-world systems exhibit nonsmooth behavior, arising from phenomena such as impacts, switching, or dry friction. Piecewise smooth (PWS) systems provide a natural framework for modeling such dynamics, where the system's evolution is governed by different sets of equations in different regions of the phase space [2, 5]. While the analysis of singular points in smooth systems provides a foundation, the presence of discontinuities introduces new complexities and challenges. In particular, the concept of a "generalized singular point" arises when the vector field on either side of a discontinuity is such that a traditional equilibrium point does not exist, yet the system's behavior in the vicinity of the discontinuity is of critical importance [3].

The study of generalized singular points in PWS has garnered increasing attention in recent years, driven by the need to understand the complex dynamics that can arise in these systems. Unlike smooth systems, PWS systems can exhibit behaviors such as sliding modes, where trajectories are constrained to move along the discontinuity boundary, and discontinuity-induced bifurcations, where small changes in parameters can lead to abrupt changes in the system's qualitative behavior. Understanding these phenomena is crucial for the design and control of engineering systems involving nonsmooth elements.

In this paper, we delve into the classification of generalized singular points in two-dimensional PWS. We begin by presenting a comprehensive classification scheme based on the sign of the vector field on either side of the discontinuity, extending the work of Han and Zhang [3] and others [7, 8]. We then analyze the effect of a small perturbation parameter, μ, on the existence, type, and stability of these generalized singular points, focusing on piecewise linear systems. Specifically, we investigate how the introduction of μ can lead to the appearance or disappearance of singular points, as well as changes in their stability properties. We start with an analysis of a one-dimensional piecewise linear system, providing a clear illustration of the key concepts. We then extend our analysis to the two-dimensional case, providing a detailed categorization based on the signs of the relevant parameters. Our findings provide a valuable framework for understanding the dynamics of piecewise smooth systems and the role of generalized singular points in shaping their behavior.

2 Preliminaries

In this study, we present principles and algorithms for analyzing singular points in piecewise systems and apply these principles and algorithms to a practical example involving vibration sensing and dynamic oscillations.
Consider a planar piecewise smooth system of the following form:

$$\begin{cases} \dot{x} = f(x,y) + \alpha\mu = f(x,y;\mu), \\ \dot{y} = g(x,y) + \beta\mu = g(x,y;\mu), \end{cases} \tag{1}$$

where

$$f(x,y) = \begin{cases} f^+(x,y), & x \geq 0, \\ f^-(x,y), & x < 0, \end{cases} \qquad g(x,y) = \begin{cases} g^+(x,y), & x \geq 0, \\ g^-(x,y), & x < 0. \end{cases}$$

In these equations, $f^\pm$ and $g^\pm$ belong to C^∞, and μ is a small parameter. This system consists of two C^∞ subsystems, referred to as the right and left subsystems, respectively, and they are described as follows:

$$\begin{cases} \dot{x} = f^+(x,y) + \alpha^+\mu = f^+(x,y;\mu), \\ \dot{y} = g^+(x,y) + \beta^+\mu = g^+(x,y;\mu), \end{cases} \qquad \begin{cases} \dot{x} = f^-(x,y) + \alpha^-\mu = f^-(x,y;\mu), \\ \dot{y} = g^-(x,y) + \beta^-\mu = g^-(x,y;\mu). \end{cases} \tag{2}$$

If we consider (1) with $\mu = 0$, it simplifies to the unperturbed system:

$$\dot{x} = f(x,y), \quad \dot{y} = g(x,y). \tag{3}$$

In other words

$$\dot{x} = f^+(x,y), \quad \dot{y} = g^+(x,y), \tag{4}$$

and

$$\dot{x} = f^-(x,y), \quad \dot{y} = g^-(x,y). \tag{5}$$

The singular point of equation (3) is considered a singular point of the right or left subsystems of equations (4) and (5), provided that $A \in \mathbb{R}^2_\pm = \{(x,y) \mid \pm x > 0\}$. If $A \notin \mathbb{R}^2_+ \cup \mathbb{R}^2_-$, we encounter a generalized singular point, the definition of which is given below.

Definition 1 (Generalized Singular Point)**.** *A point A on the y-axis is called a generalized singular point (GSP) of* (1) *if* $f^+(A).f^-(A) \leq 0$.

This definition follows the standard framework for discontinuous systems as established in Filippov's theory [2].

2.1 Classification of generalized singular point

Based on the condition $f^+(A_0)f^-(A_0) \leq 0$, GSP $A_0 = (0,y)$ can be classified into several types as follows:

A. If $f^+(A_0;0) = 0$ and $f^-(A_0;0) = 0$, then we call the GSP of type A.

B. If $f^+(A_0;0) = 0$ and $f^-(A_0;0) > 0$, then we call the GSP of type B.

C. If $f^+(A_0;0) = 0$ and $f^-(A_0;0) < 0$, then we call the GSP of type C.

D. If $f^+(A_0;0) > 0$ and $f^-(A_0;0) = 0$, then we call the GSP of type D.

E. If $f^+(A_0;0) > 0$ and $f^-(A_0;0) < 0$, then we call the GSP of type E.

F. If $f^+(A_0;0) < 0$ and $f^-(A_0;0) = 0$, then we call the GSP of type F.

G. If $f^+(A_0;0) < 0$ and $f^-(A_0;0) > 0$, then we call the GSP of type G.

This classification is consistent with the literature, particularly the work of Han and Zhang [3], where types E and G correspond to crossing and sliding scenarios, respectively, depending on the direction of the vector field.

Based on the signs of g^+ and g^-, each type of GSP can be further refined. For instance, type A is subdivided into five categories based on the signs of g^+, g^-, and f_y^+. Each subcategory corresponds to distinct phase portraits, constructed using standard techniques from smooth dynamical systems [1–3, 9, 10].

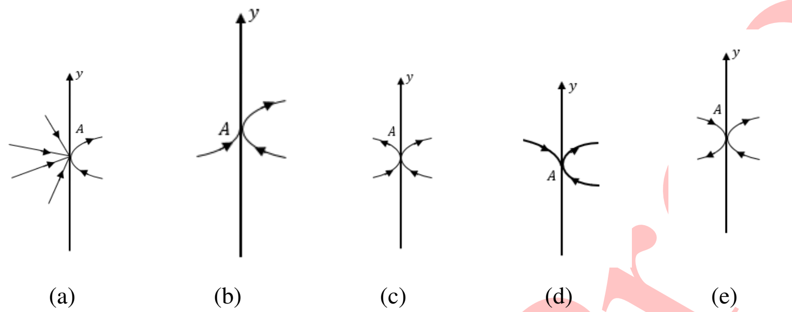


Figure 1: $f^+(A_0;0)=0$, $f^-(A_0;0)=0$, $g^+(A_0;0)>0$, and $f_y^+(A_0;0)\geq 0$

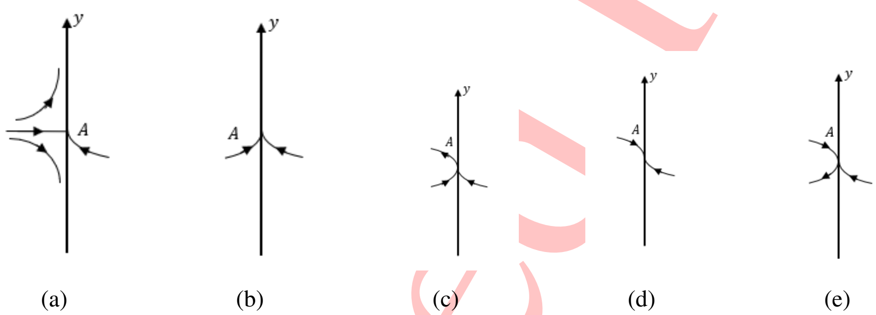


Figure 2: $f^+(A_0;0)=0$, $f^-(A_0;0)=0$, $g^+(A_0;0)>0$, and $f_y^+(A_0;0)<0$

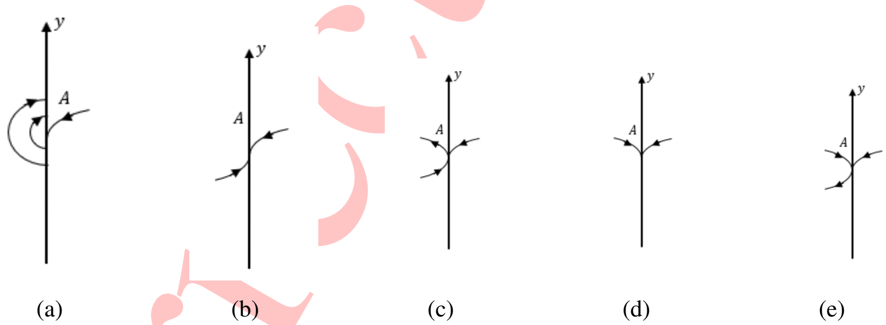


Figure 3: $f^+(A_0;0)=0$, $f^-(A_0;0)=0$, $g^+(A_0;0)<0$, and $f_y^+(A_0;0)\geq 0$

A1. $f^+(A_0;0)=0$, $f^-(A_0;0)=0$, $g^+(A_0;0)>0$, and $f_y^+(A_0;0)\geq 0$, see Figure.1.

A2. $f^+(A_0;0)=0$, $f^-(A_0;0)=0$, $g^+(A_0;0)>0$, and $f_y^+(A_0;0)<0$, see Figure.2.

A3. $f^+(A_0;0)=0$, $f^-(A_0;0)=0$, $g^+(A_0;0)<0$, and $f_y^+(A_0;0)\geq 0$, see Figure.3.

A4. $f^+(A_0;0)=0$, $f^-(A_0;0)=0$, $g^+(A_0;0)>0$, and $f_y^+(A_0;0)<0$, see Figure.4.

A5. $f^+(A_0;0)=0$, $f^-(A_0;0)=0$, $g^+(A_0;0)=0$, and $f_y^+(A_0;0)\geq 0$, see Figure.5.

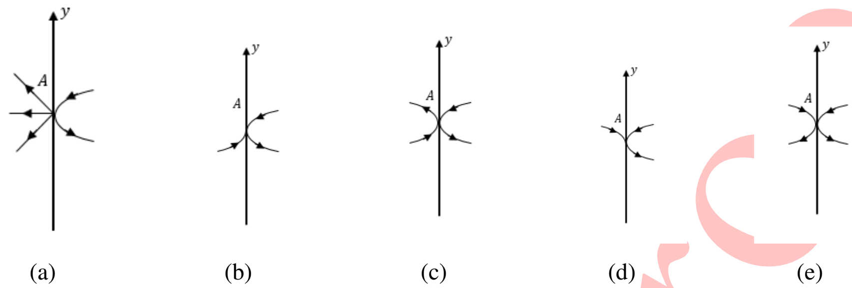


Figure 4: $f^+(A_0;0)=0$, $f^-(A_0;0)=0$, $g^+(A_0;0)<0$, and $f_y^+(A_0;0)<0$

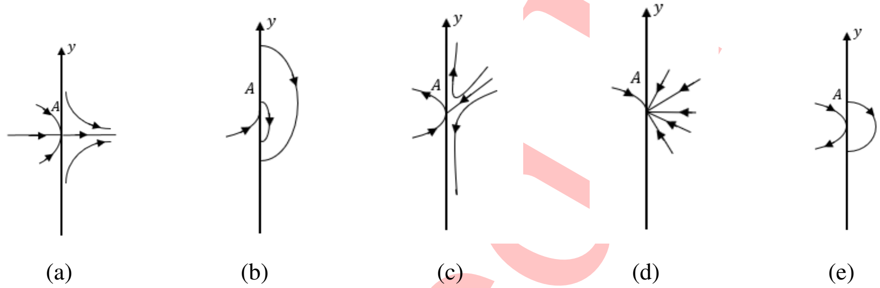


Figure 5: $f^+(A_0;0)=0$, $f^-(A_0;0)=0$, $g^+(A_0;0)>0$, and $f_y^+(A_0;0)\geq 0$

In the Figures 1 -5, when f and g are both zero in either the right or left subsystems, we have a one-sided singular point. The phase portrait of this point is constructed based on the phase portrait of a standard singular point, as described in [10]. In certain cases, generalized singular points can also be singular points of both the right and left subsystems; these may even be of different types. The phase portraits for these cases are constructed based on information from [1,3,9,10].

Now, let us consider the piecewise continuous linear system described below, assuming it has a GSP at $A_0=(0,0)$. We will investigate how introducing the parameter μ affects the type, number, and stability of this GSP. Furthermore, we restrict our analysis to cases where the functions $f^{\pm}$, $g^{\pm}$ are linear.

3 One-dimensional piecewise system

Consider the following one-dimensional linear system:

$$\dot{x}=\begin{cases} f^+(x;\mu)=a^+x+\alpha^+\mu, & x\geq 0,\\ f^-(x;\mu)=a^-x+\alpha^-\mu, & x<0. \end{cases} \tag{6}$$

At the point $A_0=0$, the system exhibits:

$$f^+(0;\mu)=\alpha^+\mu, \quad f^-(0;\mu)=\alpha^-\mu.$$

According to Definition 1 and the sign analysis of $f^{\pm}(0;\mu)$, four distinct cases arise based on the signs of α^+ and α^-. Representative sign tables are provided below; the remaining tables follow analogous patterns and are omitted for brevity.

Table 1: Signs for $\alpha^+ > 0$, $\alpha^- > 0$. No GSP for $\mu \neq 0$

μ	< 0	> 0
f^-	$-$	$+$
f^+	$-$	$+$
f^+f^-	$+$	$+$

Table 2: Signs for $\alpha^+ > 0$, $\alpha^- < 0$. GSP exists for all μ.

μ	< 0	> 0
f^-	$+$	$-$
f^+	$-$	$+$
f^+f^-	$-$	$-$

According to Definition 1 and Table 1, the GSP disappears when the parameter μ is non-zero because $f^+(0;\mu) \times f^-(0;\mu) > 0$.

Table 2 indicate that the GSP $A_0 = 0$ is preserved in the face of changes in μ. In Table 2, the GSP A_0 changes from stable to unstable as μ passes from negative to positive. From this table, it can be inferred that the GSP in the PWS is preserved by using opposite signs for α^+ and α^-.

4 Two-dimensional piecewise system

We now consider the two-dimensional system:

$$\dot{x} = \begin{cases} f^+(x,y;\mu) = a^+x + b^+y + \alpha^+\mu, & x \geq 0, \\ f^-(x,y;\mu) = a^-x + b^-y + \alpha^-\mu, & x < 0, \end{cases} \tag{7}$$

$$\dot{y} = \begin{cases} g^+(x,y;\mu), & x \geq 0, \\ g^-(x,y;\mu), & x < 0. \end{cases} \tag{8}$$

The existence and type of GSP at $A_0 = (0, y_0)$ are determined by the sign of $f^+(0,y;\mu) \cdot f^-(0,y;\mu)$. Let $\mu_1 = -\frac{b^-y}{\alpha^-}$ and $\mu_2 = -\frac{b^+y}{\alpha^+}$ be the roots of f^- and f^+ on the y-axis.

Proposition 1. *Consider the right and left systems (7). If $b^+y = 0$ and $b^-y = 0$, then:*

1. *If α^+ and α^- have the same sign, a GSP of type A exists only at $\mu = 0$ and disappears otherwise.*
2. *If α^+ and α^- have opposite signs, a GSP exists for all μ. Specifically, if $\alpha^+ > 0$ and $\alpha^- < 0$, the GSP is of type G for $\mu < 0$ and type E for $\mu > 0$ (and vice versa).*

Proof. Since $b^+y = 0$ and $b^-y = 0$, we have $\mu_1 = \mu_2 = 0$. The sign of $f^\pm(0,y;\mu)$ depends only on $\alpha^\pm\mu$. The two structurally distinct cases are represented in the following tables.

From Table 3, $f^+f^- > 0$ for $\mu \neq 0$, so no GSP exists except at $\mu = 0$ (type A). From Table 4, $f^+f^- < 0$ for all μ, so a GSP persists: type G ($\mu < 0$) and type E ($\mu > 0$).

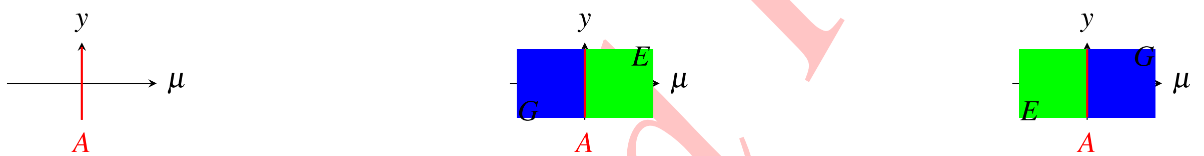
The cases $\alpha^+ < 0, \alpha^- < 0$ and $\alpha^+ < 0, \alpha^- > 0$ yield symmetric results. For brevity, the tables for these results are omitted, as they follow the same sign logic and do not introduce new dynamical configurations.

Table 3: Signs for $\alpha^+ > 0$, $\alpha^- > 0$

μ	< 0	> 0
f^-	$-$	$+$
f^+	$-$	$+$
f^+f^-	$+$	$+$

Table 4: Signs for $\alpha^+ > 0$, $\alpha^- < 0$

μ	< 0	> 0
f^-	$+$	$-$
f^+	$-$	$+$
f^+f^-	$-$	$-$



(a) $\alpha^+.\alpha^- > 0$ (b) $\alpha^+ > 0$ and $\alpha^- < 0$ (c) $\alpha^+ < 0$ and $\alpha^- > 0$

Figure 6: The bifurcation diagram in the case where $b^+ = 0$ and $b^- = 0$

Table 5: Signs for $\alpha^+ > 0$, $\alpha^- > 0$

μ	$< \mu_1$	(μ_1, μ_2)	$> \mu_2$
f^-	$-$	$+$	$+$
f^+	$-$	$-$	$+$
f^+f^-	$+$	$-$	$+$

Figure 6 shows a bifurcation diagram illustrating the relationship between the variables μ and y. By selecting a point on the coordinate axis and observing its color, the type of point can be identified. The area on the left, shaded in green, represents GSPs of type E, while the area on the right, shaded in blue, indicates GSPs of type G. The red vertical line represents GSPs of type A. □

The classification into types E and G follows directly from the sign conditions in Definition 1 and aligns with established conventions in nonsmooth dynamics [2, 3].

Proposition 2. *Consider system (7) with $b^+ = 0$ and $b^- y > 0$. Let $\mu_1 = -\frac{b^- y}{\alpha^-}$, $\mu_2 = 0$.*

1. *If α^+ and α^- have the same sign, a GSP exists only for $\mu \in (\mu_1, \mu_2)$.*
2. *If α^+ and α^- have opposite signs, a GSP exists for $\mu \notin (\mu_1, \mu_2)$.*

At $\mu = \mu_1$, the GSP is of type D; at $\mu = \mu_2$, it is of type B.

Proof. The sign configurations are captured in the following tables.

In Table 5, $f^+f^- < 0$ only in (μ_1, μ_2), so a GSP exists there (type E). In Table 6, $f^+f^- < 0$ for $\mu < \mu_2$ and $\mu > \mu_1$, yielding GSPs of type G and E, respectively.

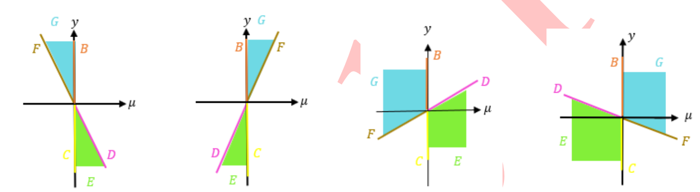
The remaining sign combinations ($\alpha^+ < 0$) are symmetric and omitted for conciseness. □

Table 6: Signs for $\alpha^+ > 0$, $\alpha^- < 0$

μ	$< \mu_2$	(μ_2, μ_1)	$> \mu_1$
f^-	$+$	$+$	$-$
f^+	$-$	$+$	$+$
$f^+ f^-$	$-$	$+$	$-$

Table 7: Signs for $\alpha^+ > 0$, $\alpha^- > 0$

μ	$< \mu_2$	(μ_2, μ_1)	$> \mu_1$
f^-	$-$	$-$	$+$
f^+	$-$	$+$	$+$
$f^+ f^-$	$+$	$-$	$+$



(a) $\alpha^+ > 0$ and $\alpha^- > 0$ (b) $\alpha^+ < 0$ and $\alpha^- < 0$ (c) $\alpha^+ > 0$ and $\alpha^- < 0$ (d) $\alpha^+ < 0$ and $\alpha^- > 0$

Figure 7: $b^- > 0$ and $b^+ = 0$

Proposition 3. *Consider system* (7) *with* $b^+ = 0$ *and* $b^- y < 0$. *Let* $\mu_1 = -\frac{b^- y}{\alpha^-}$, $\mu_2 = 0$.

1. *If* α^+ *and* α^- *have the same sign, a GSP exists only for* $\mu \in (\mu_1, \mu_2)$.
2. *If* α^+ *and* α^- *have opposite signs, a GSP exists for* $\mu \notin (\mu_1, \mu_2)$.

At $\mu = \mu_1$, *the GSP is of type D; at* $\mu = \mu_2$, *it is of type C.*

Proof. When $\alpha^+ > 0, \alpha^- > 0$, Table 7 shows a GSP only in (μ_2, μ_1) (type E). The case $\alpha^+ > 0, \alpha^- < 0$ yields a GSP outside (μ_1, μ_2). All other cases are symmetric and follow the same pattern. □

Figure 7 shows the bifurcation diagram related to Propositions 2 and 3. In Figure 7, the equation of the line FD is $\mu = \dfrac{-b^- y}{\alpha^-}$. In the figure below, items *a* and *b* correspond to Proposition 2, while items *c* and *d* correspond to Proposition 3.

Proposition 4. *Consider system* (7) *with* $b^+ y > 0$ *and* $b^- = 0$. *Let* $\mu_1 = 0$, $\mu_2 = -\frac{b^+ y}{\alpha^+}$.

1. *If* α^+ *and* α^- *have the same sign, a GSP exists only for* $\mu \in (\mu_1, \mu_2)$.
2. *If* α^+ *and* α^- *have opposite signs, a GSP exists for* $\mu \notin (\mu_1, \mu_2)$.

At $\mu = \mu_1$, *the GSP is of type D; at* $\mu = \mu_2$, *it is of type C.*

Proof. Table 8 shows a GSP only in (μ_2, μ_1) (type E). The opposite-sign case is structurally similar to Table 6 and is omitted to avoid redundancy. □

Table 8: Signs for $\alpha^+ > 0$, $\alpha^- > 0$.

μ	$< \mu_2$	(μ_2, μ_1)	$> \mu_1$
f^-	$-$	$-$	$+$
f^+	$-$	$+$	$+$
f^+f^-	$+$	$-$	$+$

Table 9: Signs for $\alpha^+ > 0$, $\alpha^- > 0$

μ	$< \mu_1$	(μ_1, μ_2)	$> \mu_2$
f^-	$-$	$+$	$+$
f^+	$-$	$-$	$+$
f^+f^-	$+$	$-$	$+$

Table 10: Signs for $\alpha^+ > 0$, $\alpha^- < 0$, $\mu_1 < \mu_2$

μ	$< \mu_1$	(μ_1, μ_2)	$> \mu_2$
f^-	$+$	$-$	$-$
f^+	$-$	$-$	$+$
f^+f^-	$-$	$+$	$-$

Table 11: Signs for $\alpha^+ > 0$, $\alpha^- < 0$, $\mu_2 < \mu_1$

μ	$< \mu_2$	(μ_2, μ_1)	$> \mu_1$
f^-	$+$	$+$	$-$
f^+	$-$	$+$	$+$
f^+f^-	$-$	$+$	$-$

Proposition 5. *Consider system* (7) *with* $b^+y > 0$ *and* $b^-y < 0$*. Let* $\mu_1 = -\frac{b^-y}{\alpha^-}$, $\mu_2 = -\frac{b^+y}{\alpha^+}$.

1. *If* α^+ *and* α^- *have the same sign, a GSP exists only for* $\mu \in (\mu_1, \mu_2)$.
2. *If* α^+ *and* α^- *have opposite signs, a GSP exists for* $\mu \notin (\mu_1, \mu_2)$.

The specific GSP types depend on the ordering of μ_1 *and* μ_2.

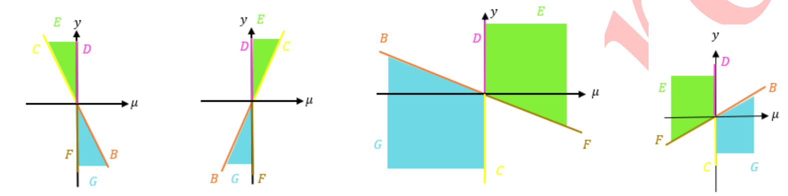
Proof. Table 9 shows a GSP only between μ_1 and μ_2 (type E). Tables 10 and 11 show GSPs outside the interval, with types depending on the order of μ_1, μ_2. Other sign combinations are symmetric and omitted. □

Proposition 6. *Consider system* (7) *with* $b^+y < 0$ *and* $b^- = 0$.
1. If α^+ *and* α^- *have the same signs then for* μ*, which is between* μ_1 *and* μ_2*, the GSP exists and for other values of* μ*, we don't have the GSP. The GSP is type F in* μ_1 *and between* μ_1 *and* μ_2 *is type G and in* μ_2 *is type B.*
2. If α^+ *and* α^- *have different signs then for* μ *that is not between* μ_1 *and* μ_2*, the GSP exists and for* μ*, which is between* μ_1 *and* μ_2*, the GSP does not exist. Here, the GSP is type F in* μ_1*, is type C in* μ_2 *and if* $\alpha^+ > 0$ *for* $\mu < \mu_1$ *is type G and for* $\mu > \mu_2$ *is type E. IF* $\alpha^+ < 0$ *For* $\mu < \mu_2$ *is type E and for* $\mu > \mu_1$ *is type G.*

Proof. The GSP exists only in (μ_1, μ_2) (type G). Other cases are symmetric. □

Table 12: Signs for $\alpha^+ > 0$, $\alpha^- > 0$

μ	$< \mu_1$	(μ_1, μ_2)	$> \mu_2$
f^-	$-$	$+$	$+$
f^+	$-$	$-$	$+$
$f^+ f^-$	$+$	$-$	$+$



(a) $\alpha^+ > 0$ and $\alpha^- > 0$ (b) $\alpha^+ < 0$ and $\alpha^- < 0$ (c) $\alpha^+ > 0$ and $\alpha^- < 0$ (d) $\alpha^+ < 0$ and $\alpha^- > 0$

Figure 8: $b^- = 0$ and $b^+ > 0$

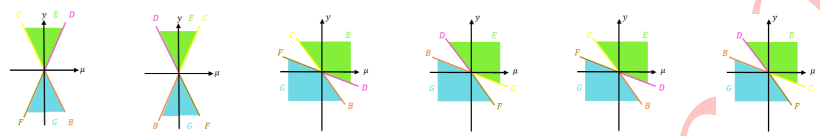
Table 13: Signs for $\alpha^+ > 0$, $\alpha^- < 0$, $\mu_1 < \mu_2$

μ	$< \mu_1$	(μ_1, μ_2)	$> \mu_2$
f^-	$+$	$-$	$-$
f^+	$-$	$-$	$+$
$f^+ f^-$	$-$	$+$	$-$

Proposition 7. *Consider system* (7) *with* $b^+y < 0$ *and* $b^-y > 0$.
1. If α^+ *and* α^- *have the same signs then for* μ, *which is between* μ_1 *and* μ_2, *the GSP exists and for other values of* μ, *we don't have the GSP. The GSP is type F in* μ_1 *and between* μ_1 *and* μ_2 *is type G and in* μ_2 *is type B.*
2. If α^+ *and* α^- *have different signs then for* μ *that is not between* μ_1 *and* μ_2, *the GSP exists and for* μ, *which is between* μ_1 *and* μ_2, *the GSP does not exist.*
If $\alpha^+ > 0$ *and* $\alpha^- < 0$, μ_1 *and* μ_2 *will both be positive. So two states will occur. If* $0 < \mu_1 < \mu_2$, *then the GSP is type F in* μ_1 *and in* μ_2 *is type C and is type G for* $\mu < \mu_1$ *and is type E for* $\mu > \mu_2$. *If* $0 < \mu_2 < \mu_1$, *the GSP is type B in* μ_2 *and in* μ_1 *is type D; and it is type G for* $\mu < \mu_2$ *and is type E for* $\mu > \mu_1$.
If $\alpha^+ < 0$ *and* $\alpha^- > 0$, μ_1 *and* μ_2 *will both be negative. So two states will occur. If* $\mu_1 < \mu_2 < 0$, *then the GSP is type D in* μ_1 *and in* μ_2 *is type B and it is type E for* $\mu < \mu_1$ *and is type G for* $\mu > \mu_2$.
If $\mu_2 < \mu_1 < 0$, *then the GSP is type C in* μ_2 *and in* μ_1 *is type F and it is type E for* $\mu < \mu_2$ *and is type G for* $\mu > \mu_1$.

Proof. The GSP exists outside (μ_1, μ_2). Other subcases are analogous. □

Proposition 8. *Consider system* (7) *with* $b^+y > 0$ *and* $b^-y > 0$.
1. If α^+ *and* α^- *have the same signs then for* μ, *which is between* μ_1 *and* μ_2, *the GSP exists and for other values of* μ, *we don't have the GSP. If* α^+ *and* α^- *are both positive, then* μ_1 *and* μ_2 *are negative. Two states* $\mu_1 < \mu_2$ *and* $\mu_2 < \mu_1$ *occur. If* $\mu_1 < \mu_2$ *then the GSP is type F in* μ_1, *and between* μ_1 *and* μ_2 *is type G and, in* μ_2 *is type B. If* $\mu_2 < \mu_1$, *then the GSP is type C in* μ_2, *and between* μ_1 *and* μ_2 *is type E, and in* μ_1 *is type D.*



(a) $\alpha^+ > 0$ and $\alpha^- > 0$ (b) $\alpha^+ < 0$ and $\alpha^- < 0$ (c) $\alpha^+ > 0$ and $\alpha^- < 0$ (d) $\alpha^+ > 0$ and $\alpha^- < 0$ (e) $\alpha^+ > 0$ and $\alpha^- < 0$ (f) $\alpha^+ > 0$ and $\alpha^- < 0$

Figure 9: $b^- < 0$ and $b^+ > 0$

Table 14: Signs for $\alpha^+ > 0$, $\alpha^- < 0$

μ	$< \mu_2$	(μ_2, μ_1)	$> \mu_1$
f^-	$+$	$+$	$-$
f^+	$-$	$+$	$+$
$f^+ f^-$	$-$	$+$	$-$

If α^+ and α^- are both negative, then μ_1 and μ_2 are positive. Two states $\mu_1 < \mu_2$ and $\mu_2 < \mu_1$ occur. If $\mu_1 < \mu_2$, then the GSP is type D in μ_1, and between μ_1 and μ_2 is type E, and in μ_2 is type C. If $\mu_2 < \mu_1$, then the GSP is type B in μ_2 and between μ_1 and μ_2 is type G and in μ_1 is type F.
2. If α^+ and α^- have different signs for μ that is not between μ_1 and μ_2, the GSP exists and for μ, which is between μ_1 and μ_2, the GSP disappears. If $\alpha^+ > 0$ and $\alpha^- < 0$, the GSP is type G for $\mu < \mu_2$, is type B for $\mu = \mu_2$, is type D for $\mu = \mu_1$ and is type E for $\mu > \mu_1$. And if $\alpha^+ < 0$ and $\alpha^- > 0$, the GSP is type E for $\mu < \mu_1$, is type D for $\mu = \mu_1$, is type D for $\mu = \mu_2$, and is type G for $\mu > \mu_2$.

Proof. The GSP exists for $\mu < \mu_2$ (type G) and $\mu > \mu_1$ (type E). Remaining cases follow the same logic. □

Proposition 9. *Consider system (7) with $b^+ y < 0$ and $b^- y < 0$.*
1. If α^+ and α^- have the same signs then for μ, which is between μ_1 and μ_2, the GSP exists and for other values of μ, we don't have the GSP. If α^+ and α^- are both positive, then μ_1 and μ_2 are positive. Two states $0 < \mu_1 < \mu_2$ and $0 < \mu_2 < \mu_1$ occur. If $0 < \mu_1 < \mu_2$ then the GSP is type F in μ_1, and between μ_1 and μ_2 is type G and, in μ_2 is type B. If $0 < \mu_2 < \mu_1$, then the GSP is type C in μ_2, and between μ_1 and μ_2 is type E, and in μ_1 is type D.

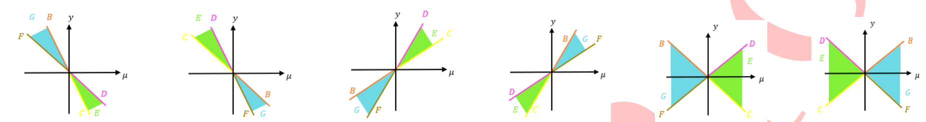
If α^+ and α^- are both negative, then μ_1 and μ_2 are negative. Two states $\mu_1 < \mu_2$ and $\mu_2 < \mu_1$ occur. If $\mu_1 < \mu_2 < 0$, then the GSP is type D in μ_1, and between μ_1 and μ_2 is type E, and in μ_2 is type C. If $\mu_2 < \mu_1$, then the GSP is type B in μ_2 and between μ_1 and μ_2 is type G and in μ_1 is type F.
2. If α^+ and α^- have different signs for μ that is not between μ_1 and μ_2, the GSP exists and for μ, which is between μ_1 and μ_2, the GSP disappears. If $\alpha^+ > 0$ and $\alpha^- < 0$, the GSP is type G for $\mu < \mu_1$, is type F for $\mu = \mu_1$, is type C for $\mu = \mu_2$ and is type E for $\mu > \mu_2$. And if $\alpha^+ < 0$ and $\alpha^- > 0$, the GSP is type E for $\mu < \mu_2$, is type C for $\mu = \mu_2$, is type F for $\mu = \mu_1$, and is type G for $\mu > \mu_1$.

Proof. The GSP exists for $\mu < \mu_1$ (type G) and $\mu > \mu_2$ (type E). Other configurations are symmetric and omitted for brevity. □

In the following two remarks, some points about the first components, ($\dot{x}$), of the two-dimensional piecewise linear system are considered.

Table 15: Signs for $\alpha^+ > 0$, $\alpha^- < 0$

μ	$< \mu_1$	(μ_1, μ_2)	$> \mu_2$
f^-	$+$	$-$	$-$
f^+	$-$	$-$	$+$
$f^+ f^-$	$-$	$+$	$-$



(a) $\alpha^+ > 0$ and $\alpha^- > 0$ (b) $\alpha^+ > 0$ and $\alpha^- > 0$ (c) $\alpha^+ < 0$ and $\alpha^- < 0$ (d) $\alpha^+ < 0$ and $\alpha^- < 0$ (e) $\alpha^+ > 0$ and $\alpha^- < 0$ (f) $\alpha^+ < 0$ and $\alpha^- > 0$

Figure 10: $b^- > 0$ and $b^+ > 0$

Remark 1. *The results presented in this section remain valid beyond the linear case. Specifically, they extend to nonlinear systems of the form*

$$\dot{x} = \begin{cases} f^+(x,y;\mu) = h(x,y) + i(y) + \alpha^+ \mu, & x \geq 0, \\ f^-(x,y;\mu) = h(x,y) + i(y) + \alpha^- \mu, & x < 0, \end{cases} \tag{9}$$

provided that $h(0,y) = 0$. This condition ensures that the structure of the discontinuity and the dependence on μ are preserved along the switching manifold.

Remark 2. *All the above propositions also hold for systems where the vector field admits a Taylor expansion near the origin, namely*

$$f^+(x,y;\mu) = a^+ x + b^+ y + \alpha^+ \mu + O(x^2, y^2),$$

and

$$f^-(x,y;\mu) = a^- x + b^- y + \alpha^- \mu + O(x^2, y^2),$$

where the perturbation is confined to the terms $\alpha^\pm \mu$. In this setting, the critical parameter values are given by

$$\mu_1 = \frac{-f^+(0,y;0)}{\alpha^+}, \qquad \mu_2 = \frac{-f^-(0,y;0)}{\alpha^-},$$

and the analysis proceeds analogously to the linear case, as higher-order terms do not affect the sign of $f^\pm(0,y;\mu)$ in a sufficiently small neighborhood of $\mu = 0$.

5 Wien bridge oscillator

The Wien bridge oscillator is a classic electronic circuit that generates sinusoidal waveforms and is widely used across a broad frequency spectrum. It finds applications in precise capacitance measurements (in relation to resistance and frequency) and in audio-frequency signal generation.

The circuit comprises four resistors (R_f, R_2, R_1, R_s) and two capacitors (C_1, C_2). While the ideal model exhibits symmetry about the origin, practical implementations often introduce asymmetries. Our

Table 16: Variable definitions for the Wien bridge oscillator

Physical quantity	Symbol
Voltage across capacitor C_1	V_{C_1}
Voltage across capacitor C_2	V_{C_2}
Op-amp output voltage	$V_0 = f(V_{C_2})$

framework for analyzing generalized singular points (GSPs) in piecewise smooth systems is well-suited to study such asymmetric variants. Here, we consider a modified Wien bridge circuit with an additional bias voltage E_B.

The governing equations for the capacitor voltages V_{C_1} and V_{C_2} are

$$\begin{aligned} R_1 C_1 \dot{V}_{C_1} &= -V_{C_1} - V_{C_2} + V_0, \\ C_1 \dot{V}_{C_1} - C_2 \dot{V}_{C_2} &= \frac{V_{C_2} - E_B}{R_2}, \end{aligned} \tag{10}$$

where $V_0 = f(V_{C_2})$ is the output voltage of the operational amplifier.

Following Kriegsmann's piecewise-linear model [4], the amplifier characteristic is:

$$f(V_{C_2}) = \begin{cases} -E, & \text{if } \alpha V_{C_2} < -E, \\ \alpha V_{C_2}, & \text{if } |\alpha V_{C_2}| \le E, \\ E, & \text{if } \alpha V_{C_2} > E. \end{cases} \tag{11}$$

Introducing the dimensionless variables

$$X := \alpha \frac{V_{C_2}}{E}, \quad Y := \alpha \frac{V_{C_1}}{E}, \quad x_B := \alpha \frac{E_B}{E}, \tag{12}$$

with $E > 0$, system (10) transforms into the following planar piecewise-smooth system:

$$\begin{aligned} \dot{X} &= f(X) - Y, \\ \dot{Y} &= g(X) - \delta, \end{aligned} \tag{13}$$

where

$$f(X) = \begin{cases} t(X+1) - T, & X < -1, \\ TX, & |X| \le 1, \\ t(X-1) + T, & X > 1, \end{cases} \qquad g(X) = \begin{cases} d(X+1) - d, & X < -1, \\ dX, & |X| \le 1, \\ d(X-1) + d, & X > 1, \end{cases}$$

and the parameters are defined as

$$\begin{aligned} t &= -\left(\frac{1}{R_1 C_1} + \frac{1}{R_1 C_2} + \frac{1}{R_2 C_2} \right) < 0, \\ T &= t + \frac{\alpha}{R_1 C_2}, \\ d &= \frac{1}{R_1 R_2 C_1 C_2} > 0, \quad \delta = d x_B. \end{aligned} \tag{14}$$

Table 17: Sign table for $y > 0$ (Case 1)

μ	$< \mu_2$	(μ_2, μ_1)	$> \mu_1$
f^-	$-$	$-$	$+$
f^+	$+$	$-$	$-$
f^+f^-	$-$	$+$	$-$

This system has two switching boundaries at $X = \pm 1$. If one shifting coordinates to each boundary (e.g., $x = X + 1$, $y = Y - T$ near $X = -1$) then for the left boundary ($X = -1$), the transformed system is

$$\dot{x} = \begin{cases} tx - y, & x < 0, \\ Tx - y, & x \geq 0, \end{cases} \qquad \dot{y} = \begin{cases} dx - d(1 + x_B), & x < 0, \\ dx - d(1 + x_B), & x \geq 0, \end{cases} \tag{15}$$

and similarly, for the right boundary ($X = 1$), we obtain

$$\dot{x} = \begin{cases} Tx - y, & x < 0, \\ tx - y, & x \geq 0, \end{cases} \qquad \dot{y} = \begin{cases} dx + d(1 - x_B), & x < 0, \\ dx + d(1 - x_B), & x \geq 0. \end{cases} \tag{16}$$

In both cases, $f^+(0,y)f^-(0,y) > 0$ when $\mu = 0$, so no GSP exists on the switching manifold. To investigate how a perturbation parameter μ affects the existence of GSPs, we introduce μ into the vector field as follows.

Case 1: Perturbed system near $X = -1$:

$$\dot{x} = \begin{cases} tx - y + \mu, & x < 0, \\ Tx - y - \mu, & x \geq 0, \end{cases} \qquad \dot{y} = \begin{cases} dx - d(1 + x_B), & x < 0, \\ dx - d(1 + x_B), & x \geq 0. \end{cases} \tag{17}$$

On the switching line $x = 0$, we have

$$f^-(0,y;\mu) = -y + \mu, \quad f^+(0,y;\mu) = -y - \mu.$$

Thus, $\mu_1 = y$ and $\mu_2 = -y$. Assuming $|x_B| < 1$, we have $g^{\pm}(0,y;\mu) < 0$.

For $y > 0$, we have $f^-(0,y;0) < 0$, $f^+(0,y;0) < 0$, with $\alpha^- = +1 > 0$ and $\alpha^+ = -1 < 0$. This corresponds to Proposition 9 (since $b^+ < 0$, $b^- < 0$). The sign configuration is

For $y < 0$, the situation aligns with Proposition 8, yielding

The corresponding bifurcation diagram is shown in Figure 11.

Table 18: Sign table for $y < 0$ (Case 1)

μ	$< \mu_1$	(μ_1, μ_2)	$> \mu_2$
f^-	$-$	$+$	$+$
f^+	$+$	$+$	$-$
$f^+ f^-$	$-$	$+$	$-$

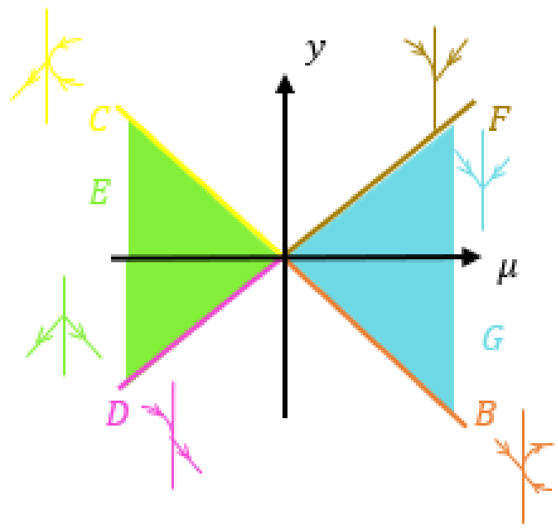


Figure 11: Bifurcation diagram for system (17). The lines $\mu = y$ and $\mu = -y$ divide the (y, μ)-plane into regions with distinct GSP types. Here, $b^+ < 0$ and $b^- < 0$.

Table 19: Sign table for $y > 0$ (Case 2)

μ	$< \mu_2$	(μ_2, μ_1)	$> \mu_1$
f^-	$+$	$+$	$-$
f^+	$+$	$-$	$-$
$f^+ f^-$	$+$	$-$	$+$

Case 2: Perturbed system near $X = 1$:

$$\dot{x} = \begin{cases} Tx - y - 3\mu, & x < 0, \\ tx - y - \mu, & x \geq 0, \end{cases}$$
$$\dot{y} = \begin{cases} dx + d(1 - x_B), & x < 0, \\ dx + d(1 - x_B), & x \geq 0. \end{cases} \tag{18}$$

Here

$$f^-(0, y; \mu) = -y - 3\mu, \quad f^+(0, y; \mu) = -y - \mu,$$

so $\mu_1 = -y/3$ and $\mu_2 = -y$.

For $y > 0$, both $\alpha^- = -3 < 0$ and $\alpha^+ = -1 < 0$, which corresponds to Proposition 9. The sign table is:

Table 20: Sign table for $y < 0$ (Case 2)

μ	$< \mu_1$	(μ_1, μ_2)	$> \mu_2$
f^-	$+$	$-$	$-$
f^+	$+$	$+$	$-$
f^+f^-	$+$	$-$	$+$

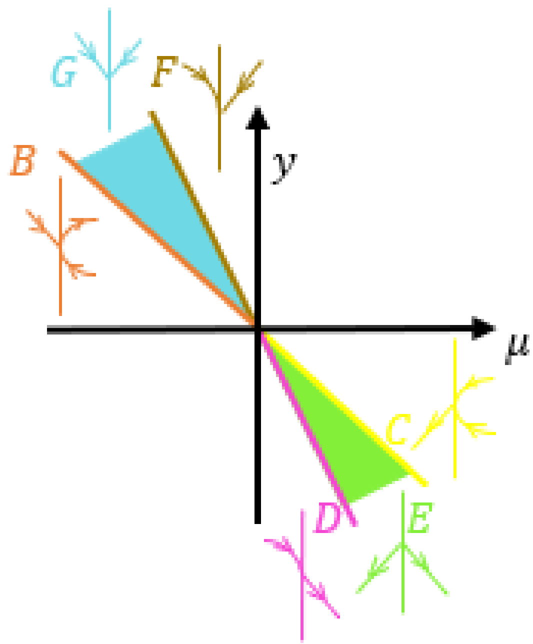


Figure 12: Bifurcation diagram for system (18). The lines $\mu = -y/3$ and $\mu = -y$ partition the parameter space. Again, $b^+ < 0$ and $b^- < 0$.

For $y < 0$, the configuration matches Proposition 8, with the bifurcation diagram is shown in Figure 12. These examples demonstrate how our classification framework can be applied to real-world nonsmooth systems, revealing the rich bifurcation structure induced by small perturbations.

6 Conclusion

This paper investigated the effect of a perturbation parameter μ on generalized singular points in linear piecewise systems. We analyzed the presence or absence of singular points when $\mu = 0$ versus $\mu \neq 0$, and derived conditions for the persistence or annihilation of GSPs under parameter variation. The detailed classification provides a foundation for understanding bifurcations in nonsmooth systems, including sliding and crossing transitions.

Future work may extend this classification to higher-dimensional systems, non-planar discontinuity manifolds, or systems with multiple switching boundaries. Additionally, connecting this framework to control applications involving sliding modes could yield practical insights.

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