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Generalized Reynolds operators and extensions of Lie-Yamaguti algebra bundle

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Abstract. A Lie-Yamaguti algebra bundle is a type of algebra bundles with fibres being Lie-Yamaguti algebras, and appears naturally from geometric considerations in the work of M. Kikkawa. The aim of the present paper is to introduce the notion of generalized Reynolds operators, $\mathcal{O}$-operators and Nijenhuis operators in the context of Lie-Yamaguti algebra bundle and find their applications. We also study abelian extensions of Lie-Yamaguti algebra bundles and investigate its relationship with its cohomology.

Keywords: Vector bundle, Lie-Yamaguti algebra, Non-associative algebra, Cohomology.
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1 Introduction

Algebra bundles are vector bundles with fibres a type of algebras, and they play a crucial role in geometry and physics. For example, associative algebra bundles [6], Lie algebra bundles [14,16], etc. A Lie-Yamaguti algebra bundle is a type of algebra bundles with fibres being Lie-Yamaguti algebras, and appears naturally from geometric considerations in the work of M. Kikkawa. In [10] we introduced Lie-Yamaguti algebra bundles and defined cohomology groups of a Lie-Yamaguti algebra bundle with coefficients in a representation.

Various notions of operators, like, Rota-Baxter operators, $\mathcal{O}$-operators and generalized Reynolds operators have been studied on a large class of algebras in order to address problems arising from mathematical physics. In 1960, G. Baxter first introduced the notion of Rota-Baxter operators

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for associative algebras. A. Das [8] studied deformations of associative Rota-Baxter operators. The Rota-Baxter operators have several applications in probability [1], combinatorics [2, 11, 19] and quantum field theory [5]. In the 1980s, the notion of Rota-Baxter operator of weight 0 was introduced in terms of the classical Yang-Baxter equation for Lie algebras. Later on, B. A. Kupershmidt [15] defined the notion of $\mathcal{O}$-operators as generalized Rota-Baxter operators to understand classical Yang-Baxter equations and related integrable systems. In [17], the authors studied $\mathcal{O}$-operators on hom-Lie algebras. Recently, in [4], the authors studied deformations of $\mathcal{O}$-operators on Lie triple systems. See [20], for Lie-Yamaguti algebra case. Reynolds numbers were introduced by O. Reynolds [18] in his study of fluctuation theory in fluid dynamics to classify fluid flow. In [13], Kampé de Fériet and S. I. Pai coined the concept of the Reynolds operator as a mathematical object in general. Generalized Reynolds operators (also called twisted Rota-Baxter operators) are algebraic analogue of twisted Poisson structure and was introduced by K. Uchino [23] in the context of associative algebras (see [7] for the Lie algebra case). The notion of a Nijenhuis operator on a Lie algebra was used in [9] to characterize infinitesimal deformations which are trivial. We refer [21, 22], for similar study in the context of Lie-Yamaguti algebras. It is then natural to investigate such operators in the context of Lie-Yamaguti algebra bundles.

In the present article, we introduce generalized Reynolds operators, $\mathcal{O}$-operators and Nijenhuis operators on Lie-Yamaguti algebra bundle and discuss their utility. We also introduce abelian extensions of Lie-Yamaguti algebra bundles, and investigate their relationship with suitable cohomology group, generalizing the work of M. Kikkawa for Lie-Yamaguti algebras.

Organization of the paper: In §2, we set up notations, recall some known definitions and results. In §3, we introduce the notion of $\mathcal{O}$-operator, generalized Reynolds operator and Nijenhuis operator on a Lie-Yamaguti algebra bundle and show that they give rise to new Lie-Yamaguti algebra bundle out of the given one. Finally, in §4, we study (abelian) extensions of Lie-Yamaguti algebra bundles and establish its connection to cohomology.

2 Preliminaries

The aim of this section is to recall some basic definitions and set up notations to be followed throughout the paper. Let $\mathbb{K}$ be a given field.

Definition 1. *A Lie-Yamaguti Algebra* $(\mathfrak{g}, [\ ,\ ], \{\ ,\ ,\ \})$ *is a vector space* $\mathfrak{g}$ *equipped with a* $\mathbb{K}$*-bilinear and a trilinear operation*

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \quad \text{and} \quad \{\ ,\ ,\ \} : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

such that for all $x, y, z, u, v, w \in \mathfrak{g}$ *the following relations hold:*

$$[x, y] = -[y, x]; \tag{LY1}$$

$$\{x, y, z\} = -\{y, x, z\}; \tag{LY2}$$

$$\Sigma_{\circlearrowleft(x,y,z)}([[x, y], z] + \{x, y, z\}) = 0; \tag{LY3}$$

$$\Sigma_{\circlearrowleft(x,y,z)}\{[x,y],z,u\}=0; \tag{LY4}$$

$$\{x,y,[u,v]\}=[\{x,y,u\},v]+[u,\{x,y,v\}]; \tag{LY5}$$

$$\{x,y,\{u,v,w\}\}=\{\{x,y,u\},v,w\}+\{u,\{x,y,v\},w\}+\{u,v,\{x,y,w\}\}. \tag{LY6}$$

Here, $\Sigma_{\circlearrowleft(x,y,z)}$ denotes the sum over cyclic permutations of x, y, and z.

Let M be a smooth manifold (Hausdorff and second countable, hence, paracompact). Let $C^\infty(M)$ be the algebra of smooth functions on M. For a (smooth) vector bundle $p: L \to M$, often denoted by $\xi = (L, p, M)$, we denote the space of smooth sections of L by ΓL. It is well-known that ΓL is a $C^\infty(M)$-module. For any $m \in M$, we denote the fibre of the vector bundle ξ over m by L_m or sometimes by ξ_m. Henceforth, we will work in the smooth category and with $\mathbb{K} = \mathbb{R}$.

Next, we recall the notion of a Lie-Yamaguti algebra bundle, and its associated cohomology groups.

Definition 2. *Let $\xi = (L, p, M)$ be a (real) vector bundle and $Hom(\xi^{\otimes k}, \xi)$ be the real vector space of vector bundle maps from $\xi^{\otimes k}$ to the vector bundle ξ, $k \geq 1$. Observe that $Hom(\xi^{\otimes k}, \xi)$ is a vector bundle over M. Let $\langle\ ,\cdots,\ \rangle$ be a section of the bundle $Hom(\xi^{\otimes k}, \xi)$. We call such a section a k-field of ($\mathbb{K}$-multilinear) brackets in ξ. Thus, a k-field of brackets in ξ is a smooth assignment*

$$m \mapsto (\langle\ ,\cdots,\ \rangle_m : \xi_m \times \cdots \times \xi_m \to \xi_m)$$

of multilinear operation on ξ_m, $m \in M$.

Definition 3. *A Lie-Yamaguti algebra bundle is a vector bundle $\xi = (L, p, M)$ together with a 2-field and a 3-field of brackets*

$$m \mapsto [\ ,\ ]_m \quad and \quad m \mapsto \{\ ,\ ,\ \}_m, \quad m \in M$$

which make each fibre ξ_m, $m \in M$ a Lie-Yamaguti algebra.

Definition 4. *Let $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}}, \{\ ,\ ,\ \}_{\mathfrak{g}})$ be a given Lie-Yamaguti algebra. A locally trivial Lie-Yamaguti algebra bundle is a vector bundle $\xi = (L, p, M)$ together with a 2-field and a 3-field of brackets*

$$m \mapsto [\ ,\ ]_m \quad and \quad m \mapsto \{\ ,\ ,\ \}_m, \quad m \in M$$

such that M admits an open covering $\{U_i\}$ equipped with local trivializations $\{\psi_i : U_i \times \mathfrak{g} \to p^{(-1)}(U_i)\}$ for which each $\psi_{i,m}$, $m \in M$ (ψ_i restricted to each fibre ξ_m) is a Lie-Yamaguti algebra isomorphism.

Remark 1. *Thus, for a Lie-Yamaguti algebra bundle as defined above each fibre $\xi_m = p^{-1}(m)$, $m \in M$, together with the binary operation $[\ ,\ ]_m$ and the ternary operation $\{\ ,\ ,\ \}_m$ is a Lie-Yamaguti algebra isomorphic to $\mathfrak{g}$, and the assignments*

$$m \mapsto [\ ,\ ]_m, \quad m \mapsto \{\ ,\ ,\ \}_m$$

varies smoothly over M.

In other words, a locally trivial Lie-Yamaguti algebra bundle over M is a vector bundle over M such that each fibre of the bundle has a Lie-Yamaguti algebra structure isomorphic to $\mathfrak{g}$. For more details about Lie-Yamaguti algebra bundles and examples, we refer the readers to [10].

Definition 5. *Let $\xi = (L, p, M)$ and $\xi' = (L', p', M')$ be two Lie-Yamaguti algebra bundles. A homomorphism $\phi : \xi \to \xi'$ is a vector bundle morphism $(\tilde{\phi}, \phi)$, where $\tilde{\phi} : L \to L'$ is the morphism between total spaces and $\phi : M \to M'$ is a morphism between base spaces such that $\tilde{\phi}|_{L_m} : L_m \to L'_{\phi(m)}$ is a Lie-Yamaguti algebra homomorphism for any $m \in M$.*

A homomorphism $\phi : \xi \to \xi'$ of two Lie-Yamaguti algebra bundles over the same base space M is a vector bundle morphism $\phi : \xi \to \xi'$ such that $\phi|_{\xi_m} : \xi_m \to \xi'_m$ is a Lie-Yamaguti algebra morphism for all $m \in M$. Moreover, if $\phi|_{\xi_m}$ is a linear bijection then $\xi = (L, p, M)$ is said to be isomorphic to $\xi' = (L', p', M)$.

Definition 6. *A Lie-Yamaguti algebra bundle ξ is said to be trivial if it is isomorphic to a product Lie-Yamaguti algebra bundle.*

Next, we recall from [10, Definition 4.1, pg 16] the notion of representation of Lie-Yamaguti algebra bundles.

Definition 7. *Let $\xi = (L, p, M)$ be a Lie-Yamaguti algebra bundle and $\eta = (E, q, M)$ be a vector bundle. A representation of ξ on the vector bundle η consists of vector bundle morphisms*

$$\rho : \xi \to End(\eta), \quad D,\ \theta : \xi \otimes \xi \to End(\eta)$$

such that these maps restricted to each fibre

$$\rho_m = \rho|_{\xi_m} : \xi_m \to End(\eta_m),\ D_m = D|_{\xi_m},\ \theta_m = \theta|_{\xi_m} : \xi_m \times \xi_m \to End(\eta_m),$$

satisfy the conditions

$$D_m(a,b) + \theta_m(a,b) - \theta_m(b,a) = [\rho_m(a), \rho_m(b)]_m - \rho_m([a,b]); \tag{RLYB1}$$
$$\theta_m(a,[b,c]_m) - \rho_m(b)\theta_m(a,c) + \rho_m(c)\theta_m(a,b) = 0; \tag{RLYB2}$$
$$\theta_m([a,b]_m,c) - \theta_m(a,c)\rho_m(b) + \theta_m(b,c)\rho_m(a) = 0; \tag{RLYB3}$$
$$\theta_m(c,d)\theta_m(a,b) - \theta_m(b,d)\theta_m(a,c) - \theta_m(a,\{b,c,d\}_m) + D_m(b,c)\theta_m(a,d) = 0; \tag{RLYB4}$$
$$[D_m(a,b), \rho_m(c)]_m = \rho_m(\{a,b,c\}_m); \tag{RLYB5}$$
$$[D_m(a,b), \theta_m(c,d)]_m = \theta_m(\{a,b,c\}_m, d) + \theta_m(c, \{a,b,d\}_m); \tag{RLYB6}$$

for any $m \in M$ and $a, b, c, d \in \xi_m$.

We shall denote a representation of a Lie-Yamaguti algebra bundle ξ on a vector bundle η as described above by $(\eta;\ \rho,\ D,\ \theta)$. A representation $(\eta;\ \rho,\ D,\ \theta)$ of a Lie-Yamaguti algebra bundle ξ is also called a ξ-module.

Remark 2. *Like a representation of a Lie-Yamaguti algebra [24], given a representation $(\eta;\ \rho,\ D,\ \theta)$ of a Lie-Yamaguti algebra bundle ξ, we have for every $m \in M$*

$$D_m([a,b]_m, c) + D_m([b,c]_m, a) + D_m([c,a]_m, b) = 0, \tag{RLYB7}$$

for any $a,\ b,\ c \in \xi_m$.

Example 1. Given a Lie-Yamaguti algebra bundle ξ over M, we may consider ξ as a ξ-module which gives us the adjoint representation of ξ on itself. Explicitly, for each $m \in M$, $\rho_m,\ D_m,\ \theta_m$ are given by

$$\rho_m(a) : b \mapsto [a,b]_m; \quad D_m(a,b) : c \mapsto \{a,b,c\}_m; \quad \theta_m(a,b) : c \mapsto \{c,a,b\}_m,$$

for any $a,\ b,\ c \in \xi_m$.

Given a Lie-Yamaguti algebra bundle together with a representation we construct a new Lie-Yamaguti algebra bundle as follows.

Example 2. Let $\xi = (L,p,M)$ be a given Lie-Yamaguti algebra bundle and let $(\eta;\ \rho,\ D,\ \theta)$ be a representation of ξ. Then, $\xi \oplus \eta$ becomes a Lie-Yamaguti algebra bundle with respect to the following 2 and 3-fields of brackets

$$[x+u, y+v]_m^{\ltimes} := [x,y]_m + \rho_m(x)v - \rho_m(y)u \tag{1}$$

$$\{x+u, y+v, z+w\}_m^{\ltimes} := \{x,y,z\}_m + D_m(x,y)w - \theta_m(y,z)u \tag{2}$$

for all $x,y,z \in \xi_m$ and $u,v,w \in \eta_m$. This bundle is called the semi-direct product bundle of ξ and η and is denoted by $\xi \ltimes \eta$.

Moreover, a representation of η of a Lie-Yamaguti algebra bundle ξ is characterized by the semi-direct product construction in the following sense.

Proposition 1. *Let $\xi = (L,p,M)$ be a given Lie-Yamaguti algebra bundle and $\eta = (E,q,M)$ be a vector bundle together with vector bundle morphisms $\rho : \xi \to End(\eta)$, $D,\ \theta : \xi \otimes \xi \to End(\eta)$. Then, $(\eta;\ \rho,\ D,\ \theta)$ is a representation of ξ if and only if the Whitney sum bundle $\xi \oplus \eta$ becomes a Lie-Yamaguti algebra bundle with respect to the following 2-fields and 3-fields of brackets*

$$[x+u, y+v]_m^{\ltimes} := [x,y]_m + \rho_m(x)v - \rho_m(y)u \tag{3}$$

$$\{x+u, y+v, z+w\}_m^{\ltimes} := \{x,y,z\}_m + D_m(x,y)w - \theta_m(y,z)u \tag{4}$$

for all $x,y,z \in \xi_m$ and $u,v,w \in \eta_m$.

We now briefly recall from [10] the definition of cohomology groups of a Lie-Yamaguti algebra bundle with coefficients in a given representation.

Definition 8. *Let $\xi = (L,p,M)$ be a Lie-Yamaguti algebra bundle and $(\eta;\rho,D,\theta)$ be a ξ-module. Let $C^n(\xi;\eta)$ be the space of all vector bundle morphisms $f : \xi^{\otimes n} \to \eta$ such that for each $m \in M$ it satisfies $f_m(x_1,\dots,x_{2i-1},x_{2i},\dots,x_n) = 0$ whenever $x_{2i-1} = x_{2i}$, where $i = 1,\dots,[n/2]$. Let $C^0(\xi;\eta)$ be the subspace spanned by the diagonal elements $(f,f) \in C^1(\xi;\eta) \times C^1(\xi,\eta)$ and for $p \geq 1$, set*

$$C^{(2p,2p+1)}(\xi;\eta) := C^{2p}(\xi;\eta) \times C^{2p+1}(\xi;\eta).$$

Any element $(f,g) \in C^{(2p,2p+1)}(\xi;\eta)$ is called a $(2p,2p+1)$-cochain. For $p \geq 1$, we have a coboundary operator

$$\delta = (\delta_I, \delta_{II}) : C^{(2p,2p+1)}(\xi;\eta) \longrightarrow C^{(2p+2,2p+3)}(\xi;\eta).$$

Additionally, for $p = 1$ we have another operator

$$\delta^* = (\delta_I^*, \delta_{II}^*) : C^{(2,3)}(\xi;\eta) \longrightarrow C^{(3,4)}(\xi;\eta).$$

Furthermore, there also exist an operator $\delta : C^0(\xi;\eta) \to C^{(2,3)}(\xi;\eta)$. For an explicit description of the operators δ and δ^* we refer the readers to [10, Definition 5.1, pg 18]. Thus, we obtain a cochain complex depicted as follows (cf [10]):

$$\begin{array}{ccccccc} C^0(\xi;\eta) & \xrightarrow{\delta} & C^{(2,3)}(\xi;\eta) & \xrightarrow{\delta} & C^{(4,5)}(\xi;\eta) & \xrightarrow{\delta} & \cdots \\ & & \big\downarrow{\scriptstyle \delta^*} & & & & \\ & & C^{(3,4)}(\xi;\eta) & & & & \end{array}$$

Definition 9. *Let ξ be a Lie-Yamaguti algebra bundle and η be a representation of ξ. For $p \geq 2$, define the $(2p, 2p+1)$-cohomology group of ξ with coefficients in η as follows:*

$$H^{(2p,2p+1)}(\xi;\eta) := \frac{Z^{(2p,2p+1)}(\xi;\eta)}{B^{(2p,2p+1)}(\xi;\eta)},$$

where $Z^{(2p,2p+1)}(\xi;\eta) = \mathrm{Ker}(\delta)$ and $B^{(2p,2p+1)}(\xi;\eta) = \mathrm{Im}(\delta)$. For $p = 1$

$$H^{(2,3)}(\xi;\eta) = \frac{Z^{(2,3)}(\xi;\eta)}{B^{(2,3)}(\xi;\eta)},$$

where $Z^{(2,3)}(\xi;\eta) := \mathrm{Ker}(\delta) \cap \mathrm{Ker}(\delta^*)$ and $B^{(2,3)}(\xi;\eta) = \{\delta(f,f) | f \in C^1(\xi;\eta)\}$.

Let $\xi = (L, p, M)$ be a Lie-Yamaguti algebra bundle and $(\eta; \rho, D, \theta)$ be a ξ-module. Also, let $\tau = (f, g) \in Z^{(2,3)}(\xi;\eta)$ be a given $(2,3)$-cocycle. Then, one can construct a new Lie-Yamaguti algebra bundle as described below.

Example 3. Consider the vector bundle $\xi \oplus \eta$ and define a 2-field of brackets and a 3-field of brackets as follows: For any $m \in M$

$$[x+u, y+v]_m^\tau := [x,y]_m + \rho_m(x)v - \rho_m(y)u + f_m(x,y) \tag{5}$$

$$\{x+u, y+v, z+w\}^\tau := \{x,y,z\}_m + D_m(x,y)w - \theta_m(y,z)u + g_m(x,y,z) \tag{6}$$

for all $x, y, z \in \xi_m$ and $u, v, w \in \eta_m$. Then, using the fact that τ is a cocycle it can be checked that equipped with these fields of brackets the bundle $\xi \oplus \eta$ becomes a Lie-Yamaguti algebra bundle. We call this new Lie-Yamaguti algebra bundle the twisted semi-direct product of ξ and η with respect to $\tau = (f, g)$, and is denoted by $\xi \ltimes_\tau \eta$.

3 Generalized Reynolds operators on Lie-Yamaguti algebra bundle

The aim of this section is to introduce generalized Reynolds operators on a Lie-Yamaguti algebra bundle and show that they are connected to the twisted semi-direct product bundle as defined in Example 3. We obtain few special classes of generalized Reynolds operators on a Lie-Yamaguti algebra bundle which are provided by Reynolds operators, $\mathcal{O}$-operators and Rota-Baxter operators on a Lie-Yamaguti algebra bundle. The results of this section generalize existing results related to such operators in the context of a type of algebra, for example, Lie algebra, Lie triple system or Lie-Yamaguti algebra.

Let $\xi = (L, p, M)$ be a Lie-Yamaguti algebra bundle and $(\eta; \rho, D, \theta)$ be a ξ-module. Also, let $\tau = (f, g) \in Z^{(2,3)}(\xi; \eta)$ be a given $(2, 3)$-cocycle.

Definition 10. *A vector bundle morphism $R : \eta \to \xi$ is said to be a generalized Reynolds operator if for each $m \in M$ the restriction $R_m : \eta_m \to \xi_m$ of R satisfies the following:*

$$[R_m u, R_m v]_m = R_m\Big(\rho_m(R_m u)(v) - \rho_m(R_m v)(u) + f_m(R_m u, R_m v)\Big)$$

$$\begin{aligned}\{R_m u, R_m v, R_m w\}_m = R_m\Big(&D_m(R_m u, R_m v)w + \theta_m(R_m v, R_m w)u \\ &-\theta_m(R_m u, R_m w)v + g_m(R_m u, R_m v, R_m w)\Big)\end{aligned}$$

for all $u, v, w \in \eta_m$.

Remark 3. *In the above definition if we take η as the adjoint representation ξ (cf. Example 1) and $\tau = (-[\ ,\ ], -\{\ ,\ ,\ \})$ then, we obtain the definition of a Reynolds operator on ξ. as defined below.*

Definition 11. *A vector bundle morphism $R : \xi \to \xi$ is called a Reynolds operator if R restricted to each fiber, $R_m : \xi_m \to \xi_m$, $m \in M$ satisfies the following: for all $u, v, w \in \xi_m$*

$$[R_m u, R_m v]_m = R_m\Big([R_m u, v]_m + [u, R_m v]_m - [R_m u, R_m v]_m\Big)$$

$$\begin{aligned}\{R_m u, R_m v, R_m w\}_m = R_m\Big(&\{R_m u, R_m v, w\}_m + \{u, R_m v, R_m w\}_m \\ &+\{R_m u, v, R_m w\}_m - \{R_m u, R_m v, R_m w\}_m\Big).\end{aligned}$$

Next, in absence of any twisting by a $(2, 3)$-cocycle (that is, taking τ to be 0) in the definition of generalized Reynolds operator, we obtain the definition of an $\mathcal{O}$-operator on a Lie-Yamaguti algebra bundle ξ.

Definition 12. *A vector bundle morphism $R : \eta \to \xi$ is said to be a $\mathcal{O}$-operator if for each $m \in M$ the restriction $R_m : \eta_m \to \xi_m$ of R satisfies the following: for all $u, v, w \in \eta_m$*

$$[R_m u, R_m v]_m = R_m\Big(\rho_m(R_m u)(v) - \rho_m(R_m v)(u)\Big)$$

$$\begin{aligned}\{R_mu, R_mv, R_mw\}_m &= R_m\Big(D_m(R_mu, R_mv)w + \theta_m(R_mv, R_mw)u \\ &\quad -\theta_m(R_mu, R_mw)v\Big).\end{aligned}$$

Remark 4. *In the above definition if we take η as the adjoint representation ξ (cf. Example 1), then we obtain the definition of a Rota-Baxter operator on ξ. Thus, O-operators are the generalization of Rota-Baxter operators.*

Next, we prove a characterization of a generalized Reynolds operator on a locally trivial Lie-Yamaguti algebra bundle. Let $\xi = (L, p, M)$ be a locally trivial Lie-Yamaguti algebra bundle and $(\eta; \rho, D, \theta)$ be a ξ-module. Let $\tau = (f, g) \in Z^{(2,3)}(\xi; \eta)$ be a given $(2, 3)$-cocycle and $R : \eta \to \xi$ be a vector bundle morphism. Define

$$\text{Graph }(R) = \{R_mu + u \in \xi_m \oplus \eta_m : u \in \eta_m, \forall\ m \in M\} \subset \bigcup_m (\xi_m \oplus \eta_m).$$

Then, it is easy to see that Graph (R) is a sub-bundle of $\xi \oplus \eta$, called the graph of R.

Proposition 2. *A vector bundle morphism $R : \eta \to \xi$ is a generalized Reynolds operator if and only if its graph Graph $(R) = \{R_mu + u : u \in \eta_m, \forall\ m \in M\}$ is a Lie-Yamaguti algebra sub-bundle of the twisted semi-direct product Lie-Yamaguti algebra bundle $\xi \ltimes_\tau \eta$ as defined in Example 3.*

Proof. Let $(Ru, u), (Rv, v), (Rw, w) \in$ Graph (R). Since $\xi \ltimes_\tau \eta$ is a Lie-Yamaguti algebra bundle then, for each $m \in M$, R_m satisfies

$$[R_mu + u, R_mv + v]_m = [R_mu, R_mv]_m + \rho_m(R_mu)v - \rho_m(R_mv)u + f_m(R_mu, R_mv)$$

$$\begin{aligned}\{R_mu + u, R_mv + v, R_mw + w\} = \{R_mu, R_mv, R_mw\}_m &+ D_m(R_mu, R_mv)w - \theta_m(R_mv, R_mw)u \\ &+ g_m(R_mu, R_mv, R_mw).\end{aligned}$$

Assuming that Graph (R) is a sub-algebra bundle of $\xi \ltimes_\tau \eta$, we have:

$$[R_mu, R_mv]_m = R\Big(\rho_m(R_mu)v - \rho_m(R_mv)u + f_m(R_mu, R_mv)\Big)$$

$$\{R_mu + u, R_mv + v, R_mw + w\} = R_m\Big(D_m(R_mu, R_mv)w - \theta_m(R_mv, R_mw)u + g_m(R_mu, R_mv, R_mw)\Big).$$

It follows that R is a generalized Reynolds operator. Similarly, if R is a generalized Reynolds operator, a reverse computation shows that the sub-bundle Graph (R) of $\xi \ltimes_\tau \eta$ is a Lie-Yamaguti algebra bundle. □

Finally, we introduce Nijenhuis operator on a Lie-Yamaguti algebra bundle. In [9], I. Dorfman used the notion of a Nijenhuis operator to characterize an infinitesimal deformation of a Lie algebra which is trivial. In [21,22], the authors studied infinitesimal deformations and extensions of Lie-Yamaguti algebras and used Nijenhuis operators on Lie-Yamaguti algebras to characterize trivial infinitesimal deformations. We introduce Nijenhuis operators on Lie-Yamaguti algebra bundles and show that they give rise to generalized Reynolds operators.

Definition 13. *Let $\xi = (L, p, M)$ be a Lie-Yamaguti algebra bundle. A vector bundle morphism $N : \xi \to \xi$ is called a Nijenhuis operator on ξ if for all $m \in M$, $N|_{\xi_m} := N_m$ satisfies:*

$$\begin{aligned}
[N_m u, N_m v]_m &= N_m\Big([N_m u, v]_m + [u, N_m v]_m - N_m[u, v]_m\Big),\\
\{N_m u, N_m v, N_m w\}_m &= N_m\Big(\{N_m u, N_m v, w\}_m + \{N_m u, v, N_m w\}_m\\
&\quad +\{u, N_m v, N_m w\}_m\Big) - N_m^2\Big(\{N_m u, v, w\}_m\\
&\quad +\{u, N_m v, w\}_m + \{u, v, N_m w\}_m\Big)\\
&\quad +N_m^3\{u, v, w\}_m
\end{aligned}$$

for all $u, v, w \in \xi_m$.

Now, we will show that a Nijenhuis operator on Lie-Yamaguti algebra bundle gives rise to a generalized Reynolds operator on a Lie-Yamaguti algebra bundle.

Let $\xi = (L, p, M)$ be a Lie-Yamaguti algebra bundle. We denote its 2-field of brackets and 3-field of brackets by

$$m \mapsto [\ ,\ ]_m, \quad m \mapsto \{\ ,\ ,\ \}_m, m \in M.$$

Let $N : \xi \to \xi$ be a given Nijenhuis operator. On ξ define a new 2-field of brackets and a 3-field of brackets

$$m \mapsto [\ ,\ ]_m^N, \quad m \mapsto \{\ ,\ ,\ \}_m^N$$

as follows. For any $m \in M$

$$[u, v]_m^N := [N_m u, v]_m + [u, N_m v]_m - N_m[u, v]_m, \tag{7}$$

$$\begin{aligned}
\{u, v, w\}_m^N &:= \{N_m u, N_m y, w\}_m + \{N_m u, v, N_m w\}_m\\
&\quad +\{u, N_m v, N_m w\}_m + N_m\Big(\{N_m u, v, w\}_m\\
&\quad +\{u, N_m v, w\}_m + \{u, v, N_m w\}_m\Big)\\
&\quad +N_m^2\{u, v, w\}_m.
\end{aligned}$$

for all $u, v, w \in \xi_m$. Then, a routine verification shows that the bundle $(\xi, [\ ,\ ]^N, \{\ ,\ ,\ \}^N)$ is a Lie-Yamaguti algebra bundle which we call a deformed Lie-Yamaguti algebra bundle and denote it by ξ^N.

Lemma 1. *Let N be a Nijenhuis operator on a Lie-Yamaguti algebra bundle $\xi = (L, p, M)$. We consider ξ as a vector bundle and define the vector bundle maps $\rho^N : \xi^N \to \mathrm{End}(\xi)$, and $D^N, \theta^N : \xi \otimes \xi \to \mathrm{End}(\xi)$ as follows. For each $m \in M$*

$$\rho_m^N(u)a := [N_m u, a], \quad D_m^N(u, v)a := \{N_m u, N_m v, a\}, \tag{8}$$

$$\theta_m^N(u, v)a := \{a, N_m u, N_m v\} \tag{9}$$

for all $u, v \in \xi_m^N$ and $a \in \xi_m$. Then, the vector bundle ξ is a representation of the deformed Lie-Yamaguti algebra bundle ξ^N with the representation maps being (ρ^N, D^N, θ^N).

Proof. We need to show that the vector bundle morphisms ρ^N, D^N, θ^N restricted to each fibres satisfy the relations (RLYB1)-(RLYB6). To check that the relation (RLYB1) and (RLYB2) holds, note that

$$
\begin{aligned}
& D^N_m(u,v)a - \theta^N_m(v,u)a + \theta^N_m(u,v)a \\
&\quad + \rho^N_m([u,v]^N_m)a - \rho^N_m(u)\rho^N_m(v)a + \rho^N_m(v)\rho^N_m(u)a \\
&\qquad = \{N_m u, N_m v, a\}_m - \{a, N_m v, N_m u\}_m + \{a, N_m u, N_m v\}_m \\
&\qquad\quad + [[N_m u, N_m v]_m, a]_m - [N_m u, [N_m v, a]_m]_m \\
&\qquad\quad + [N_m v, [N_m u, a]_m]_m \\
&\qquad = 0. \\
& D^N_m([u,v]^N_m, w)a + D^N_m([v,w]^N_m, u)a + D^N_m([w,u]^N_m, v)a \\
&\qquad = [N_m u, N_m v]_m, N_m w, a_m + \{[N_m v, N_m w]_m, N_m u, a\}_m \\
&\qquad\quad + \{[N_m w, N_m u]_m, N_m v, a\}_m \\
&\qquad = 0.
\end{aligned}
$$

Verification of the remaining four relations are similar, so we omit the details. Hence, $(\xi, \rho^N, D^N, \theta^N)$ is a representation of the deformed Lie-Yamaguti algebra bundle ξ^N. □

Next we show that Nijenhuis operators on Lie-Yamaguti algebra bundle can also be viewed as Reynolds operators on the deformed Lie-Yamaguti algebra bundle.

Theorem 1. *Let N be a Nijenhuis operator on a Lie-Yamaguti algebra bundle $\xi = (L, p, M)$. Define vector bundle maps $f^N : \xi^N \otimes \xi^N \to \xi$ and $g^N : \xi^N \otimes \xi^N \otimes \xi^N \to \xi$ as follows. For any $m \in M$*

$$f^N_m(u,v) := -N_m[u,v] \tag{10}$$

$$
\begin{aligned}
g^N_m(u,v,w) \;\; := \;\; & -N\Big(\{N_m u, v, w\}_m + \{u, N_m v, w\}_m \\
& + \{u, v, N_m w\}_m - N_m\{u,v,w\}_m\Big)
\end{aligned} \tag{11}
$$

for all $u, v \in \xi^N_m$ and $w \in \xi$. Then $(f^N, g^N) \in Z^{(2,3)}(\xi^N, \xi)$, that is, (f^N, g^N) is a $(2,3)$-cocycle of ξ^N with coefficients in $(\xi, \rho^N, D^N, \theta^N)$. Moreover, the identity morphism $\xi \to \xi^N$ is a generalized Reynolds operator on ξ^N with respect to the representation $(\xi, \rho^N, D^N, \theta^N)$ and with $\tau = (f^N, g^N)$.

Proof. To show that $(f^N, g^N) \in C^{(2,3)}(\xi^N; \xi)$ is a $(2,3)$-cocycle, we need to verify that $\delta(f^N, g^N) = 0 = \delta^*(f^N, g^N)$. Let $m \in M$, then for any $a, b, c, d, e \in \xi^N_m$ we observe that

$$
\begin{aligned}
\delta_I(f^N_m)(a,b,c,d) &= \rho^N_m(c)g^N_m(a,b,d) + \rho^N_m(d)g^N_m(a,b,c) + g^N_m(a,b,[c,d]^N_m) \\
&\quad + D^N(a,b)f^N_m(c,d) - f^N_m(\{a,b,c\}^N_m, d) - f^N_m(c, \{a,b,d\}^N_m) \\
&= 0.
\end{aligned}
$$

$$\delta_I I(g^N_m)(a,b,c,d,e) = -\theta^N_m(d,e)g^N_m(a,b,c) + \theta^N_m(c,e)g^N_m(a,b,d) + D^N_m(a,b)g^N_m(c,d,e)$$

$$
\begin{aligned}
&- D_m^N(c,d)g_m^N(a,b,e) - g_m^N(\{a,b,c\}_m^N,d,e) - g_m^N(c,\{a,b,d\}_m^N,e)\\
&- g_m^N(c,d,\{a,b,e\}_m^N) + g_m^N(a,b,\{c,d,e\}_m^N)\\
=\ & 0.
\end{aligned}
$$

By a similar computation one can check that

$$\delta_I^*(f_m^N) = 0 = \delta_{II}^*(g_m^N).$$

As a result, we see that (f_m^N, g_m^N) is a $(2,3)$-cocycle. Moreover, it is straightforward to verify that the identity vector bundle map $Id : \xi \to \xi^N$ is a generalized Reynolds operator (cf. Definition 10) with respect to the representation $(\xi, \rho^N, D^N, \theta^N)$, and with $\tau = (f^N, g^N)$. □

4 Extensions of Lie-Yamaguti algebra bundles

C. Chevally and S. Eilenberg [3] showed that extensions of algebras can be interpreted in terms of certain Hochschild cohomology group. Later, D. K. Harrison [12] showed that certain Harrison cohomology group of commutative algebras can be related to extensions of commutative algebras. In the same spirit, Yamaguti showed that that $(2,3)$-cohomology of a Lie-Yamaguti algebra with coefficients in a representation may be interpreted in terms of isomorphism classes of extensions of the Lie-Yamaguti algebra. The aim of this section is to introduce the notion of extension of Lie-Yamaguti algebra bundles and relate isomorphism classes of extensions in terms of $(2,3)$-cohomology of such bundle as introduced in the previous section. We begin with some definitions.

Definition 14. *Let $\xi = (L,p,M)$ be a Lie-Yamaguti algebra bundle. An ideal of ξ is a sub-bundle η of the vector bundle ξ such that for all $m \in M$, $v \in \eta_m$, a, $b \in \xi_m$*

$$[v,a]_m \in \eta_m \quad \text{and} \quad \{v,a,b\}_m \in \eta_m,\ \{a,b,v\}_m \in \eta_m.$$

*An ideal η of ξ is said to be **abelian** if for all $m \in M$, $u, v \in \eta_m$ and $a \in \xi_m$,*

$$[u,a]_m = 0, \quad \{u,v,a\}_m = \{u,a,v\}_m = \{a,u,v\}_m = 0.$$

Definition 15. *Let $\tilde{\xi} = (\tilde{L}, \tilde{p}, M)$ and $\eta = (E,q,M)$ be Lie-Yamaguti algebra bundles. An extension of Lie-Yamaguti algebra bundle over M is a short exact sequence in the category of Lie-Yamaguti algebra bundles over M*

$$0 \longrightarrow \eta \xrightarrow{\ i\ } \tilde{\xi} \xrightarrow{\ j\ } \xi \longrightarrow 0.$$

*We call $\tilde{\xi}$ an extension of ξ by η, and denote it by $Ext_{\tilde{\xi}}$. An extension as above is said to be **abelian** if η is an abelian ideal of $\tilde{\xi}$.*

Throughout in this section, we will consider only abelian extensions.

A **section** of the vector bundle map $j : \tilde{\xi} \to \xi$ is a vector bundle map $\sigma : \xi \to \tilde{\xi}$ such that $j \circ \sigma = id_{\xi}$. Note that if σ is a splitting as above then $\tilde{\xi}$ may be viewed as as a Whitney sum of ξ and η. Here, we identify η with its image $i(\eta)$.

In other words, $\tilde{\xi} = \xi \oplus \eta$. Since we are concerned with smooth vector bundles over smooth manifolds (paracompact), any short exact sequence splits.

We now introduce the notion of equivalence of two extensions.

Definition 16. *Two extensions of ξ by η, say $Ext_{\tilde{\xi}} : 0 \to \eta \xrightarrow{i} \tilde{\xi} \xrightarrow{j} \xi \to 0$ and $Ext_{\hat{\xi}} : 0 \to \eta \xrightarrow{\hat{i}} \hat{\xi} \xrightarrow{\hat{j}} \xi \to 0$, are said to be equivalent, if there exists a Lie-Yamaguti algebra bundle isomorphism $f : \tilde{\xi} \to \hat{\xi}$ such that the following diagram is commutative.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \eta & \xrightarrow{i} & \tilde{\xi} & \xrightarrow{j} & \xi & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \eta & \xrightarrow[\hat{i}]{} & \hat{\xi} & \xrightarrow[\hat{j}]{} & \xi & \longrightarrow & 0 \end{array}$$

Note that, any extension $\tilde{\xi}$ of ξ by η, is isomorphic to $\xi \oplus \eta$ as vector bundles.

Next, we show that any given extension $Ext_{\tilde{\xi}} : 0 \to \eta \xrightarrow{i} \tilde{\xi} \xrightarrow{j} \xi \to 0$ induces a representation of the Lie-Yamaguti algebra bundle on the vector bundle η. It turns out that equivalent extensions induce the same representation on η.

Let $Ext_{\tilde{\xi}} : 0 \to \eta \xrightarrow{i} \tilde{\xi} \xrightarrow{j} \xi \to 0$ be a given extension of ξ by η, that is,

$$0 \longrightarrow \eta \xrightarrow{i} \tilde{\xi} \xrightarrow{j} \xi \longrightarrow 0$$

and $\sigma : \xi \to \tilde{\xi}$ be a section of $j : \tilde{\xi} \to \xi$. Let us denote the associated 2-field and 3-field of brackets of the Lie-Yamaguti algebra bundle $\tilde{\xi}$ by

$$m \to [\ ,\ ]^{\sim}_m \ \text{ and } \ m \to \{\ ,\ ,\ \}^{\sim}_m, \ m \in M,$$

respectively.

Define vector bundle morphisms $\rho : \xi \to \mathrm{End}(\eta)$ and $D, \theta : \xi \otimes \xi \to \mathrm{End}(\eta)$ fibre-wise as follows. Let $m \in M$. For any $a, b \in \xi_m$ and $v \in \eta_m$

$$\rho_m(a)(v) := [\sigma_m(a), v]^{\sim}_m \tag{12}$$
$$D_m(a,b)(v) := \{\sigma_m(a), \sigma_m(b), v\}^{\sim}_m \tag{13}$$
$$\theta_m(a,b)(v) := \{v, \sigma_m(a), \sigma_m(b)\}^{\sim}_m. \tag{14}$$

Proposition 3. *The above data yield a representation $(\eta;\ \rho,\ D,\ \theta)$ of ξ. Furthermore,*

1. *The definition of ρ, D, and θ does not depend on the choice of the section σ, that is, given any two sections of j, say σ^1, and σ^2, we have*

$$\begin{array}{rcl} \sigma^1_m(a), v]^{\sim} & = & [\sigma^2_m(a), v]^{\sim} \\ \{\sigma^1_m(a), \sigma^1_m(b), v\}^{\sim} & = & \{\sigma^2_m(a), \sigma^2_m(b), v\}^{\sim} \\ \{v, \sigma^1_m(a), \sigma^1(b)\}^{\sim} & = & \{v, \sigma^2_m(a), \sigma^2_m(b)\}^{\sim}. \end{array}$$

2. *Equivalent extensions induce the same representation on η, that is, given any two equivalent extensions, say $Ext_{\tilde{\xi}}$ and $Ext_{\hat{\xi}}$ with induced representations being $(\eta;\ \rho,\ D,\ \theta)$ and $(\eta;\ \rho',\ D',\ \theta')$ respectively. Then $\rho=\rho',\ D=D',\ \theta=\theta'$.*

Proof. Let $m\in M$, $a,b,c,d\in\xi_m$ and $v\in\eta_m$. Let $\sigma:\xi\to\tilde{\xi}$ be a given section of $j:\tilde{\xi}\to\xi$. Then from (LY3) of $\tilde{\xi}_m$ we get

$$\sum_{\circlearrowleft(\sigma_m(a),\sigma_m(b),v)}[[\sigma_m(a),\sigma_m(b)]^{\sim},v]^{\sim}+\sum_{\circlearrowleft(\sigma_m(a),\sigma_m(b),v)}\{\sigma_m(a),\sigma_m(b),v\}^{\sim}=0$$

which reduces to (RLYB1):

$$D_m(a,b)+\theta_m(a,b)-\theta_m(b,a)=[\rho_m(a),\rho_m(b)]_m-\rho_m([a,b]_m).$$

By (LY5) of $\tilde{\xi}$ we get the following equality

$$\begin{aligned}&\{\sigma_m(a),v,[\sigma_m(b),\sigma_m(c)]_m^{\sim}\\&=[\{\sigma_m(a),v,\sigma_m(b)\}_m^{\sim},\sigma_m(c)]_m^{\sim}+[\sigma_m(b),\{\sigma_m(a),v,\sigma_m(c)\}_m^{\sim}]_m^{\sim}\end{aligned}$$

which reduces to (RLYB2):

$$\theta_m(a,[b,c]_m)=\rho_m(b)\theta_m(a,c)-\rho_m(c)\theta_m(a,b).$$

By (LY4) of $\tilde{\xi}_m$ we get

$$\sum_{\circlearrowleft(\sigma_m(a),\sigma_m(b),v)}\{[\sigma_m(a),\sigma_m(b)]_m^{\sim},v,\sigma_m(c)\}_m^{\sim}=0$$

which reduces to (RLYB3):

$$\theta_m([a,b]_m,c)=\theta_m(a,c)\rho_m(b)-\theta_m(b,c)\rho_m(a).$$

By (LY6) of $\tilde{\xi}_m$ we get

$$\begin{aligned}&\{v,\sigma_m(a),\{\sigma_m(b),\sigma_m(c),\sigma_m(d)\}_m^{\sim}\}_m^{\sim}\\&\quad=\{\{v,\sigma_m(a),\sigma_m(b)\}_m^{\sim},\sigma_m(c),\sigma_m(d)\}_m^{\sim}\\&\qquad+\{\sigma_m(b),\{v,\sigma_m(a),\sigma_m(c)\_m^{\sim},\sigma_m(d)\}_m^{\sim}\\&\qquad+\{\sigma_m(b),\sigma_m(c),\{v,\sigma_m(a),\sigma_m(d)\}_m^{\sim}\}_m^{\sim}\end{aligned}$$

which reduces to (RLYB4):

$$\theta_m(a,\{b,c,d\}_m)=\theta_m(c,d)\theta_m(a,b)-\theta_m(b,d)\theta_m(a,c)+D_m(b,c)\theta_m(a,d).$$

By (LY5) of $\tilde{\xi}_m$ we get

$$\begin{aligned}&\{\sigma_m(a),\sigma_m(b),[\sigma_m(c),v]_m^{\sim}\}_m^{\sim}\\&=[\{\sigma_m(a),\sigma_m(b),\sigma_m(c)\}_m^{\sim},v]_m^{\sim}+[\sigma_m(c),\{\sigma_m(a),\sigma_m(b),v\}_m^{\sim}]_m^{\sim}\end{aligned}$$

which reduces to (RLYB5):

$$D_m(a,b)\rho_m(c) = \rho_m(c)D_m(a,b) + \rho_m(\{a,b,c\}_m).$$

By (LY6) of $\tilde{\xi}_m$ we get

$$\begin{aligned}
&\{\sigma_m(a), \sigma_m(b), \{v, \sigma_m(c), \sigma_m(d)\}^{\sim}_m\}^{\sim}_m \\
&\quad = \{\{\sigma_m(a), \sigma_m(b), v\}^{\sim}_m, \sigma_m(c), \sigma_m(d)\}^{\sim}_m \\
&\qquad + \{v, \{\sigma_m(a), \sigma_m(b), \sigma_m(c)\}^{\sim}_m, \sigma_m(d)\}^{\sim}_m \\
&\qquad + \{v, \sigma_m(c), \{\sigma_m(a), \sigma_m(b), \sigma_m(d)\}^{\sim}_m\}^{\sim}_m
\end{aligned}$$

which reduces to (RLYB6):

$$D_m(a,b)\theta_m(c,d) = \theta_m(c,d)D_m(a,b) + \theta_m(\{a,b,c\}_m, d) + \theta_m(c, \{a,b,d\}_m).$$

Therefore, $(\eta; \rho,\ D,\ \theta)$ is a representation of ξ. Hence any extension of ξ by η gives a representation of ξ on η.

Next we show that the definition of θ is independent of the choice of the section. The proofs that the definitions of ρ and D do not depend on the choice of the section σ are similar, hence, we omit the details.

Let $\sigma,\ \sigma' : \xi \to \tilde{\xi}$ be two sections of $j : \tilde{\xi} \to \xi$. Let $m \in M$. Then, for any $a \in \xi_m$

$$j(\sigma_m(a) - \sigma'_m(a)) = 0.$$

Therefore, $\sigma_m(a) - \sigma'_m(a) \in \mathrm{Ker}(j) = \eta_m$, so that $\sigma_m(a) = \sigma'_m(a) + v_a$ for some $v_a \in \eta_m$. Since we are considering abelian extension, for any $v \in \eta_m,\ a,\ b \in \xi_m$ we have

$$\begin{aligned}
\{v, \sigma_m(a), \sigma_m(b)\}^{\sim}_m &= \{v, \sigma'_m(a) + v_a, \sigma'_m(b) + v_b\}^{\sim}_m \\
&= \{v, \sigma'_m(a), \sigma'_m(b) + v_b\}^{\sim}_m + \{v, v_a, \sigma'_m(b) + v_b\}^{\sim}_m \\
&= \{v, \sigma'_m(a), \sigma'_m(b)\}^{\sim}_m + \{v, \sigma'_m(a), v_b\}^{\sim}_m \\
&= \{v, \sigma'_m(a), \sigma'_m(b)\}^{\sim}_m.
\end{aligned}$$

Finally, we show that two equivalent extensions of ξ by η induce the same representation. Suppose that $Ext_{\tilde{\xi}}$ and $Ext_{\hat{\xi}}$ are two equivalent extensions of ξ. Let us denote the associated 2-field and 3-field of brackets of the Lie-Yamaguti algebra bundle $\hat{\xi}$ by

$$m \to [\ ,\ ]^{\wedge}_m \ \text{ and } \ m \to \{\ ,\ ,\ \}^{\wedge}_m,\ m \in M,$$

respectively. Let $f : \tilde{\xi} \to \hat{\xi}$ be a Lie-Yamaguti algebra isomorphism satisfying $f \circ i = \hat{i}$ and $\hat{j} \circ f = j$. Thus, the following diagram is commutative.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \eta & \xrightarrow{\ i\ } & \tilde{\xi} & \xrightarrow{\ j\ } & \xi & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & \eta & \xrightarrow[\ \hat{i}\ ]{} & \hat{\xi} & \xrightarrow[\ \hat{j}\ ]{} & \xi & \longrightarrow & 0
\end{array}$$

Let $\sigma : \xi \to \tilde{\xi}$ and $\sigma' : \xi \to \hat{\xi}$ be sections of j and $\hat{j}$ respectively. Then, for any $a \in \xi_m,\ m \in M$ we have

$$\hat{j} \circ f(\sigma_m(a)) = j \circ (\sigma_m(a)) = a = \hat{j} \circ (\sigma'_m(a))$$
$$\Rightarrow \hat{j}(f(\sigma_m(a)) - \sigma'_m(a)) = 0.$$

This implies $f(\sigma_m(a)) - \sigma'_m(a) \in \mathrm{Ker}(\hat{j}_m) = \eta_m$, that is, $f(\sigma_m(a)) = \sigma'_m(a) + v_a$ for some $v_a \in \eta_m$. Thus, we have for any $a, b \in \xi_m$ and $v \in \eta_m$

$$f\big(\{v, \sigma_m(a), \sigma_m(b)\}^{\sim}_m\big) = \{f(v), f(\sigma_m(a)), f(\sigma_m(b))\}^{\wedge}_m = \{v, \sigma'_m(a), \sigma'_m(b)\}^{\wedge}_m.$$

Note that $f(v) = v$ follows from the commutativity of first box in the diagram. Therefore, equivalent extensions induce the same θ. Similarly one can show that equivalent extensions induce the same D and ρ. □

As of now, we have seen that any extension $Ext_{\tilde{\xi}}$

$$0 \longrightarrow \eta \xrightarrow{i} \tilde{\xi} \xrightarrow{j} \xi \longrightarrow 0$$

of ξ by η, induces a representation $(\eta; \rho, D, \theta)$ of ξ, where the vector bundle maps ρ, D, and θ are defined by (12)- (14) in terms of a section $\sigma : \xi \to \tilde{\xi}$ of $j : \tilde{\xi} \to \xi$. Therefore, we have a cochain complex of the Lie-Yamaguti algebra bundle ξ with coefficients in the induced representation $(\eta; \rho, D, \theta)$ of ξ as discussed in Section 2.

Our next goal is to attach a $(2,3)$-cocycle of this cochain complex to $Ext_{\tilde{\xi}}$. Fix a section $\sigma : \xi \to \tilde{\xi}$ of $j : \tilde{\xi} \to \xi$. Define two maps; $f : \xi \otimes \xi \to \eta$ and $g : \xi \otimes \xi \otimes \xi \to \eta$ in the following way. Let $m \in M$. Denote by f_m and g_m the resulting bilinear and trilinear maps obtained by restricting f and g to the fibres $(\xi \otimes \xi)_m$ and $(\xi \otimes \xi \otimes \xi)_m$ respectively. For all $a_1,\ a_2,\ a_3 \in \xi_m$, define

$$f_m(a_1, a_2) := [\sigma_m(a_1), \sigma_m(a_2)]^{\sim}_m - \sigma_m([a_1, a_2]_m) \tag{15}$$
$$g_m(a_1, a_2, a_3) := \{\sigma_m(a_1), \sigma_m(a_2), \sigma_m(a_3)\}^{\sim}_m - \sigma_m(\{a_1, a_2, a_3\}_m) \tag{16}$$

Note that $(f, g) \in C^{(2,3)}(\xi; \eta)$.

Proposition 4. *For any given abelian extension $Ext_{\tilde{\xi}}$ of ξ by η, the cochain $(f, g) \in C^{(2,3)}(\xi; \eta)$ as defined above is a $(2,3)$-cocycle.*

Proof. To show that (f, g) is a $(2,3)$-cocycle, we need to show

$$\delta(f, g) = 0 \quad \text{and} \quad \delta^*(f, g) = 0,$$

that is, $\delta_I f = 0,\ \delta_{II} g = 0$ and $\delta^*_I f = 0,\ \delta^*_{II} g = 0$. Recall that the representation induced by the given extension are given by the vector bundle morphisms ρ, D, and θ, where for $a, b \in \xi_m$ and $v \in \eta_m$, $m \in M$,

$$\rho_m(a)(v) = [\sigma_m(a), v]^{\sim}_m$$

$$D_m(a,b)(v) = \{\sigma_m(a), \sigma_m(b), v\}^{\sim}_m$$
$$\theta_m(a,b)(v) = \{v, \sigma_m(a), \sigma_m(b)\}^{\sim}_m.$$

Let $a_i \in \xi_m$, $1 \leq i \leq 5$. By the definitions of δ and δ^* we obtain the following equality:

$$\begin{aligned}
&(\delta_I)_m f_m(a_1,a_2,a_3,a_4)\\
&= -\rho_m(a_3)g_m(a_1,a_2,a_4) + \rho_m(a_4)g_m(a_1,a_2,a_3) + g_m(a_1,a_2,[a_3,a_4]_m)\\
&\quad + D_m(a_1,a_2)f_m(a_3,a_4) - f_m(\{a_1,a_2,a_3\}_m,a_4) - f(a_3,\{a_1,a_2,a_4\}_m)\\
&= 0,
\end{aligned}$$

where we have used the definition of representation as given above and (LY5). Similarly, we obtain using (LY6)

$$\begin{aligned}
&(\delta_{II})_m g_m(a_1,a_2,a_3,a_4,a_5)\\
&\quad = -\theta_m(a_4,a_5)g_m(a_1,a_2,a_3) + \theta_m(a_3,a_5)g_m(a_1,a_2,a_4)\\
&\qquad + D_m(a_1,a_2)g_m(a_3,a_4,a_5) - D_m(a_3,a_4)g_m(a_1,a_2,a_5)\\
&\qquad - g_m(\{a_1,a_2,a_3\}_m,a_4,a_5) - g_m(a_3,\{a_1,a_2,a_4\}_m,a_5)\\
&\qquad - g_m(a_3,a_4,\{a_1,a_2,a_5\}_m) + g_m(a_1,a_2,\{a_3,a_4,a_5\}_m)\\
&\quad = 0.
\end{aligned}$$

Moreover, from the above definition of representation, (LY3), and (LY4) we get

$$\begin{aligned}
(\delta_I^*)_m f_m(a_1,a_2,a_3) &= -\sum_{\circlearrowleft(a_1,a_2,a_3)} \rho_m(a_1)f_m(a_2,a_3) + \sum_{\circlearrowleft(a_1,a_2,a_3)} f_m([a_1,a_2]_m,a_3)\\
&\quad + \sum_{\circlearrowleft(a_1,a_2,a_3)} g_m(a_1,a_2,a_3)\\
&= 0,
\end{aligned}$$

$$\begin{aligned}
(\delta_{II}^*)_m g_m(a_1,a_2,a_3,a_4) &= \theta_m(a_1,a_4)f_m(a_2,a_3) + \theta_m(a_2,a_4)f_m(a_3,a_4)\\
&\quad + \theta_m(a_3,a_4)f_m(a_1,a_2) + g_m([a_1,a_2]_m,a_3,a_4)\\
&\quad + g_m([a_2,a_3]_m,a_1,a_4) + g_m([a_3,a_1]_m,a_2,a_4)\\
&= 0.
\end{aligned}$$

Thus, $(f,g) \in C^{(2,3)}(\xi;\eta)$ is a $(2,3)$-cocycle. □

By a routine calculation we obtain the following result.

Corollary 1. *If $\sigma, \sigma' : \xi \to \tilde{\xi}$ are any two chosen sections of $j : \tilde{\xi} \to \xi$ and (f,g), (f',g') are the corresponding cocycles as obtained in Proposition 4, then (f,g) and (f',g') are cohomologous. Hence, the extension $Ext_{\tilde{\xi}}$ of ξ by η determines uniquely an element of $H^{(2,3)}(\xi;\eta)$.*

On the other hand, given a Lie-Yamaguti algebra bundle ξ equipped with a representation $(\eta;\rho,D,\theta)$, any $(2,3)$-cocycle in $Z^{(2,3)}(\xi;\eta)$ determines an abelian extension of ξ by η which is unique up to equivalence.

Let $\xi=(L,p,M)$ be a given Lie-Yamaguti algebra bundle and $(\eta;\rho,D,\theta)$ be a representation of ξ. Also, let $(f,g)\in Z^{(2,3)}(\xi;\eta)$. Then, we have the following result.

Lemma 2. *The vector bundle $\tilde{\xi}=\xi\oplus\eta$ becomes a Lie-Yamaguti algebra bundle, where the associated 2-field and 3-field*

$$m\mapsto [\ ,\ ]^{\sim}_m,\quad m\mapsto \{\ ,\ ,\ \}^{\sim}_m,\quad m\in M$$

are given by

$$[a_1+w_1,a_2+w_2]^{\sim}_m:=[a_1,a_2]_m+f_m(a_1,a_2)+\rho_m(a_1)(w_2)-\rho_m(a_2)(w_1)$$

$$\begin{aligned}\{a_1+w_1,a_2+w_2,a_3+w_3\}^{\sim}_m\\ :=\{a_1,a_2,a_3\}_m+g_m(a_1,a_2,a_3)+D_m(a_1,a_2)(w_3)\\ -\theta_m(a_1,a_3)(w_2)+\theta_m(a_2,a_3)(w_1)\end{aligned}$$

where $a_1,a_2,a_3\in\xi_m$ and $w_1,w_2,w_3\in\eta_m$. It is convenient to denote this Lie-Yamaguti algebra bundle by $\xi\oplus_{(f,g)}\eta$ to emphasize that it is induced by the given cocycle.

Proof. Clearly the assignments

$$m\mapsto [\ ,\ ]^{\sim}_m,\quad m\mapsto \{\ ,\ ,\ \}^{\sim}_m,\quad m\in M$$

as defined in the statement are smooth. So, it is enough to show that for any $m\in M$, $\tilde{\xi}_m$ is a Lie-Yamaguti algebra. Let $m\in M$. It is easy to see that (LY1) and (LY2) holds for $[\ ,\ ]^{\sim}_m$ and $\{\ ,\ ,\ \}^{\sim}_m$ defined above. To verify (LY6) proceed as follows.

$$\begin{aligned}\big\{a_1+w_1,&a_2+w_2,\{b_1+v_1,b_2+v_2,b_3+v_3\}^{\sim}_m\big\}^{\sim}_m\\ &=\big\{a_1+w_1,\ a_2+w_2,\ \{b_1,b_2,b_3\}_m+g_m(b_1,b_2,b_3)\\ &\quad+D_m(b_1,b_2)(v_3)-\theta_m(b_1,b_3)(v_2)+\theta_m(b_2,b_3)(v_1)\big\}^{\sim}_m\\ &=\{a_1,a_2,\{b_1,b_2,b_3\}_m\}_m+g_m(a_1,a_2,\{b_1,b_2,b_3\})\\ &\quad+D_m(a_1,a_2)g_m(b_1,b_2,b_3)+D_m(a_1,a_2)D_m(b_1,b_2)(v_3)\\ &\quad-D_m(a_1,a_2)\theta_m(b_1,b_3)(v_2)+D_m(a_1,a_2)\theta_m(b_2,b_3)(v_1)\\ &\quad-\theta_m(a_1,\{b_1,b_2,b_3\}_m)(w_2)+\theta_m(a_2,\{b_1,b_2,b_3\}_m)(w_1)\end{aligned}$$

$$\begin{aligned}\big\{\{a_1+w_1,&a_2+w_2,b_1+v_1\}^{\sim}_m,b_2+v_2,b_3+v_3\big\}^{\sim}_m\\ &=\big\{\{a_1,a_2,b_1\}_m+g_m(a_1,a_2,b_1)+D_m(a_1,a_2)(v_1)\\ &\quad-\theta_m(a_1,b_1)(w_2)+\theta_m(a_2,b_1)(w_1),\ b_2+v_2,\ b_3+v_3\big\}^{\sim}_m\\ &=\{\{a_1,a_2,b_1\}_m,b_2,b_3\}_m+g_m(\{a_1,a_2,b_1\}_m,b_2,b_3)\end{aligned}$$

$$
\begin{aligned}
&+ D_m(\{a_1, a_2, b_1\}_m, b_2)(v_3) - \theta_m(\{a_1, a_2, b_1\}_m, b_3)(v_2)\\
&+ \theta_m(b_2, b_3)g_m(a_1, a_2, b_1) + \theta_m(b_2, b_3)D_m(a_1, a_2)(v_1)\\
&- \theta_m(b_2, b_3)\theta_m(a_1, b_1)(w_2) + \theta_m(b_2, b_3)\theta_m(a_2, b_1)(w_1)
\end{aligned}
$$

$$
\begin{aligned}
&\big\{b_1 + v_1, \{a_1 + w_1, a_2 + w_2, b_2 + v_2\}^{\sim}_m, b_3 + v_3\big\}^{\sim}_m\\
&\quad = \big\{b_1 + v_1,\ \{a_1, a_2, b_2\}_m + g_m(a_1, a_2, b_2) + D_m(a_1, a_2)(v_2)\\
&\qquad - \theta_m(a_1, b_2)(w_2) + \theta_m(a_2, b_2)(w_1), b_3 + v_3\big\}^{\sim}_m\\
&\quad = \{b_1, \{a_1, a_2, b_2\}_m, b_3\}_m + g_m(b_1, \{a_1, a_2, b_2\}_m, b_3)\\
&\qquad + D_m(b_1, \{a_1, a_2, b_2\}_m)(v_3) + \theta_m(\{a_1, a_2, b_2\}_m, b_3)(v_1)\\
&\qquad - \theta_m(b_1, b_3)g_m(a_1, a_2, b_2) - \theta_m(b_1, b_3)D_m(a_1, a_2)(v_2)\\
&\qquad + \theta_m(b_1, b_3)\theta_m(a_1, b_2)(w_2) - \theta_m(b_1, b_3)\theta_m(a_2, b_2)(w_1)
\end{aligned}
$$

$$
\begin{aligned}
&\big\{b_1 + v_1, b_2 + v_2, \{a_1 + w_1, a_2 + w_2, b_3 + v_3\}^{\sim}_m\big\}^{\sim}_m\\
&\quad = \big\{b_1 + v_1,\ b_2 + v_2,\ \{a_1, a_2, b_3\}_m + g_m(a_1, a_2, b_3)\\
&\qquad + D_m(a_1, a_2)(v_3) - \theta_m(a_1, a_3)(w_2) + \theta_m(a_2, a_3)(w_1)\big\}^{\sim}_m\\
&\quad = \{b_1, b_2, \{a_1, a_2, b_3\}_m\}_m + g_m(b_1, b_2, \{a_1, a_2, b_3\}_m)\\
&\qquad + D_m(b_1, b_2)g_m(a_1, a_2, b_3) + D_m(b_1, b_2)D_m(a_1, a_2)(v_3)\\
&\qquad - D_m(b_1, b_2)\theta_m(a_1, a_3)(w_2) + D_m(b_1, b_2)\theta_m(a_2, a_3)(w_1)\\
&\qquad - \theta_m(b_1, \{a_1, a_2, b_3\})(v_2) + \theta_m(b_2, \{a_1, a_2, b_3\})(v_1)
\end{aligned}
$$

Using (RLYB6), (RLYB4) and the definition of coboundary maps we can show

$$
\begin{aligned}
&\big\{a_1 + w_1, a_2 + w_2, \{b_1 + v_1, b_2 + v_2, b_3 + v_3\}^{\sim}_m\big\}^{\sim}_m\\
&\quad = \big\{\{a_1 + w_1, a_2 + w_2, b_1 + v_1\}^{\sim}_m, b_2 + v_2, b_3 + v_3\big\}^{\sim}_m\\
&\qquad + \big\{b_1 + v_1, \{a_1 + w_1, a_2 + w_2, b_2 + v_2\}^{\sim}_m, b_3 + v_3\big\}^{\sim}_m\\
&\qquad + \big\{b_1 + v_1, b_2 + v_2, \{a_1 + w_1, a_2 + w_2, b_3 + v_3\}^{\sim}_m\big\}^{\sim}_m
\end{aligned}
$$

giving us (LY6). Other relations, (LY3), (LY4), (LY5) can also be obtained in the same way. Thus making $\xi \oplus_{(f,g)} \eta$ a Lie-Yamaguti algebra bundle. □

Observe that the Lie-Yamaguti algebra brackets of the fibres of $\xi \oplus_{(f,g)} \eta$ makes η an abelian ideal in $\xi \oplus_{(f,g)} \eta$ and we have the following extension of ξ by η:

$$
0 \longrightarrow \eta \xrightarrow{\ i\ } \xi \oplus_{(f,g)} \eta \xrightarrow{\ j\ } \xi \longrightarrow 0
$$

where i is the inclusion map and j is the projection map. Furthermore, if $(h,k) \in Z^{(2,3)}(\xi;\eta)$ is another cocycle. Then, we have the following result.

Lemma 3. *Two extensions* $0 \to \eta \to \xi \oplus_{(f,g)} \eta \to \xi \to 0$ *and* $0 \to \eta \to \xi \oplus_{(h,k)} \to \xi \to 0$ *are equivalent iff* $(f,g),\ (h,k) \in Z^{(2,3)}(\xi;\eta)$ *are cohomologous.*

Proof. Let the two extensions $0 \to \eta \xrightarrow{i} \xi \oplus_{(f,g)} \eta \xrightarrow{p} \xi \to 0$ and $0 \to \eta \xrightarrow{i} \xi \oplus_{(h,k)} \eta \xrightarrow{p} \xi \to 0$ be equivalent through a Lie-Yamaguti algebra isomorphism

$$\gamma : \xi \oplus_{(f,g)} \eta \to \xi \oplus_{(h,k)} \eta$$

Then for each $m \in M$ we have the following equivalence of abelian extension.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \eta_m & \longrightarrow & \xi_m \oplus_{(f_m,g_m)} \eta_m & \longrightarrow & \xi_m & \longrightarrow & 0 \\ & & {\scriptstyle id}\downarrow & & {\scriptstyle \gamma_m}\downarrow & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \eta_m & \longrightarrow & \xi_m \oplus_{(h_m,k_m)} \eta_m & \longrightarrow & \xi_m & \longrightarrow & 0 \end{array}$$

To show that (f,g) and (h,k) are cohomologous it is enough to show for each $m \in M$, (f_m, g_m) and (h_m, k_m) are cohomologous, that is,

$$(f_m, g_m) - (h_m, k_m) \in B^{(2,3)}(\xi_m; \eta_m)$$

We define a map $\lambda_m : \xi_m \to \eta_m$ by $\lambda_m(a) = \gamma_m(a) - a$ by which one can show

$$f_m - h_m = (\delta_I)_m(\lambda_m) \quad \text{and} \quad g_m - k_m = (\delta_{II})_m(\lambda_m)$$

Conversely, assume that for each $m \in M$, (f_m, g_m) and (h_m, k_m) are in the same cohomology class, that is, $(f_m, g_m) - (h_m, k_m) = (\delta)_m(\lambda_m)$. Then, $\gamma_m : \xi_m \oplus_{(f,g)} \eta_m \to \xi_m \oplus_{(h,k)} \eta_m$ defined by

$$\gamma_m(a + v) = a + \lambda_m(a) + v$$

gives the required isomorphism. □

By summarizing the above observations we have the following theorem.

Theorem 2. *To each equivalence class of abelian extensions of ξ by η there corresponds an element of $H^{(2,3)}(\xi; \eta)$. Suppose ξ is a given Lie-Yamaguti algebra bundle over M equipped with a representation $(\eta; \rho,\ D, \theta)$. To each cohomology class $[(f,g)] \in H^{(2,3)}(\xi; \eta)$, there is an extension of ξ by η*

$$0 \longrightarrow \eta \xrightarrow{\ i\ } \xi \oplus_{(f,g)} \eta \xrightarrow{\ j\ } \xi \longrightarrow 0$$

which is unique up to equivalence of extensions.

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