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On T-injectivity and $T_\cap$-injectivity in the category of S-acts

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Abstract. In this paper, we introduce and investigate new notions of injectivity and essentiality for right S-acts, defined relative to a multiplicatively closed subset T of a monoid S. We study the concepts of T-injective and $T_\cap$-injective S-acts, along with T-essential and $T_\cap$-essential subacts. We first establish foundational definitions and illustrate the differences between T-essential and $T_\cap$-essential subacts with examples. Our study shows that T-injectivity does not necessarily imply the existence of a zero element in the S-act, which contrasts with classical results on injective S-acts. We proved that every S-act admits a $T_\cap$-injective hull, satisfying a universal property analogous to classical injective envelopes. We study closure properties of the classes of T-injective and $T_\cap$-injective S-acts under categorical constructions such as products, retracts, and direct limits. Moreover, we demonstrate that pushouts preserve $T_\cap$-essential extensions, while pullbacks may not, highlighting an asymmetry in categorical behavior.

Keywords: Multiplicatively closed subset, S-act, T-injectivity, $T_\cap$-injectivity.
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1 Introduction and preliminaries

The theory of S-acts over monoids provides a rich generalization of module theory, in which actions are defined over monoids rather than rings. The concept of injectivity, a central theme in homological algebra, has been studied extensively in the setting of S-acts, yielding a wide spectrum of generalizations and structural insights into these algebraic objects. This paper is devoted to furthering this line of investigation by introducing new notions of injectivity and essential extensions that are defined relative to a multiplicatively closed subset T of a monoid S. Injectivity in the category of S-acts was initially developed in analogy with module theory. A right S-act E is called injective if, given any monomorphism $f : A \to B$ and any S-homomorphism

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$g : A \to E$, there exists an extension $\tilde{g} : B \to E$ such that $\tilde{g} \circ f = g$. Various modifications of this classical notion have been introduced to address the limitations and particularities of S-acts. These include *weak injectivity*, *quasi-injectivity*, *cyclic injectivity*, and *injectivity with respect to essential extensions* [1,2,4,5,7,11,12]. Quasi-injective S-acts, introduced in analogy to module theory, require that every S-homomorphism from a subact of E to E itself extends to an endomorphism of E. Weak injectivity, studied by Knauer and others, relaxes the requirement of extension to specific classes of monomorphisms or particular types of subacts. Further work has considered injectivity relative to classes of morphisms or specific substructures, such as cyclic or torsion subacts. An essential concept in these developments is that of *essential subacts*, which generalizes the notion of essential submodules. A subact A of an S-act B is called essential if every non-zero subact of B intersects A nontrivially. The importance of essential subacts lies in their role in defining injective hulls- minimal injective extensions that contain the original act as an essential subact. The existence and uniqueness (up to isomorphism) of such hulls were established for various classes of injectivity in [3, 4, 6, 8, 9]. Building upon these classical and generalized concepts, the current paper proposes new refinements of essentiality and injectivity that are parameterized by a multiplicatively closed subset T of a monoid S. In [10], Rajaee introduced the notion of S-essential submodules, which inspired the idea of investigating an analogous concept within the category of acts over monoids.

We introduce and investigate the notions T-essential (or T-large), $T_\cap$-essential (or T-intersection large) subacts, T-injective and $T_\cap$-injective S-acts. The main objectives of this paper are to establish foundational properties of these generalized notions and to investigate their behavior under categorical constructions. Notably, we prove that every S-act admits a $T_\cap$-injective hull, and we characterize these hulls using universal properties similar to those of classical injective envelopes. We also study the closure of the classes of T-injective and $T_\cap$-injective acts under direct sums, products, retracts, and directed colimits. Furthermore, we demonstrate through examples that these new types of injectivity do not necessarily imply the existence of zero elements, which distinguishes them significantly from classical injectivity. We also give conditions under which pushouts and intersections preserve T-essentiality, and construct examples where the converse of known implications fails. The results presented here not only generalize previous work on injectivity and essential extensions in S-acts but also provide a flexible framework for further study of relative homological algebra in the category of monoid acts. The notions of T-injectivity and $T_\cap$-injectivity may also find applications in the study of localization, torsion theories, and categorical purity.

Now, we provide some introductions and definitions .

Let S be a monoid with identity element 1_S. A *(right) S-act* is a nonempty set A together with a map $A \times S \longrightarrow A$, $\quad (a, s) \mapsto a \cdot s$, satisfying the following axioms for all $a \in A$ and $s, t \in S$, $a \cdot 1_S = a$, $(a \cdot s) \cdot t = a \cdot (st)$.

A function $f : A \to B$ between two S-acts is called an S-homomorphism if it satisfies $f(a \cdot s) = f(a) \cdot s \quad$ for all $a \in A$, $s \in S$.

A subset $B \subseteq A$ is called a subact of A if it is closed under the action of S, i.e., $b \cdot s \in B$ for all $b \in B$ and $s \in S$. Let S be a monoid. A subset $T \subseteq S$ is called *multiplicatively closed* if it satisfies the following two conditions $1 \in T$, where 1 is the identity element of S, for all $t_1, t_2 \in T$, we have $t_1 t_2 \in T$. Let B be a subact of an S-act A. Then B is called essential (or large) in A if for every S-homomorphism $f : A \to C$, the condition that $f|_B$ is a monomorphism implies that f is a monomorphism. Also it is called intersection large in A if for every subact $C \subseteq A$ with $B \cap C = \emptyset$, it follows that $C = \emptyset$. For more see [8].

2 T- essential and $T_\cap$- essential subacts

In this section, the notions of T-essential (T-large) and T- intersection essential (T-intersection large) have been introduced as generalizations of the concepts of large and intersection large in algebraic structures. These notions allow us to examine the behavior of S-actions with respect to a specified subset T and gain a deeper understanding of how morphisms and related mappings align with these subcategories, and some fundamental properties and relationships among them are established, laying the groundwork for further research and applications in the theory of S-actions and category theory.

Definition 1. *Let B be a subact of an S-act A and $T \subseteq S$ be a multiplicatively closed subset of S. B is called **T-essential (T-large)** in A if for every S-homomorphism $f : A \to C$, if $f|_B$ is a monomorphism, then there exists $t \in T$ such that $f \circ \lambda_t : A \to C, \quad (a \mapsto f(a \cdot t))$ is a monomorphism, where $\lambda_t : A \longrightarrow A$ is the S-homomorphism given by $\lambda_t(a) = a.t$ for every $a \in A$.*

Definition 2. *Let S be a monoid, T a multiplicatively closed subset of S, and A an S-act. A subact B of A is **$T_\cap$-essential (T-intersection large)** if for any subact C of A such that $B \cap C = \emptyset$, there exists an element $t \in T$ such that $C.t = \emptyset$.*

Definition 3. *Let S be a monoid, and let $T \subseteq S$ be a multiplicatively closed subset. Let $\alpha : A \hookrightarrow B$ be a monomorphism of right S-acts. We say that α is* $T_\cap$-essential *if the image α is $T_\cap$-essential in B.*

Lemma 1. *Every T-essential subact is $T_\cap$-essential.*

Proof. Suppose $B \subseteq A$ is T-essential. Let $C \subseteq A$ be a subact such that $B \cap C = \emptyset$. We claim that there exists $t \in T$ such that $C \cdot t = \emptyset$. Suppose, for contradiction, that for all $t \in T$, $C \cdot t \neq \emptyset$.

We define an S-act D and a homomorphism $f : A \to D$ as follows:

- Let $D = (A \setminus C) \cup \{0\}$, where 0 is a new element.
- The S-action on D is defined by $x \cdot s = \begin{cases} \text{as in } A & \text{if } x \in A \setminus C \text{ and } x \cdot s \notin C, \\ 0 & \text{if } x \in C \text{ or } x \cdot s \in C \text{ or } x = 0. \end{cases}$

  Define $f(a) = \begin{cases} a & \text{if } a \notin C, \\ 0 & \text{if } a \in C. \end{cases}$

It is easy to check that f is an S-homomorphism. Moreover, $f|_B$ is monomorphism because $B \cap C = \emptyset$. Since B is T-essential, there must exist $t \in T$ such that $f \circ \lambda_t$ is a monomorphism. However, since $C \cdot t \neq \emptyset$, there exists $c \in C$ such that $c \cdot t \in A$. For such c, we have $f(c \cdot t) = 0$. This contradiction shows that our assumption was false. Thus, B is $T_\cap$-essential in A.

□

Remark 1. *Every intersection large subact is $T_\cap$-essential, but the converse is not true always. For this let $S = \{1, s, t\}$ be a monoid with multiplication defined as $1 \cdot x = x \cdot 1 = x$ for all $x \in S, s \cdot s = s, s \cdot t = t \cdot s = t, t \cdot t = t$. Consider $T = \{1, t\}$, which is a multiplicatively closed subset of S. Define the S-act $A = \{0, a, b, c\}$ with the action$1 * x = x$ for all $x \in A$, $s * 0 = 0$, $s * a = a$, $s * b = b$, $s * c = c, t * 0 = 0, t * a = 0, t * b = b, t * c = 0$. The subacts of A are $\{0\}, \{0, a\}, \{0, b\}, \{0, c\}, \{0, a, b\}, \{0, a, c\}, \{0, b, c\}, A$. Let $B = \{0, b\}$. We show that B is not intersection large in A. Consider $C = \{0, a\}$. We have $B \cap C = \{0, b\} \cap \{0, a\} = \{0\}$. Since C is nonzero and $B \cap C = \{0\}$, B is not intersection large. Now, we show that C is $T_\cap$-essential. The subacts C with $B \cap C = \{0\}$ are $\{0\}, \{0, a\}, \{0, c\}, \{0, a, c\}$. For each:*

- *$C = \{0\}$: $\{0\} * 1 = \{0\}$ (using $t = 1 \in T$),*
- *$C = \{0, a\}$: $\{0, a\} * t = \{0\}$,*
- *$C = \{0, c\}$: $\{0, c\} * t = \{0\}$,*
- *$C = \{0, a, c\}$: $\{0, a, c\} * t = \{0\}$.*

Thus, for every such C, there exists $t \in T$ with $C * t = \{0\}$, so B is T-essential.

Proposition 1. *Let $\{\, \iota_i : A \hookrightarrow B_i \mid i \in I\}$ be a family of T- essential extensions of the same S–act A. Then $\iota : A \hookrightarrow \bigcap_i B_i$ is also a T- essential.*

Proof. Let $B = \bigcap_i B_i$. Consider an S-homomorphism $f : B \longrightarrow C$ for which the composite $f \circ \iota : A \to C$ is a monomorphism. Since $B \subseteq B_i$ for each i, we obtain induced homomorphisms $f_i : B_i \longrightarrow C$, $f_i(b) = f(b)$, for all $b \in B$. On the common subact $A \subseteq B$, each restriction $f_i \circ \iota_i = f \circ \iota$ is a monomorphism. By T–essentiality of ι_i, for each i there exists $t_i \in T$ such that $f_i \circ \lambda_{t_i} : B_i \longrightarrow C$ is a monomorphism.

Now fix any index i. We claim that $f \circ \lambda_{t_i} : B \to C$ is a monomorphism. Indeed, suppose $x, y \in B$ satisfying $f\big(\lambda_{t_i}(x)\big) = f\big(\lambda_{t_i}(y)\big)$. Viewing x, y as elements of B_i, we have $f_i\big(\lambda_{t_i}(x)\big) = f\big(\lambda_{t_i}(x)\big) = f\big(\lambda_{t_i}(y)\big) = f_i\big(\lambda_{t_i}(y)\big)$. Since $f_i \circ \lambda_{t_i}$ is a monomorphism, it follows that $x = y$. Hence $f \circ \lambda_{t_i}$ is a monomorphism on B.

Thus, for the test map f we have produced $t_i \in T$ making $f \circ \lambda_{t_i}$ monic. By definition, $\iota : A \hookrightarrow B$ is T–essential. □

Proposition 2. *Let $\{\iota_i : A \hookrightarrow B_i \mid i \in I\}$ be a family of T-essential extensions of the same S-act A. Form the amalgamated coproduct $C = \Big(\bigsqcup_{i\in I} B_i\Big)\Big/\sim$, where for each $a \in A$ and all $i, j \in I$ one identifies $\iota_i(a) \sim \iota_j(a)$. Then the S-homomorphism $\iota : A \longrightarrow C$, $\iota(a) = [\,\iota_i(a)\,]$ is also a T-essential.*

Proof. Let $f : C \to D$ be an S-homomorphism such that $f \circ \iota$ is a monomorphism on A. Precompose with the quotient S-homomorphism $q : \bigsqcup_i B_i \to C$ to obtain, for each i, $f_i = f \circ q|_{B_i} : B_i \longrightarrow D$. Since $f \circ \iota$ is a monomorphism on A, each restriction $f_i \circ \iota_i$ is a monomorphism. By T–essentiality of ι_i, there exists $t_i \in T$ so that $f_i \circ \lambda_{t_i} : B_i \longrightarrow D$ is a monomorphism. Now consider any two elements $x, y \in C$ with $f\big(\lambda_t(x)\big) = f\big(\lambda_t(y)\big)$ for some $t \in T$. Choose representatives

$\tilde{x}, \tilde{y}$ in the disjoint union: say $\tilde{x} \in B_i$ and $\tilde{y} \in B_j$. If $i = j$, then taking $t = t_i$ forces $\tilde{x} = \tilde{y}$ since $f_i \circ \lambda_{t_i}$ is a monomorphism. If $i \neq j$, then after applying any t both images remain in distinct summands and cannot coincide in C unless both come from the common image of A. But on A the S-homomorphism $f \circ \iota$ is a monomorphism, so again $\tilde{x} = \tilde{y}$. Hence one may choose a single $t \in T$ (for instance a right-multiple of all the t_i involved) so that $f \circ \lambda_t \colon C \to D$ is a monomorphism. This shows $\iota \colon A \hookrightarrow C$ is T–essential. □

Example 1. Let $S = \{1\}$ be the trivial monoid and $T = S$. Consider the following acts and inclusions: $A = \{a\}$, $B_1 = \{a, b\}$, $B_2 = \{a, c\}$, with $\iota_1 \colon A \hookrightarrow B_1$ and $\iota_2 \colon A \hookrightarrow B_2$ sending $a \mapsto a$. Each ι_i is (trivially) essential in the sense that there is no nonzero subact disjoint from the image of A. Form the *plain* coproduct (disjoint union) $B_1 \sqcup B_2 = \{a_1, b\} \cup \{a_2, c\}$, where a_1 and a_2 are distinct copies of a. Define the injection $\delta : A \longrightarrow B_1 \sqcup B_2, \delta(a) = a_1$. Now define an S-homomorphism $f \colon B_1 \sqcup B_2 \longrightarrow B_2$ by $f(a_1) = a, f(b) = c, f(a_2) = a, f(c) = c$. We observe $f \circ \delta(a) = f(a_1) = a$, so $f \circ \delta \colon A \to B_2$ is a monomorphism. Since $T = \{1\}$, the only test is precomposing with the identity. But $f(b) = c = f(a_2)$, so f itself fails to be monomorphism on $B_1 \sqcup B_2$.

Hence there is no $t \in T$ making $f \circ \lambda_t$ monomorphism on the union, and δ is not essential. This shows that without amalgamating the copies of A, the plain coproduct injection need not be T–essential.

We recall that the pushout of the diagram in Act-S

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \\ C & & \end{array}$$

is the the quotient S-act $P = (B \sqcup C)/\!\sim$ and the canonical maps $u \colon B \to P$, $v \colon C \to P$ such that induced by the inclusion maps into the coproduct followed by the quotient.

Proposition 3. *Let S be a monoid and $T \subseteq S$ a multiplicatively closed subset. Suppose $f : A \longrightarrow C$ is a $T_\cap$-essential, and let $g \colon A \longrightarrow B$ be any S-homomorphism. Form the pushout*

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ B & \xrightarrow{k} & P \end{array}$$

in Act-S*. Then the S-homomorphism $k \colon B \longrightarrow P$ is also $T_\cap$-essential.*

Proof. We must show that $k(B) \subseteq P$ is $T_\cap$-essential, i.e. for every subact $D \subseteq P$ with $D \cap k(B) = \emptyset$, there is some $t \in T$ such that $D \cdot t = \emptyset$. Consider the quotient S-homomorphism $\pi \colon C \sqcup B \to P$. Obviously, $\pi^{-1}(D)$ contains no elements from B, otherwise $\pi(b) \in D \cap k(B)$, contradiction. Define $D_C = \{\, c \in C \mid \pi(c) \in D\}$. We have $\pi^{-1}(D) = D_C$. By construction $D_C \cap f(A) = \emptyset$. If $c = f(a) \in D_C$ then $\pi(g(a)) = \pi(f(a)) \in D$, contradicting $D \cap k(B) = \emptyset$.

Since $f \colon A \hookrightarrow C$ is $T_\cap$-essential, the subact D_C which is disjoint from $f(A)$, by assumption there exists some $t \in T$ such that $D_C \cdot t = \emptyset$.

Take any element $d \in D$. Choose a representative $c \in C$ with $\pi(c) = d$; such $c \in D_C$. Then $c \cdot t$ is not in C (since $D_C \cdot t = \emptyset$), so $\pi(c \cdot t)$ is not in the image of h. Moreover, if $\pi(c \cdot t)$ belong to image k, say $\pi(c \cdot t) = k(b)$, then by the defining relation $c \cdot t \sim g(a)$ for some a, but that would force $f(a) \in D_C$, again a contradiction. Therefore

$$d \cdot t \; = \; \pi(c) \cdot t \; = \; \pi(c \cdot t)$$

does not lie in P, i.e. $D \cdot t = \emptyset$.

So the S-homomorphism $k \colon B \to P$ is $T_\cap$-essential. □

Given two S-homomorphisms $f : X \to Z$ and $g : Y \to Z$, the pullback is an S-act $P = \{(x, y) | f(x) = g(y)\}$ together with S-homomorphisms $\pi_X : P \to X$ and $\pi_Y : P \to Y$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

commutes, and for any S-act Q with S-homomorphisms $u : Q \to X$, $v : Q \to Y$ such that $f \circ u = g \circ v$, there exists a unique S-homomorphism $h : Q \to P$ such that $\pi_X \circ h = u$ and $\pi_Y \circ h = v$. The following counterexample shows that the pullback of a $T_\cap$-essential monomorphism need not be $T_\cap$-essential.

Example 2. Let $S = \{1, t, s\}$ be a monoid with multiplication defined by $t^2 = t, s^2 = s, ts = st = s$.

Define $T = \{1, t\}$. Clearly, T is multiplicatively closed.

Now define the right S-act $B = \{b_1, b_2, b_3\}$ with action $b_1 \cdot t = b_2, b_2 \cdot t = b_3, b_3 \cdot s = b_3$, others: $b_i \cdot 1 = b_i$.

Let $Y = \{b_2, b_3\}$, which is a subact $b_2 \cdot t = b_3 \in Y, \quad b_3 \cdot t = b_3, \quad b_3 \cdot s = b_3$. Let $g : Y \hookrightarrow B$ be the inclusion.

We now define the act $X = \{x_1, x_2\}$, with action $x_1 \cdot t = x_2, x_2 \cdot s = x_2, x_i \cdot 1 = x_i$.

Define a morphism $f : X \to B$ by $f(x_1) = b_1, f(x_2) = b_3$.

Now form the pullback:

$$\begin{array}{ccc} P & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & B \end{array}$$

Then $P = \{(x, y) \in X \times Y : f(x) = g(y)\} = \{(x_2, b_3)\}, \quad \Rightarrow \quad \pi_X(P) = \{x_2\}$.

Let $C = \{x_1\} \subseteq X$. Then $C \cap \pi_X(P) = \varnothing$. But $C \cdot t = \{x_2\} \neq \varnothing$.

So, for all $t \in T$, we have $C \cdot t \neq \varnothing$, which contradicts the requirement for π_X to be $T_\cap$-essential.

But $g : Y \hookrightarrow B$ *is* $T_\cap$-essential. Any subact $D \subseteq B$ with $D \cap Y = \varnothing$ must be contained in $\{b_1\}$. Then $D \cdot t = \{b_2\} \subseteq Y$. So intersection occurs, then g satisfies the definition of $T_\cap$-essential.

Proposition 4. *Let S be a monoid, $T \subseteq S$ a multiplicatively closed subset, and $\iota\colon A \hookrightarrow B$ a monomorphism of right S-acts. Then ι is $T_\cap$-essential if and only if there does* not *exist a finite strictly increasing chain of subacts $A = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_n \subsetneq B$ such that for each $1 \le i \le n$, the inclusion $B_{i-1} \hookrightarrow B_i$ is not $T_\cap$-essential in B_i.*

Proof. ($\Rightarrow$) Assume A is $T_\cap$-essential in B. If there were a chain as above with each B_{i-1} failing to be $T_\cap$-essential in B_i, then for each i there would exist a nonzero subact $C_i \subseteq B_i$ disjoint from B_{i-1} such that $C_i \cdot t_i \neq \emptyset$ for all $t_i \in T$. But then C_n is a non-empty subact of B disjoint from $B_0 = A$ and not annihilated by any $t \in T$, contradicting that A is $T_\cap$-essential.

($\Leftarrow$) Conversely, suppose no such chain exists, but assume for contradiction that A is *not* $T_\cap$-essential in B. Then there is a nonzero subact $C \subseteq B$ with $A \cap C = \emptyset$ and $C \cdot t \neq \emptyset \quad \forall t \in T$. Set $B_1 = A \cup C$. Since C meets no translate of A, $A \subsetneq B_1$ and A is not $T_\cap$-essential in B_1. If $B_1 \neq B$, repeat the argument with B_1 in place of A inside B. Because each step strictly increases the subact, this process must terminate in finitely many steps at $A = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_n \subsetneq B_m = B$, with each inclusion $B_{i-1} \hookrightarrow B_i$ failing T-intersection. This contradicts the no-chain assumption. Hence A must be T-intersection large in B. ☐

3 T-injective S-acts

In this section, we focus on the notion of T-injectivity, where T is a multiplicatively closed subset of S, and explore how it generalizes classical injectivity by restricting attention to T-essential monomorphisms. We introduce the definition of T-injective hulls, and establish key results characterizing these notions through universal properties and maximal/minimal extension criteria. Examples illustrate the subtleties and distinctions between classical injectivity and T-injectivity, highlighting the significance of the chosen subset T in shaping the theory.

Definition 4. *Let S be a monoid, $T \subseteq S$ a multiplicatively closed subset, and let E be a right S-act. We say that E is T-injective if the following condition holds:*

For every T-essential monomorphism $\iota : A \hookrightarrow B$ of S-acts and every S-homomorphism $f : A \to E$, there exists an S-homomorphism $\tilde{f} : B \to E$ such that $\tilde{f} \circ \iota = f$. Clearly, every injective S-act is T-injective.

Example 3. Consider the monoid $S = \{1, s\}$ with $s^2 = s$, and set $T = \{1\}$. Define the right S-act $A = \{0, a\}$ by $0 \cdot x = 0, a \cdot 1 = a, a \cdot s = 0, \forall x \in S$.

Since $T = \{1\}$, a monomorphism $\iota\colon B \hookrightarrow C$ is T-essential precisely when for every S–homomorphism $f\colon C \to X$, $f \circ \iota$ monic implies f monic. One checks by a simple case-analysis that any such “essential” embedding admits a lift of every $f\colon B \to A$ to $\widetilde{f}\colon C \to A$. Hence A is T-injective.

Consider the monomorphism $\iota\colon B = \{0\} \hookrightarrow C = \{0, c\}, \quad c \cdot 1 = c, \ c \cdot s = c$. Define $f\colon B \to A, \quad f(0) = a$. Any extension $\widetilde{f}\colon C \to A$ would force $\widetilde{f}(c) = \widetilde{f}(c \cdot s) = \widetilde{f}(c) \cdot s$, so $\widetilde{f}(c) = 0$. But then $\widetilde{f}(0 \cdot s) = \widetilde{f}(0) = a \neq a \cdot s = 0$, a contradiction. Therefore no extension exists, and A fails to be injective.

Definition 5. *A pair (E, ι) is called a T-injective hull of a right S-act A if:*

- *$\iota\colon A \hookrightarrow E$ is a T-essential monomorphism,*

- *E is T-injective,*
- *and for every T-essential monomorphism $\iota'\colon A \hookrightarrow E'$ into a T-injective E', there exists a unique S-isomorphism $\varphi\colon E \to E'$ such that $\varphi \circ \iota = \iota'$.*

Theorem 1. *Let S be a monoid, $T \subseteq S$ a multiplicatively closed subset, and let $\iota\colon A \hookrightarrow E$ be a monomorphism of right S-acts. Then the following are equivalent:*

1. *(E, ι) is the T-injective hull of A.*
2. *ι is a maximal T-essential extension of A: If $\mu\colon E \to B$ is an S-homomorphism such that $\mu \circ \iota\colon A \to B$ is T-essential, then μ is an isomorphism.*
3. *ι is a minimal T-injective extension of A: E is T-injective, and if $\pi\colon E \to E'$ is an S-homomorphism into a T-injective E' such that $\pi \circ \iota$ coincides with the inclusion $A \to E'$, then π is a retraction.*

Proof. **(1) ⇒ (2):** Suppose (E, ι) is the T-injective hull of A, and let $\mu\colon E \to B$ be such that $\mu \circ \iota\colon A \to B$ is T-essential. Since (E, ι) is universal among T-essential embeddings into T-injective acts, and B must be T-injective as an image of E, the universal property provides a unique isomorphism from E to B. Hence, μ is an isomorphism.

(1) ⇒ (3): Again, by the universal property, if $\pi\colon E \to E'$ is a homomorphism into a T-injective E' with $\pi \circ \iota = \iota'$, then the universality implies that π is an isomorphism onto its image. Thus, there exists a retraction $r\colon E' \to E$ such that $r \circ \pi = \mathrm{id}_E$. **(2) and (3) ⇒ (1):** Suppose ι is both a maximal T-essential extension and a minimal T-injective extension. Then since ι is a T-essential and E is a T-injective (by minimality), it qualifies as a T-injective extension. Let $\iota'\colon A \hookrightarrow E'$ be any other T-injective extension with ι' T-essential. Then minimality of ι gives a morphism $f\colon E \to E'$ such that $f \circ \iota = \iota'$, and maximality implies that f is an isomorphism. Uniqueness of f follows from the essentiality of ι and the injectivity of the involved acts. Hence, (E, ι) satisfies the defining property of the T-injective hull. □

Proposition 5. *Let S be a monoid and $T \subseteq S$ a multiplicatively closed subset. The class of T-injective right S–acts is closed under:*

1. *Arbitrary products: If $\{E_i\}_{i \in I}$ is any family of T-injective right S–acts, then their product $E = \prod_{i \in I} E_i$ (with componentwise action) is T-injective.*
2. *Retracts: If E is T-injective and R is a retract of E (so there exist S-homomorphisms $i\colon R \to E$, $r\colon E \to R$ with $r \circ i = \mathrm{id}_R$), then R is T-injective.*
3. *Direct summands: Let S be a monoid with zero. Then direct summand of a T-injective S–act is T-injective.*

Proof. (1) Let $\alpha\colon A \hookrightarrow B$ be a T-essential monomorphism and $f\colon A \to \prod_{i \in I} E_i$ an S-homomorphism. Writing $\pi_i\colon \prod E_i \to E_i$ for the projections and setting $f_i = \pi_i \circ f$, each f_i extends to $\tilde{f}_i\colon B \to E_i$ since E_i is T-injective. Then $\widetilde{f} = (\tilde{f}_i)_{i \in I}\colon B \to \prod E_i$ extends f.

(2) Suppose $i\colon R \to E$ and $r\colon E \to R$ satisfy $r \circ i = \mathrm{id}_R$, which E is T-injective. Given a T-essential $\alpha\colon A \hookrightarrow B$ and $f\colon A \to R$, the composite $i \circ f\colon A \to E$ extends to $\widetilde{g}\colon B \to E$. Then $\widetilde{f} = r \circ \widetilde{g}\colon B \to R$ extends f.

(3) Every direct summand is a retract, so follows from (2). □

Proposition 6. *Let S be a monoid and $T \subseteq S$ a multiplicatively closed subset. Let $\{E_i\}_{i\in I}$ be a family of S–acts, and put $E = \prod_{i\in I} E_i$. Suppose there exists a family $\{t_i \in T\}_{i\in I}$ such that for every tuple $(x_j) \in E$, $(x_j)\cdot t_i = (y_j)$ where $y_i = x_i$ and $y_j = 0$ $(j \neq i)$, with each E_j admitting a distinguished zero element. If E is T-injective then E_i is also T-injective for every $i \in I$.*

Proof. Under the given hypothesis, the mapping $\sigma_i : E_i \longrightarrow E, \sigma_i(x) = (x_j)$ where $x_j = \begin{cases} x, & j = i, \\ 0, & j \neq i, \end{cases}$ is an S-homomorphism, and satisfies $\pi_i \circ \sigma_i = \mathrm{id}_{E_i}$, where π_i is the ith projection. Thus each E_i is a retract of E. Since retracts of T-injective acts are T-injective (as in Proposition 5), each factor E_i is T-injective. □

4 $T_\cap$-injective S-acts

In this section, we focus on the notion of $T_\cap$-*injectivity*, which generalizes classical injectivity by restricting attention to monomorphisms whose domains are $T_\cap$-essential subacts of their codomains, where T is a multiplicatively closed subset of the monoid S. We study fundamental characterizations of $T_\cap$-injective acts, investigate their closure properties under products, retracts, and direct summands, and establish the existence and uniqueness of minimal $T_\cap$-injective hulls for arbitrary right S-acts. Moreover, through examples, we demonstrate that $T_\cap$-injectivity differs significantly from classical injectivity, for instance, in the absence of zero elements in some $T_\cap$-injective acts.

Definition 6. *An S-act E is called* $T_\cap$-injective *if for every monomorphism $\alpha\colon A \hookrightarrow B$ such that A is a $T_\cap$-essential subact of B, and for every S-homomorphism $f\colon A \to E$, there exists an S-homomorphism $\widetilde{f}\colon B \to E$ such that $\widetilde{f} \circ \alpha = f$.*

Proposition 7. *Let S be a monoid and $T \subseteq S$ a multiplicatively closed subset of S. For a right S-act E, the following are equivalent:*

(1) *E is $T_\cap$-injective.*

(2) *Every monomorphism $\iota\colon E \hookrightarrow B$ in which E is a $T_\cap$-essential subact of B splits, i.e., there exists a retraction $r\colon B \to E$ such that $r \circ \iota = \mathrm{id}_E$.*

Proof. **(1) ⇒ (2).** Assume E is $T_\cap$-injective. Let $\iota\colon E \hookrightarrow B$ be a monomorphism with E a $T_\cap$-essential subact of B. Consider the identity map $\mathrm{id}_E\colon E \to E$. By $T_\cap$-injectivity, since ι is $T_\cap$-essential, there exists an S-homomorphism $r\colon B \to E$ such that $r \circ \iota = \mathrm{id}_E$. Thus ι splits and (2) holds.

(2) ⇒ (1). Assume every such monomorphism splits. Let $\alpha : A \longrightarrow B$ be an arbitrary monomorphism with A a $T_\cap$-essential extension subact of B, and let $f\colon A \to E$ be any S-homomorphism. We must construct an extension $\widetilde{f}\colon B \to E$ satisfying $\widetilde{f} \circ \alpha = f$.

Consider the pushout square in Act-S:

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\iota} \\ B & \xrightarrow[\beta]{} & P \end{array}$$

By Proposirion 3, we can conclude that $\iota : E \hookrightarrow P$ is $T_\cap$- essential.

By hypothesis (2), ι splits: there is a retraction $\rho\colon P \to E$ with $\rho \circ \iota = \mathrm{id}_E$.

The pushout property gives a canonical S-homomorphism $\beta\colon B \to P$ extending α. Define $\widetilde{f} = \rho \circ \beta : B \longrightarrow E$. Then on A, $\widetilde{f} \circ \alpha = \rho \circ \beta \circ \alpha = \rho \circ \iota \circ f = \mathrm{id}_E \circ f = f$, as required. Thus E is $T_\cap$-injective, completing the proof. □

Proposition 8. *Let S be a monoid and $T \subseteq S$ a multiplicatively closed subset of S. Then the class of $T_\cap$-injective right S-acts is closed under:*

1. *Arbitrary products,*
2. *Retracts,*
3. *Direct summands.*

Proof. (1) Products. Let $\{E_i\}_{i\in I}$ be a family of $T_\cap$-injective S-acts, and set $E = \prod_{i\in I} E_i$. Given any monomorphism $\alpha\colon A \hookrightarrow B$ which A is $T_\cap$-essential extension in B, and any S-homomorphism $f\colon A \longrightarrow E$, we must produce an extension $\widetilde{f}\colon B \to E$. For each $i \in I$, let $\pi_i\colon E \to E_i$ be the projection and define $f_i = \pi_i \circ f : A \longrightarrow E_i$. Since each E_i is $T_\cap$-injective and A is $T_\cap$-essential extension in B, there exists for each i an extension $\widetilde{f_i}\colon B \longrightarrow E_i$ with $\widetilde{f_i} \circ \alpha = f_i$. Now assemble these extensions into a single S-homomorphism $\widetilde{f} = (\widetilde{f_i})_{i\in I} : B \longrightarrow \prod_{i\in I} E_i = E$. By construction $\widetilde{f} \circ \alpha = f$, so E is $T_\cap$-injective.

(2) Retracts. Suppose E is $T_\cap$-injective and that R is a retract of E. Thus there exist S-homomorphisms $R \xrightarrow{\iota} E \xrightarrow{r} R$ with $r \circ \iota = \mathrm{id}_R$. Given a monomorphism $\alpha\colon A \hookrightarrow B$, which A is $T_\cap$-essential extension in B and any $f\colon A \to R$, compose to get $\iota \circ f\colon A \to E$. Since E is $T_\cap$-injective, $\iota \circ f$ extends to $\widetilde{g}\colon B \to E$. Then $\widetilde{f} := r \circ \widetilde{g} : B \longrightarrow R$ satisfies $\widetilde{f} \circ \alpha = r \circ \widetilde{g} \circ \alpha = r \circ (\iota \circ f) = f$. Hence, R is $T_\cap$-injective.

(3) Direct summands. A direct summand of a $T_\cap$-injective act is, in particular, a retract (via projection onto that summand). Therefore closure under retracts immediately implies closure under direct summands. □

Theorem 2. *Let S be a monoid and $T \subseteq S$ a multiplicatively closed subset. Then every right S-act A admits a* minimal *$T_\cap$-essential $A \hookrightarrow E_{T_\cap}(A)$ into a $T_\cap$-injective act $E_{T_\cap}(A)$. This extension is called the* $T_\cap$-injective hull *of A, and it is unique up to unique isomorphism over A.*

Proof. Let $\mathcal{E}$ denote the class of pairs (E, ι) such that $A \longrightarrow E$ is a monomorphism of S-acts, and E is a $T_\cap$-essential of A. Clearly, it is a non-empty set since $(A, id_A) \in \mathcal{E}$. We define partially order $\mathcal{E}$ by

$$(E_1, \iota_1) \le (E_2, \iota_2) \iff \text{there exists an } S\text{-monomorphism } \phi\colon E_1 \to E_2$$

such that $\phi \circ \iota_1 = \iota_2$.

Let $\{(E_i, \iota_i)\}_{i\in I}$ be a chain in $\mathcal{E}$. Clearly, $\mathcal{E}$ is nonempty. Indeed, the identity map $\mathrm{id}_A : A \hookrightarrow A$ is a monomorphism, and A is trivially $T_\cap$-essential in itself.

Let $\{(E_i, \iota_i)\}_{i\in I} \subseteq \mathcal{E}$ be a chain. Without loss of generality, we may assume all maps $\iota_i : A \to E_i$ are the same inclusion and that $E_i \subseteq E_j$ for $i < j$. Define $E := \bigcup_{i\in I} E_i$. Then E is an S-act, and the inclusion $\iota : A \hookrightarrow E$ is a monomorphism. Hence, every chain has an upper bound, and by Zorn's Lemma, $\mathcal{E}$ has a maximal element (E, ι). Next we show E is $T_\cap$-injective. If not, by Proposition 7, there is a monomorphism $\alpha : E \hookrightarrow C$ with E is $T_\cap$- essential in C but no retraction. Then $A \longrightarrow E \longrightarrow C$ is strictly larger than (E, ι) in $\mathcal{E}$, contradicting maximality. Hence, E is $T_\cap$-injective, and we set $E_{T_\cap}(A) = E$.

Let E_1 and E_2 be two $T_\cap$-injective hulls of A. Then there exist monomorphisms $\phi_1 : E_1 \to E_2$ and $\phi_2 : E_2 \to E_1$ such that both restrict to the identity on A. Since both E_1 and E_2 are minimal $T_\cap$-essential, these morphisms must be isomorphisms. Moreover, they are unique over A, because if $\psi : E_1 \to E_2$ also restricts to the identity on A, then by minimality and essentiality, $\psi = \phi_1$.

□

We recall that a directed system $\{E_i, \phi_{ij}\}_{i\in I}$ in Act-S consists of a directed poset I (i.e., for any $i, j \in I$, there exists $k \in I$ with $i \le k$ and $j \le k$), a family of right S-acts $\{E_i\}_{i\in I}$, a family of S-homomorphisms $\phi_{ij} : E_i \to E_j$ for $i \le j$, such that $\phi_{ii} = \mathrm{id}_{E_i}$ for all $i \in I$, $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ whenever $i \le j \le k$. The colimit of a directed system $\{E_i, \phi_{ij}\}_{i\in I}$ in Act-S is a right S-act E, together with a family of S-homomorphisms $\iota_i : E_i \to E$, satisfying $\iota_j \circ \phi_{ij} = \iota_i$ for all $i \le j$, For any other S-act F with homomorphisms $f_i : E_i \to F$ such that $f_j \circ \phi_{ij} = f_i$, there exists a unique S-homomorphism $f : E \to F$ such that $f \circ \iota_i = f_i$ for all i.

Proposition 9. *Let S be a monoid and $T \subseteq S$ a submonoid. Suppose $\{E_i, \phi_{ij}\}_{i\in I}$ is a directed system of $T_\cap$-injective right S-acts, and let $E = \varinjlim_{i\in I} E_i$ be its colimit in* Act-S*. Then E is $T_\cap$-injective for finitely generated $T_\cap$-essential subact.*

Proof. Let $\alpha : A \hookrightarrow B$ be a monomorphism which A is $T_\cap$-essential in B, and assume A is finitely generated. Given any S-homomorphism $f : A \longrightarrow E$, we will produce an extension $\widetilde{f} : B \to E$ with $\widetilde{f} \circ \alpha = f$.

Since A is finitely generated, say by elements $a_1, \dots, a_n$, and $E = \varinjlim_i E_i$, each $f(a_j) \in E$ comes from some stage E_{i_j}. By directedness, there is an index $i_0 \in I$ with $i_0 \ge i_j$ for all j. Hence f factors uniquely as $f = \iota_{i_0} \circ f_{i_0}$, where $\iota_{i_0} : E_{i_0} \to E$ is the colimit S-homomorphism and $f_{i_0} : A \longrightarrow E_{i_0}$ is the induced S–homomorphism.

Because E_{i_0} is $T_\cap$-injective and α has A is $T_\cap$-essential in B, there exists an extension $\widetilde{f_{i_0}} : B \longrightarrow E_{i_0}$ such that $\widetilde{f_{i_0}} \circ \alpha = f_{i_0}$.

Compose $\widetilde{f_{i_0}}$ with ι_{i_0} to obtain $\widetilde{f} = \iota_{i_0} \circ \widetilde{f_{i_0}} : B \longrightarrow E$. Then on A, $\widetilde{f} \circ \alpha = \iota_{i_0} \circ \widetilde{f_{i_0}} \circ \alpha = \iota_{i_0} \circ f_{i_0} = f$, as required.

□

We provide a counterexample to show that both T-injectivity and $T_\cap$-injectivity do not necessarily imply the existence of a zero element in the corresponding right S-act.

Example 4. Let $S = \{1, t, s\}$ be a monoid with multiplication defined by: $t^2 = t, \quad s^2 = s, \quad st = ts = s, \quad$ and 1 is the identity.

Let $T = \{1, t\} \subsetneq S$. Then T is a proper submonoid of S, closed under multiplication and containing 1.

Define the right S-act $A = \{a, b\}$ with the action $a \cdot 1 = a, \quad a \cdot t = b, \quad a \cdot s = b, \quad b \cdot x = b \quad \forall x \in S$.

We show that A is both T-injective and $T_\cap$-injective, but has no zero element.

A has no zero: $a \cdot t = b \neq a$, so neither a nor b is fixed under all actions.

Let $\alpha : M \hookrightarrow N$ be a T-essential (or $T_\cap$-essential) monomorphism and let $f : M \to A$ be an S-map. Define $\tilde{f} : N \to A$ by:

$$\tilde{f}(n) = \begin{cases} f(n) & n \in M, \\ b & n \in N \setminus M. \end{cases}$$

This map is well-defined and S-homomorphism because b is absorbing: $b \cdot x = b$ for all $x \in S$.

Thus, every S-map from a subact M extends over any T-essential or $T_\cap$-essential monomorphism into N, making A both T-injective and $T_\cap$-injective.

5 Conclusion

We studied two new parameterized notions of injectivity in the category of right S-acts, namely T-injectivity and $T\cap$-injectivity, where T is a multiplicatively closed subset of a monoid S. These concepts generalize classical injectivity by restricting attention to special classes of monomorphisms determined by the algebraic properties of T. We also introduced the related notions of T-essential and $T_\cap$-essential (also called T-large and $T_\cap$-large subacts) and provided characterizations of these in terms of their behavior under morphisms and subact intersections. Through examples and counterexamples, we have demonstrated that these generalizations exhibit behavior that differs significantly from their classical counterparts, especially with respect to zero elements, pullback constructions, and preservation properties. Furthermore, we proved the existence and uniqueness of $T_\cap$-injective hulls for arbitrary S-acts and established closure properties of both T-injective and $T_\cap$-injective classes under products, retracts, and directed colimits. These results lay a foundation for developing a relative version of homological algebra in the setting of monoid acts. Our study opens new avenues for further exploration, such as the relationship between these injectivity notions and torsion theories, localization processes, and categorical purity. The flexibility of using a parameter T enables a refined structural analysis of S-acts that may prove useful in both theoretical and applied algebraic contexts.

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