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Super-biderivations and linear super-commuting maps on infinite-dimensional Lie superalgebras

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Abstract. Let $\mathcal{G}_\epsilon$ (resp. $\mathcal{W}_\epsilon$) with $\epsilon = 0$ or $\frac{1}{2}$ be the complete spectrum-generating superalgebra (resp. the centerless super Virasoro algebra). In this paper, the super-skewsymmetric super-biderivations on $\mathcal{G}_\epsilon$ and $\mathcal{W}_\epsilon$ are completely determined. In particular, we show that every super-skewsymmetric super-biderivation φ (resp. ϕ) of $\mathcal{G}_\epsilon$ (resp. $\mathcal{W}_\epsilon$) is inner. Based on the results of super-biderivations, we shall give the certain forms of all linear super-commuting maps on $\mathcal{G}_\epsilon$ and $\mathcal{W}_\epsilon$.

Keywords: Complete spectrum-generating superalgebra, Centerless super Virasoro algebra, Super-biderivation, Super-skewsymmetric, Linear super-commuting map.
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1 Introduction

Let $(L, \circ)$ be an algebra (not necessarily be an associative algebra), where L is a vector space and $\circ$ is a binary operation from $L \times L$ to L defined by

$$\circ : (x, y) \mapsto x \circ y$$

for all $x, y \in L$. A linear map $\phi : L \to L$ is called a derivation if it satisfies

$$\phi(x \circ y) = \phi(x) \circ y + x \circ \phi(y)$$

for all $x, y \in L$. Let $\mathfrak{g}$ be a Lie algebra, then we denote by $x \circ y = [x, y]$ for all $x, y \in \mathfrak{g}$. In this case, for $x \in \mathfrak{g}$, it is easy to see that $\sigma_x : \mathfrak{g} \to \mathfrak{g}$, $y \mapsto ad_x(y) = [x, y]$ for all $x, y \in \mathfrak{g}$, is a

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derivation of $\mathfrak{g}$, which is called an inner derivation. Denote by $Der(\mathfrak{g})$ and by $Inn(\mathfrak{g})$ the space of derivations and the space of inner derivations of $\mathfrak{g}$, respectively.

Let $\mathcal{A}$ be an associative algebra (or ring). A map $\varphi : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a biderivation of $\mathcal{A}$ if it is a derivation concerning both components, that is,

$$\varphi(ab, c) = a\varphi(b, c) + \varphi(a, c)b \tag{1}$$

and

$$\varphi(a, bc) = \varphi(a, b)c + b\varphi(a, c) \tag{2}$$

for all $a, b, c \in \mathcal{A}$. The problems of biderivations and their generalizations have been extensively studied in [3, 9, 11, 29, 34]. The concept of biderivation was transfered from associative algebras to Lie algebras in [15, 29].

For a Lie algebra $\mathfrak{g}$, a bilinear map $\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called a biderivation of $\mathfrak{g}$ if it is a derivation concerning both components, in other words, both linear maps ϕ_x and ψ_z from $\mathfrak{g}$ into itself given by $\phi_x = \varphi(x, .)$ and $\psi_z = \varphi(., z)$ are derivations of $\mathfrak{g}$, namely,

$$\varphi(x, [y, z]) = [\varphi(x, y), z] + [y, \varphi(x, z)]\,, \tag{3}$$

and

$$\varphi([x, y]\,, z) = [x, \varphi(y, z)] + [\varphi(x, z), y]\,, \tag{4}$$

for all $x, y, z \in \mathfrak{g}$. The biderivations of Lie algebras have been studied extensively, such as [6, 7, 27, 29]. It was showed that all biderivations on commutative prime rings are inner biderivations and the biderivations on a perfect and centerless Lie algebra are inner biderivations in [4, 5]. Some biderivations of specific examples of Lie algebras have been presented by many authors in [8, 15, 21, 26, 28, 29].

For a Lie algebra $\mathfrak{g}$ and $\lambda \in \mathbb{C}$, it is easy to verify that the bilinear map $\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ given by $\varphi(x, y) = \lambda\,[x, y]$ for all $x, y \in \mathfrak{g}$ is a biderivation of $\mathfrak{g}$, which is called an inner biderivation.

As a generalization of the biderivation of Lie algebra, the concept of super-biderivation of Lie superalgebra was introduced in [10] and [33], respectively. Lie superalgebras as a generalization of Lie algebras came from supersymmetry in mathematical physics. The theory of Lie superalgebras over a field of characteristic zero have seen a remarkable evolution, both in mathematics and physics (see [19, 20]).

Let S be a Lie superalgebra with $\mathbb{Z}_2$-grading $S = S_{\overline{0}} + S_{\overline{1}}$, where $S_{\overline{0}}$ and $S_{\overline{1}}$ are even and odd parts of S, respectively. We call a bilinear map $\varphi : S \times S \to S$ a super-biderivation of S if for every $x_{\overline{0}} \in S_{\overline{0}}$ the maps

$$x \mapsto \varphi(x_{\overline{0}}, x) \;\text{ and }\; x \mapsto \varphi(x, x_{\overline{0}}),$$

are even superderivations, and for every $x_{\overline{1}} \in S_{\overline{1}}$ the maps

$$x \mapsto \varphi(x_{\overline{1}}, x) \;\text{ and }\; x \mapsto \varphi(x, x_{\overline{1}})^{\sigma},$$

are even superderivations, where σ is defined by

$$(x_{\overline{0}} + x_{\overline{1}})^{\sigma} = x_{\overline{0}} - x_{\overline{1}}, \;\; x_{\overline{0}} \in S_{\overline{0}}, \; x_{\overline{1}} \in S_{\overline{1}}.$$

One can easily check that this is equivalent to

$$\varphi([x,y],z)=[x,\varphi(y,z)]+(-1)^{|y||z|}[\varphi(x,z),y], \tag{5}$$

and

$$\varphi(x,[y,z])=[\varphi(x,y),z]+(-1)^{|x||y|}[y,\varphi(x,z)], \tag{6}$$

for all $x,y,z\in S$. Here, and in what follows, we use the notation $|x|$ ($\overline{0}$ or $\overline{1}$) to denote the $\mathbb{Z}_2$-degree of a homogeneous element $x\in S$, and we always assume that x is homogeneous if $|x|$ appears in an expression. The map φ_λ with $\lambda\in\mathbb{C}$ given by

$$\varphi_\lambda(x,y)=\lambda[x,y] \quad \text{foa all} \quad (x,y)\in S\times S,$$

is a super-biderivation of S obviously. We call φ_λ an inner super-biderivation of S.

The skew-symmetric biderivations on the conformal Galilei algebra, the Lie algebra $\mathcal{W}(a,b)$, Schrodinger-Virasoro Lie algebra and the Lie algebra gca have been studied in [1,7,15,30], respectively. In [2,18,24,33] the authors proved that each super-skewsymmetric super-biderivation on the simple modular Lie superalgebras of Witt type and special type, the $N=1$ super Heisenberg-Virasoro algebra, the super-Virasoro algebra and the twisted $N=1$ Schrödinger-Neveu-Schwarz algebra is inner. In [16] and [17] the authors characterized the super-skewsymmetric super-biderivations of the $2d$ supersymmetric Galilean conformal algebra and the super Heisenberg-Virasoro algebra, respectively.

In this paper, we will study the super-skewsymmetric superbiderivations on the complete spectrum-generating superalgebra $\mathcal{G}_\epsilon$ and on the centerless super Virasoro algebra $\mathcal{W}_\epsilon$. We can find that every super-skewsymmetric super-biderivation of these Lie superalgebras is inner. As applications of these super-skewsymmetric super-biderivations, we give the form of each linear super-commuting maps on $\mathcal{G}_\epsilon$ and $\mathcal{W}_\epsilon$.

Now we present the definition of complete spectrum-generating superalgebra $\mathcal{G}_\epsilon$ in [12,25]. Let $\Gamma=\epsilon+\mathbb{Z}$, with $\epsilon=0$ or $\frac{1}{2}$. For $\epsilon\in\{0,\frac{1}{2}\}$, the complete spectrum-generating superalgebra is an infinite-dimensional Lie superalgebra

$$\mathcal{G}_\epsilon=\bigoplus_{m\in\mathbb{Z}}L_m\oplus\bigoplus_{m\in\mathbb{Z}}I_m\oplus\bigoplus_{r\in\Gamma}G_r\oplus\bigoplus_{r\in\Gamma}J_r$$

which satisfies the following super-brackets:

$$\begin{aligned}
&[L_m,L_n]=(m-n)L_{m+n}, && [L_m,I_n]=-nI_{m+n},\\
&[L_m,G_r]=\left(\tfrac{m}{2}-r\right)G_{r+m}, && [L_m,J_r]=-\left(\tfrac{m}{2}+r\right)J_{r+m},\\
&[I_m,G_r]=mJ_{r+m}, && [G_r,G_s]=2L_{r+s},\\
&[G_r,J_s]=I_{r+s}, && [I_m,I_n]=[I_m,J_r]=[J_r,J_s]=0
\end{aligned}$$

for $m,n\in\mathbb{Z}$, $r,s\in\Gamma$. Obviously, $\mathcal{G}_\epsilon$ is $\mathbb{Z}_2$-graded: $\mathcal{G}_\epsilon=\mathcal{G}_{\epsilon\overline{0}}\oplus\mathcal{G}_{\epsilon\overline{1}}$, with

$$\mathcal{G}_{\epsilon\overline{0}}=\operatorname{span}\{L_m,I_m\mid m\in\mathbb{Z}\}, \tag{7}$$

$$\mathcal{G}_{\epsilon\overline{1}}=\operatorname{span}\{G_r,J_r\mid r\in\Gamma\}. \tag{8}$$

Then, based on the results of super-biderivations of the Lie superalgebras, we shall investigate the form of all linear super-commuting maps on the Lie superalgebras. The definition of linear commuting map on associative algebras (or Lie algebras) has been introduced in [15, 30, 33]. As a generalization of the concept of commuting maps on Lie algebras, one can easily give the concept of super-commuting maps on Lie superalgebras [10, 31–33]. For a Lie superalgebra S, a map $\psi : S \to S$ is called a super-commuting if it preserves the $\mathbb{Z}_2$-grading of S and

$$[\psi(x), x] = 0 \quad \text{foa all} \quad (x, y) \in S. \tag{9}$$

A super-commuting map ψ on S is called standard if it maps the even part $S_{\overline{0}}$ of S to the center of S, and maps the odd part $S_{\overline{1}}$ of S to zero. All super-commuting maps of other forms are said to be non-standard. It was shown in [32, 33] that all linear super-commuting maps on the super-Galilean conformal algebras and the super-Virasoro algebras are standard. In [31], the authors showed that all linear super-commuting maps on the super-BMS$_3$ are non-standard.

Recall that a Lie superalgebra S is perfect if $[S, S] = S$. Note that the complete spectrum-generating superlalgebra $\mathcal{G}_\epsilon$ is perfect which can be easily checked by above definition. The centre of this Lie superalgebra is

$$Z(\mathcal{G}_\epsilon) = \text{span}\{I_0\}.$$

Now, we outline our main results in this paper. In Section 2, we recall some general results on super-biderivations of Lie superalgebras. In Section 3, we characterize the super-skewsymmetric super-biderivations of the complete spectrum-generating superalgebra $\mathcal{G}_\epsilon$. In Section 4, we investigate the super-skewsymmetric super-biderivations of the centerless super Virasoro algebra $\mathcal{W}_\epsilon$. Based on these results of super-biderivations, finally, in Section 5, we study linear super-commuting maps on $\mathcal{G}_\epsilon$ and $\mathcal{W}_\epsilon$.

Throughout this paper, we denote by $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{Z}^*$ the sets of complex numbers, integers and nonzero integers, respectively.

2 Basic results on super-biderivations of Lie superalgebras

Let S be a Lie superalgebra with $\mathbb{Z}_2$-grading $S = S_{\overline{0}} + S_{\overline{1}}$, where $S_{\overline{0}}$ and $S_{\overline{1}}$ are even and odd parts of S, respectively. A bilinear map $\varphi : S \times S \longrightarrow S$ is called super-skewsymmetric (or super-antisymmetric) if

$$\varphi(x, y) = -(-1)^{|x||y|}\varphi(y, x) \tag{10}$$

for all $x, y \in S$. In order to avoid lengthy notations, we set

$$F(x, y, u, v) = (-1)^{|u||y|} \left([\varphi(x, y), [u, v]] - [[x, y], \varphi(u, v)]\right) \tag{11}$$

for all $x, y \in S$.

The following result, which will be used later in our paper, is cited from Lemma 2.1 and Lemma 2.2 in [33].

Lemma 1. *Let φ be a super-biderivation on S. Then*

$$F(x, y, u, v) = (-1)^{|y||v|}F(x, v, u, y), \quad x, y, u, v \in S.$$

Lemma 2. *Let φ be a super-skewsymmetric super-biderivation on S.*

(1) $F(x, y, u, v) = 0$ for all $x, y, u, v \in S$.

(2) For $x, y \in S$, if $|x| + |y| = \overline{0}$, then $[\varphi(x, y), [x, y]] = 0$.

(3) Suppose S is perfect. For $x, y \in S$, if $[x, y] = 0$, then $\varphi(x, y) \in Z(S)$.

Remark 1. *Let $\mathfrak{g}$ be a Lie algebra with center $Z(\mathfrak{g})$. A bilinear map $\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called skewsymmetric (or antisymmetric) if*

$$\varphi(x, y) = -\varphi(y, x) \quad \text{foa all} \quad x, y \in \mathfrak{g}.$$

Let φ be a skewsymmetric biderivation on $\mathfrak{g}$. As direct corollaries of Lemmas 1 and 2, we have

(1) $[\varphi(x, y), [u, v]] = [[x, y], \varphi(u, v)]$ for all $x, y, u, v \in \mathfrak{g}$. In particular,

$$[\varphi(x, y), [x, y]] = 0 \quad \text{for all} \quad x, y \in \mathfrak{g}.$$

(2) Suppose $\mathfrak{g}$ is perfect. For $x, y \in \mathfrak{g}$, if $[x, y] = 0$, then $\varphi(x, y) \in Z(\mathfrak{g})$.

Similar statements with $\mathfrak{g}$ being some special Lie algebras were given in [29, 30] but with a gap (without the assumption of skewsymmetry of φ), which was first filled in [6].

Remark 2. *The superalgebra $\mathcal{G}_\epsilon$ was independently introduced as a supersymmetric extension of the Beltrami algebra [14]. The superalgebra $\mathcal{G}_\epsilon$ was studied by Marcel, Roger and Ovsienko in their investigation of the generalized Sturm-Liouville operators [22, 23]. Recently, $\mathcal{G}_\epsilon$ also appeared in the Guha's interesting work [13] on the integrable geodesic flows on the superextension of the Bott-Virasoro group.*

3 Super-skewsymmetric super-biderivations of the complete spectrum-generating superalgebra $\mathcal{G}_\epsilon$

In this section,we present a description of the super-skewsymmetric super-biderivations of the complete spectrum-generating superalgebra $\mathcal{G}_\epsilon$.

Definition 1. *For $b_{m,n,0}, d_{m,n,0}, q_{r,s,0}, v_{r,s,0}, \kappa_{m,n,1}, \kappa_{r,s,2} \in \mathbb{C}$, $m, n \in \mathbb{Z}$, $r, s \in \Gamma$, the bilinear map $\varphi_0 : \mathcal{G}_\epsilon \times \mathcal{G}_\epsilon \to \mathbb{C}I_0$ is given by*

$$\begin{aligned}
&\varphi_0(L_m, L_n) = b_{m,n,0} I_0, \\
&\varphi_0(L_m, I_n) = d_{m,n,0} I_0, \\
&\varphi_0(I_m, I_n) = \kappa_{m,n,1} I_0, \\
&\varphi_0(G_r, G_s) = q_{r,s,0} I_0, \\
&\varphi_0(G_r, J_s) = v_{r,s,0} I_0, \\
&\varphi_0(J_r, J_s) = \kappa_{r,s,2} I_0, \\
&\varphi_0(L_m, G_r) = \varphi_0(L_m, J_r) = \varphi_0(I_m, G_r) = \varphi_0(I_m, J_r) = 0,
\end{aligned}$$

Note that φ_0 is super-skewsymmetric. Now we present our main result in this section.

Theorem 1. *Every super-skewsymmetric super-biderivation φ of $\mathcal{G}_\epsilon$ is inner.*

Proof. First, we claim that $|\varphi(x,y)| = |x|+|y|$ for any homogeneous elements $x, y \in \mathcal{G}_\epsilon$. This fact will remarkably simplify our discussion. If $|x| = \overline{0}$ and y is homogeneous, from the original definition of super-biderivetions above, we know that the map $z \mapsto \varphi(x,z)$ is an even superderivation, and hence if z is homogeneous then $\varphi(x,z)$ is also homogeneous and $|\varphi(x,z)| = |z|$. In particular, by taking $z = y$, we have $|\varphi(x,y)| = |y|$. Namely, $|\varphi(x,y)| = |x| + |y|$ since $|x| = \overline{0}$. Similarly, if $|x| = \overline{1}$ and y is homogeneous, then the map $z \mapsto \varphi(x,z)$ is an odd superderivation, and hence if z is homogeneous then $\varphi(x,z)$ is also homogeneous, and $|\varphi(x,z)| = |z| + \overline{1}$. In particular, by taking $z = y$, we have $|\varphi(x,y)| = |y| + \overline{1}$. Namely, $|\varphi(x,y)| = |x| + |y|$ since $|x| = \overline{1}$.

Next, we give the proof of the theorem by the following several claims.

Claim 1: There exists $\lambda \in \mathbb{C}$ such that

$$\varphi(L_m, L_n) = \lambda\,[L_m, L_n] + \varphi_0(L_m, L_n) \text{ for all } m, n \in \mathbb{Z}.$$

Note that $|\varphi(L_m, L_n)| = |L_m| + |L_n| = \overline{0}$. For any $m, n \in \mathbb{Z}$, we can suppose that

$$\varphi(L_m, L_n) = \sum_{\alpha\in\mathbb{Z}} a_{m,n,\alpha} L_\alpha + \sum_{\beta\in\mathbb{Z}} b_{m,n,\beta} I_\beta,$$

where $a_{m,n,\alpha}, b_{m,n,\beta} \in \mathbb{C}$.

If $m = n$, then $[L_m, L_n] = 0$, by Lemma 2(3), $\varphi(L_m, L_n) \in Z(\mathcal{G}_\epsilon) = \mathbb{C}I_0$. That is, this claim holds.

Next, we assume that $m \neq n$. By Lemma 2(2), we have

$$[\varphi(L_m, L_n), [L_m, L_n]] = 0,$$

then, we get

$$\sum_{\alpha\in\mathbb{Z}} (m+n-\alpha) a_{m,n,\alpha} L_{m+n+\alpha} - \sum_{\beta\in\mathbb{Z}} \beta b_{m,n,\beta} I_{m+n+\beta} = 0,$$

it follows that

$$(m+n-\alpha) a_{m,n,\alpha} = 0; \qquad \beta b_{m,n,\beta} = 0.$$

So we have $a_{m,n,\alpha} = 0$ if $\alpha \neq m+n$; $b_{m,n,\beta} = 0$ if $\beta \neq 0$. We conclude that

$$\varphi(L_m, L_n) = a_{m,n,m+n} L_{m+n} + b_{m,n,0} I_0.$$

Furthermore, by Lemma 2(1), we get

$$[\varphi(L_m, L_n), [L_1, L_0]] = [[L_m, L_n], \varphi(L_1, L_0)],$$

and then we have

$$(m+n-1) a_{m,n,m+n} L_{m+n+1} = (m-n)(m+n-1) a_{1,0,1} L_{m+n+1},$$

then we get $a_{m,n,m+n} = (m-n)a_{1,0,1}$ if $m+n \neq 1$.

Using

$$[\varphi(L_m, L_n), [L_2, L_0]] = [[L_m, L_n], \varphi(L_2, L_0)],$$

we get the identity

$$(m+n-2)a_{m,n,m+n}L_{m+n+2} = (m-n)(m+n-2)a_{1,0,1}L_{m+n+2},$$

then we get $a_{m,n,m+n} = (m-n)a_{1,0,1}$ if $m+n \neq 2$. Thus, for any $m, n \in \mathbb{Z}$, we have $a_{m,n,m+n} = (m-n)a_{1,0,1}$. Now, taking $\lambda = a_{1,0,1}$, we have

$$\varphi(L_m, L_n) = \lambda [L_m, L_n] + b_{m,n,0}I_0 \quad \text{for all} \quad m, n \in \mathbb{Z}.$$

Because that $\varphi_0(L_m, L_n) = b_{m,n,0}I_0$, then we have

$$\varphi(L_m, L_n) = \lambda [L_m, L_n] + \varphi_0(L_m, L_n) \quad \text{for all} \quad m, n \in \mathbb{Z}.$$

Hence, this claim holds.

Claim 2: $\varphi(L_m, I_n) = (1 - \delta_{m+n,0})\lambda [L_m, I_n] + \varphi_0(L_m, I_n)$ for all $m, n \in \mathbb{Z}$.

Note that $|\varphi(L_m, I_n)| = |L_m| + |I_n| = \overline{0}$. For any $m, n \in \mathbb{Z}$, we can suppose that

$$\varphi(L_m, I_n) = \sum_{\alpha \in \mathbb{Z}} c_{m,n,\alpha}L_\alpha + \sum_{\beta \in \mathbb{Z}} d_{m,n,\beta}I_\beta,$$

where $c_{m,n,\alpha}, d_{m,n,\beta} \in \mathbb{C}$.

By Lemma 2(1), we have

$$[\varphi(L_m, I_n), [L_1, L_0]] = [[L_m, I_n], \varphi(L_1, L_0)],$$

we have that

$$\sum_{\alpha \in \mathbb{Z}} (\alpha - 1)c_{m,n,\alpha}L_{\alpha+1} + \sum_{\beta \in \mathbb{Z}} \beta d_{m,n,\beta}I_{\beta+1} = n(m+n)a_{1,0,1}I_{m+n+1},$$

then we get $c_{m,n,\alpha} = 0$ if $\alpha \neq 1$; $d_{m,n,\beta} = 0$ if $\beta \neq m+n$ and $\beta \neq 0$; $d_{m,n,m+n} = -na_{1,0,1}$ if $\beta = m+n \neq 0$.

From

$$[\varphi(L_m, I_n), [L_2, L_0]] = [[L_m, I_n], \varphi(L_2, L_0)],$$

we have that

$$\sum_{\alpha \in \mathbb{Z}} (\alpha - 2)c_{m,n,\alpha}L_{\alpha+2} + \sum_{\beta \in \mathbb{Z}} \beta d_{m,n,\beta}I_{\beta+2} = n(m+n)a_{1,0,1}I_{m+n+2},$$

then, $c_{m,n,\alpha} = 0$ if $\alpha \neq 2$; $d_{m,n,\beta} = 0$ if $\beta \neq m+n$ and $\beta \neq 0$; $d_{m,n,m+n} = -na_{1,0,1}$ if $\beta = m+n \neq 0$.

Hence, we obtain that $c_{m,n,\alpha} = 0$ for any $\alpha \in \mathbb{Z}$, then

$$\varphi(L_m, I_n) = d_{m,n,m+n} I_{m+n} + d_{m,n,0} I_0 \quad \text{if} \quad m+n \neq 0,$$
$$\varphi(L_m, I_n) = d_{m,n,0} I_0 \quad \text{if} \quad m+n = 0,$$

then we get that

$$\varphi(L_m, I_n) = (1-\delta_{m+n,0})\lambda\,[L_m, I_n] + d_{m,n,0} I_0 \quad \text{for all} \quad m,n \in \mathbb{Z}.$$

Because that $\varphi_0(L_m, I_n) = d_{m,n,0} I_0$, then we have

$$\varphi(L_m, I_n) = (1-\delta_{m+n,0})\lambda\,[L_m, I_n] + \varphi_0(L_m, I_n) \quad \text{for all} \quad m,n \in \mathbb{Z}.$$

Hence, this claim holds.

Claim 3: $\varphi(L_m, G_r) = \lambda\,[L_m, G_r]$ for all $m \in \mathbb{Z}$, $r \in \Gamma$.

Note that $|\varphi(L_m, G_r)| = |L_m| + |G_r| = \bar{1}$. For any $m \in \mathbb{Z}$, $r \in \Gamma$, we can suppose that

$$\varphi(L_m, G_r) = \sum_{\alpha \in \Gamma} g_{m,r,\alpha} G_\alpha + \sum_{\beta \in \Gamma} h_{m,r,\beta} J_\beta,$$

where $g_{m,r,\alpha}, h_{m,r,\beta} \in \mathbb{C}$.

For the case $\epsilon = 0$**,** by Lemma 2(1), we have

$$[\varphi(L_m, G_r), [L_2, L_0]] = [[L_m, G_r], \varphi(L_2, L_0)],$$

we have that

$$\sum_{\alpha \in \mathbb{Z}} (\alpha - 1) g_{m,r,\alpha} G_{\alpha+2} + \sum_{\beta \in \mathbb{Z}} (\beta + 1) h_{m,r,\beta} J_{\beta+2} = -(\frac{m}{2} - r)(m + r - 1) a_{1,0,1} G_{m+r+2},$$

then we get $h_{m,r,\beta} = 0$ if $\beta \neq -1$; $g_{m,r,\alpha} = 0$ if $\alpha \neq m+r$ and $\alpha \neq 1$; $g_{m,r,m+r} = -(\frac{m}{2} - r) a_{1,0,1}$ if $\alpha = m + r \neq 1$.

From

$$[\varphi(L_m, G_r), [L_4, L_0]] = [[L_m, G_r], \varphi(L_4, L_0)],$$

we have that

$$\sum_{\alpha \in \mathbb{Z}} (\alpha - 2) g_{m,r,\alpha} G_{\alpha+4} + \sum_{\beta \in \mathbb{Z}} (\beta + 2) h_{m,r,\beta} J_{\beta+4} = -(\frac{m}{2} - r)(m + r - 2) a_{1,0,1} G_{m+r+4},$$

then, $h_{m,r,\beta} = 0$ if $\beta \neq -2$; $g_{m,r,\alpha} = 0$ if $\alpha \neq m+r$ and $\alpha \neq 2$; $g_{m,r,m+r} = -(\frac{m}{2} - r) a_{1,0,1}$ if $\alpha = m + r \neq 2$.

Hence, we obtain that $h_{m,r,\beta} = 0$ for any $\beta \in \mathbb{Z}$ and $g_{m,r,m+r} = -(\frac{m}{2} - r) a_{1,0,1}$ if $m + r = \alpha$ with $\alpha \in \mathbb{Z}$, then $\varphi(L_m, G_r) = g_{m,r,m+r} G_{m+r}$ for all $m, r \in \mathbb{Z}$.

For the case $\epsilon = \frac{1}{2}$**,** by Lemma 2(1), we have

$$[\varphi(L_m, G_r), [L_1, L_0]] = [[L_m, G_r], \varphi(L_1, L_0)],$$

we have that

$$\sum_{\alpha\in\frac{1}{2}+\mathbb{Z}}(\alpha-\frac{1}{2})g_{m,r,\alpha}G_{\alpha+1}+\sum_{\beta\in\frac{1}{2}+\mathbb{Z}}(\beta+\frac{1}{2})h_{m,r,\beta}J_{\beta+1}=-(\frac{m}{2}-r)(m+r-\frac{1}{2})a_{1,0,1}G_{m+r+1},$$

then we get $h_{m,r,\beta}=0$ if $\beta\neq-\frac{1}{2}$; $g_{m,r,\alpha}=0$ if $\alpha\neq m+r$ and $\alpha\neq\frac{1}{2}$; $g_{m,r,m+r}=-(\frac{m}{2}-r)a_{1,0,1}$ if $\alpha=m+r\neq\frac{1}{2}$.

From

$$[\varphi(L_m,G_r),[L_3,L_0]]=[[L_m,G_r],\varphi(L_3,L_0)],$$

we have that

$$\sum_{\alpha\in\frac{1}{2}+\mathbb{Z}}(\alpha-\frac{3}{2})g_{m,r,\alpha}G_{\alpha+3}+\sum_{\beta\in\frac{1}{2}+\mathbb{Z}}(\beta+\frac{3}{2})h_{m,r,\beta}J_{\beta+3}=-(\frac{m}{2}-r)(m+r-\frac{3}{2})a_{1,0,1}G_{m+r+3},$$

then, $h_{m,r,\beta}=0$ if $\beta\neq-\frac{3}{2}$; $g_{m,r,\alpha}=0$ if $\alpha\neq m+r$ and $\alpha\neq\frac{3}{2}$; $g_{m,r,m+r}=-(\frac{m}{2}-r)a_{1,0,1}$ if $\alpha=m+r\neq\frac{3}{2}$.

Hence, we obtain that $h_{m,r,\beta}=0$ for any $\beta\in\frac{1}{2}+\mathbb{Z}$ and $g_{m,r,m+r}=-(\frac{m}{2}-r)a_{1,0,1}$ if $m+r=\alpha$ with $\alpha\in\frac{1}{2}+\mathbb{Z}$, then $\varphi(L_m,G_r)=g_{m,r,m+r}G_{m+r}$ for all $m\in\mathbb{Z}$, $r\in\frac{1}{2}+\mathbb{Z}$. Then we get that

$$\varphi(L_m,G_r)=\lambda\,[L_m,G_r]\quad\text{for all}\quad m\in\mathbb{Z},r\in\Gamma.$$

Hence, this claim holds.

Claim 4: $\varphi(L_m,J_r)=\lambda\,[L_m,J_r]$ for all $m\in\mathbb{Z}$, $r\in\Gamma$.

Note that $|\varphi(L_m,J_r)|=|L_m|+|J_r|=\overline{1}$. For any $m\in\mathbb{Z}$, $r\in\Gamma$, we can suppose that

$$\varphi(L_m,J_r)=\sum_{\alpha\in\Gamma}k_{m,r,\alpha}G_\alpha+\sum_{\beta\in\Gamma}l_{m,r,\beta}J_\beta,$$

where $k_{m,r,\alpha},l_{m,r,\beta}\in\mathbb{C}$.

For the case $\epsilon=0$**,** by Lemma 2(1), we have

$$[\varphi(L_m,J_r),[L_2,L_0]]=[[L_m,J_r],\varphi(L_2,L_0)],$$

we have that

$$\sum_{\alpha\in\mathbb{Z}}(\alpha-1)k_{m,r,\alpha}G_{\alpha+2}+\sum_{\beta\in\mathbb{Z}}(\beta+1)l_{m,r,\beta}J_{\beta+2}=(\frac{m}{2}+r)(m+r+1)a_{1,0,1}J_{m+r+2},$$

then we get $k_{m,r,\alpha}=0$ if $\alpha\neq1$; $l_{m,r,\beta}=0$ if $\beta\neq m+r$ and $\beta\neq-1$; $l_{m,r,m+r}=(\frac{m}{2}+r)a_{1,0,1}$ if $\beta=m+r\neq-1$.

From

$$[\varphi(L_m,J_r),[L_4,L_0]]=[[L_m,J_r],\varphi(L_4,L_0)],$$

we have that

$$\sum_{\alpha\in\mathbb{Z}}(\alpha-2)k_{m,r,\alpha}G_{\alpha+4}+\sum_{\beta\in\mathbb{Z}}(\beta+2)l_{m,r,\beta}J_{\beta+4}=(\frac{m}{2}+r)(m+r+2)a_{1,0,1}J_{m+r+4},$$

then, $k_{m,r,\alpha} = 0$ if $\alpha \neq 2$; $l_{m,r,\beta} = 0$ if $\beta \neq m+r$ and $\beta \neq -2$; $l_{m,r,m+r} = (\frac{m}{2}+r)a_{1,0,1}$ if $\beta = m+r \neq -2$.

Hence, we obtain that $k_{m,r,\alpha} = 0$ for any $\alpha \in \mathbb{Z}$ and $l_{m,r,m+r} = (\frac{m}{2}+r)a_{1,0,1}$ if $m+r = \beta$ with $\beta \in \mathbb{Z}$, then $\varphi(L_m, J_r) = l_{m,r,m+r}J_{m+r}$ for all $m, r \in \mathbb{Z}$.

For the case $\epsilon = \frac{1}{2}$, by Lemma 2(1), we have

$$[\varphi(L_m, J_r), [L_1, L_0]] = [[L_m, J_r], \varphi(L_1, L_0)],$$

we have that

$$\sum_{\alpha\in\frac{1}{2}+\mathbb{Z}} (\alpha - \frac{1}{2})k_{m,r,\alpha}G_{\alpha+1} + \sum_{\beta\in\frac{1}{2}+\mathbb{Z}} (\beta + \frac{1}{2})l_{m,r,\beta}J_{\beta+1} = (\frac{m}{2}+r)(m+r+\frac{1}{2})a_{1,0,1}J_{m+r+1},$$

then we get $k_{m,r,\alpha} = 0$ if $\alpha \neq \frac{1}{2}$; $l_{m,r,\beta} = 0$ if $\beta \neq m+r$ and $\beta \neq -\frac{1}{2}$; $l_{m,r,m+r} = (\frac{m}{2}+r)a_{1,0,1}$ if $\alpha = m+r \neq -\frac{1}{2}$.

From

$$[\varphi(L_m, J_r), [L_3, L_0]] = [[L_m, J_r], \varphi(L_3, L_0)],$$

we have that

$$\sum_{\alpha\in\frac{1}{2}+\mathbb{Z}} (\alpha - \frac{3}{2})k_{m,r,\alpha}G_{\alpha+3} + \sum_{\beta\in\frac{1}{2}+\mathbb{Z}} (\beta + \frac{3}{2})l_{m,r,\beta}J_{\beta+3} = (\frac{m}{2}+r)(m+r+\frac{3}{2})a_{1,0,1}J_{m+r+3},$$

then we get $k_{m,r,\alpha} = 0$ if $\alpha \neq \frac{3}{2}$; $l_{m,r,\beta} = 0$ if $\beta \neq m+r$ and $\beta \neq -\frac{3}{2}$; $l_{m,r,m+r} = (\frac{m}{2}+r)a_{1,0,1}$ if $\beta = m+r \neq -\frac{3}{2}$.

Hence, we obtain that $k_{m,r,\alpha} = 0$ for any $\alpha \in \frac{1}{2}+\mathbb{Z}$ and $l_{m,r,m+r} = (\frac{m}{2}+r)a_{1,0,1}$ if $m+r = \beta$ with $\beta \in \frac{1}{2}+\mathbb{Z}$, then $\varphi(L_m, J_r) = l_{m,r,m+r}J_{m+r}$ for all $m \in \mathbb{Z}$, $r \in \frac{1}{2}+\mathbb{Z}$. Then we get that

$$\varphi(L_m, J_r) = \lambda [L_m, J_r] \quad \text{for all} \quad m \in \mathbb{Z}, r \in \Gamma.$$

Hence, this claim holds.

Claim 5:

$$\varphi(G_r, G_s) = \lambda [G_r, G_s] + \varphi_0(G_r, G_s) \quad \text{for all} \quad r, s \in \Gamma.$$

Note that $|\varphi(G_r, G_s)| = |G_r| + |G_s| = \overline{0}$. For any $r, s \in \Gamma$, we can suppose that

$$\varphi(G_r, G_s) = \sum_{\alpha\in\mathbb{Z}} p_{r,s,\alpha}L_\alpha + \sum_{\beta\in\mathbb{Z}} q_{r,s,\beta}I_\beta,$$

where $p_{r,s,\alpha}, q_{r,s,\beta} \in \mathbb{C}$.

For the case $r+s \in \frac{1}{2}+\mathbb{Z}$, by Lemma 2(1), we have

$$[\varphi(G_r, G_s), [L_1, L_0]] = [[G_r, G_s], \varphi(L_1, L_0)],$$

we have that

$$\sum_{\alpha\in\mathbb{Z}} (\alpha - 1)p_{r,s,\alpha}L_{\alpha+1} + \sum_{\beta\in\mathbb{Z}} \beta q_{r,s,\beta}I_{\beta+1} = -2(r+s-1)a_{1,0,1}L_{r+s+1},$$

then we get $q_{r,s,\beta} = 0$ if $\beta \neq 0$; $p_{r,s,\alpha} = 0$ if $\alpha \neq 1$.

From

$$[\varphi(G_r, G_s), [L_2, L_0]] = [[G_r, G_s], \varphi(L_2, L_0)],$$

we have that

$$\sum_{\alpha\in\mathbb{Z}}(\alpha-2)p_{r,s,\alpha}L_{\alpha+2} + \sum_{\beta\in\mathbb{Z}}\beta q_{r,s,\beta}I_{\beta+2} = -2(r+s-1)a_{1,0,1}L_{r+s+2},$$

then we get $q_{r,s,\beta} = 0$ if $\beta \neq 0$; $p_{r,s,\alpha} = 0$ if $\alpha \neq 2$.

Hence, we obtain that $q_{r,s,\beta} = 0$ for any $\beta \in \mathbb{Z}^*$ and $p_{r,s,\alpha} = 0$ for all $\alpha \in \mathbb{Z}$, then $\varphi(G_r, G_s) = q_{r,s,0}I_0$ for all $r, s \in \epsilon + \mathbb{Z}$ with $r + s \in \frac{1}{2} + \mathbb{Z}$. Then we get that

$$\varphi(G_r, G_s) = q_{r,s,0}I_0 \quad \text{for all} \quad r, s \in \epsilon + \mathbb{Z} \quad \text{with} \quad r + s \in \tfrac{1}{2} + \mathbb{Z}.$$

Because that $\varphi_0(G_r, G_s) = q_{r,s,0}I_0$, then we have

$$\varphi(G_r, G_s) = \varphi_0(G_r, G_s) \quad \text{for all} \quad r, s \in \epsilon + \mathbb{Z} \quad \text{with} \quad r + s \in \tfrac{1}{2} + \mathbb{Z}.$$

For the case $r + s \in \mathbb{Z}$, by Lemma 2(1), we have

$$[\varphi(G_r, G_s), [L_1, L_0]] = [[G_r, G_s], \varphi(L_1, L_0)],$$

we have that

$$\sum_{\alpha\in\mathbb{Z}}(\alpha-1)p_{r,s,\alpha}L_{\alpha+1} + \sum_{\beta\in\mathbb{Z}}\beta q_{r,s,\beta}I_{\beta+1} = -2(r+s-1)a_{1,0,1}L_{r+s+1},$$

then we get $q_{r,s,\beta} = 0$ if $\beta \neq 0$; $p_{r,s,\alpha} = 0$ if $\alpha \neq r + s$ and $\alpha \neq 1$; $p_{r,s,r+s} = -2a_{1,0,1}$ if $\alpha = r + s \neq 1$.

From

$$[\varphi(G_r, G_s), [L_2, L_0]] = [[G_r, G_s], \varphi(L_2, L_0)],$$

we have that

$$\sum_{\alpha\in\mathbb{Z}}(\alpha-2)p_{r,s,\alpha}L_{\alpha+2} + \sum_{\beta\in\mathbb{Z}}\beta q_{r,s,\beta}I_{\beta+2} = -2(r+s-1)a_{1,0,1}L_{r+s+2},$$

then we get $q_{r,s,\beta} = 0$ if $\beta \neq 0$; $p_{r,s,\alpha} = 0$ if $\alpha \neq r + s$ and $\alpha \neq 2$; $p_{r,s,r+s} = -2a_{1,0,1}$ if $\alpha = r + s \neq 2$.

Hence, we obtain that $q_{r,s,\beta} = 0$ for any $\beta \in \mathbb{Z}^*$ and $p_{r,s,r+s} = -2a_{1,0,1}$ if $r + s = \alpha$ with $\alpha \in \mathbb{Z}$, then $\varphi(G_r, G_s) = p_{r,s,r+s}L_{r+s} + q_{r,s,0}I_0$ for all $r, s \in \epsilon + \mathbb{Z}$ with $r + s \in \mathbb{Z}$. Then we get that

$$\varphi(G_r, G_s) = \lambda\,[G_r, G_s] + q_{r,s,0}I_0 \quad \text{for all} \quad r, s \in \epsilon + \mathbb{Z} \quad \text{with} \quad r + s \in \mathbb{Z}.$$

Because that $\varphi_0(G_r, G_s) = q_{r,s,0}I_0$, then we have

$$\varphi(G_r, G_s) = \lambda\,[G_r, G_s] + \varphi_0(G_r, G_s) \quad \text{for all} \quad r, s \in \epsilon + \mathbb{Z} \quad \text{with} \quad r + s \in \mathbb{Z}.$$

Hence, we have

$$\varphi(G_r, G_s) = \lambda\,[G_r, G_s] + \varphi_0(G_r, G_s) \quad \text{for all } r, s \in \Gamma.$$

The claim holds.

Claim 6:

$$\varphi(G_r, J_s) = (1 - \delta_{r+s,0})\lambda\,[G_r, J_s] + \varphi_0(G_r, J_s) \quad \text{for all } r, s \in \Gamma.$$

Note that $|\varphi(G_r, J_s)| = |G_r| + |J_s| = \overline{0}$. For any $r, s \in \Gamma$, we can suppose that

$$\varphi(G_r, J_s) = \sum_{\alpha\in\mathbb{Z}} u_{r,s,\alpha} L_\alpha + \sum_{\beta\in\mathbb{Z}} v_{r,s,\beta} I_\beta,$$

where $u_{r,s,\alpha}, v_{r,s,\beta} \in \mathbb{C}$.

For the case $r + s \in \frac{1}{2} + \mathbb{Z}$**,** by Lemma 2(1), we have

$$[\varphi(G_r, J_s), [L_1, L_0]] = [[G_r, J_s], \varphi(L_1, L_0)],$$

we have that

$$\sum_{\alpha\in\mathbb{Z}} (\alpha - 1) u_{r,s,\alpha} L_{\alpha+1} + \sum_{\beta\in\mathbb{Z}} \beta v_{r,s,\beta} I_{\beta+1} = -(r + s) a_{1,0,1} I_{r+s+1},$$

then we get $u_{r,s,\alpha} = 0$ if $\alpha \neq 1$; $v_{r,s,\beta} = 0$ if $\beta \neq 0$.

From

$$[\varphi(G_r, J_s), [L_2, L_0]] = [[G_r, J_s], \varphi(L_2, L_0)],$$

we have that

$$\sum_{\alpha\in\mathbb{Z}} (\alpha - 2) u_{r,s,\alpha} L_{\alpha+2} + \sum_{\beta\in\mathbb{Z}} \beta v_{r,s,\beta} I_{\beta+2} = -(r + s) a_{1,0,1} I_{r+s+2},$$

then we get $u_{r,s,\alpha} = 0$ if $\alpha \neq 2$; $v_{r,s,\beta} = 0$ if $\beta \neq 0$.

Hence, we obtain that $u_{r,s,\alpha} = 0$ for all $\alpha \in \mathbb{Z}$ and $v_{r,s,\beta} = 0$ for any $\beta \in \mathbb{Z}^*$, then $\varphi(G_r, J_s) = v_{r,s,0} I_0$ for all $r, s \in \Gamma$ with $r + s \in \frac{1}{2} + \mathbb{Z}$. Then we get that

$$\varphi(G_r, J_s) = v_{r,s,0} I_0 \quad \text{for all} \quad r, s \in \Gamma \quad \text{with} \quad r + s \in \tfrac{1}{2} + \mathbb{Z}.$$

Because that $\varphi_0(G_r, J_s) = v_{r,s,0} I_0$, then we have

$$\varphi(G_r, J_s) = \varphi_0(G_r, J_s) \quad \text{for all} \quad r, s \in \Gamma \quad \text{with} \quad r + s \in \tfrac{1}{2} + \mathbb{Z}.$$

For the case $r + s \in \mathbb{Z}$**,** by Lemma 2(1), we have

$$[\varphi(G_r, J_s), [L_1, L_0]] = [[G_r, J_s], \varphi(L_1, L_0)],$$

we have that

$$\sum_{\alpha\in\mathbb{Z}} (\alpha - 1) u_{r,s,\alpha} L_{\alpha+1} + \sum_{\beta\in\mathbb{Z}} \beta v_{r,s,\beta} I_{\beta+1} = -(r + s) a_{1,0,1} I_{r+s+1},$$

then we get $u_{r,s,\alpha} = 0$ if $\alpha \neq 1$; $v_{r,s,\beta} = 0$ if $\beta \neq r+s$ and $\beta \neq 0$; $v_{r,s,r+s} = -a_{1,0,1}$ if $\beta = r+s \neq 0$.

From

$$[\varphi(G_r, J_s), [L_2, L_0]] = [[G_r, J_s], \varphi(L_2, L_0)],$$

we have that

$$\sum_{\alpha\in\mathbb{Z}}(\alpha-2)u_{r,s,\alpha}L_{\alpha+2} + \sum_{\beta\in\mathbb{Z}}\beta u_{r,s,\beta}I_{\beta+2} = -2(r+s-1)a_{1,0,1}I_{r+s+2},$$

then we get $u_{r,s,\alpha} = 0$ if $\alpha \neq 2$; $v_{r,s,\beta} = 0$ if $\beta \neq r+s$ and $\beta \neq 0$; $v_{r,s,r+s} = -a_{1,0,1}$ if $\beta = r+s \neq 0$.

Hence, we obtain that $u_{r,s,\alpha} = 0$ for all $\alpha \in \mathbb{Z}$ and $v_{r,s,r+s} = -a_{1,0,1}$ if $r+s = \beta$ with $\beta \in \mathbb{Z}^*$, then $\varphi(G_r, J_s) = v_{r,s,r+s}I_{r+s} + v_{r,s,0}I_0$ if $r+s \in \mathbb{Z}^*$; $\varphi(G_r, J_s) = v_{r,s,0}I_0$ if $r+s = 0$. Then we get that

$$\varphi(G_r, J_s) = (1-\delta_{r+s,0})\lambda[G_r, J_s] + v_{r,s,0}I_0 \quad \text{for all} \quad r, s \in \Gamma \quad \text{with} \quad r+s \in \mathbb{Z}.$$

Because that $\varphi_0(G_r, J_s) = v_{r,s,0}I_0$, then we have

$$\varphi(G_r, J_s) = (1-\delta_{r+s,0})\lambda[G_r, J_s] + \varphi_0(G_r, J_s) \quad \text{for all} \quad r, s \in \Gamma \quad \text{with} \quad r+s \in \mathbb{Z}.$$

Hence, we have

$$\varphi(G_r, J_s) = (1-\delta_{r+s,0})\lambda[G_r, J_s] + \varphi_0(G_r, J_s) \quad \text{for all } r, s \in \Gamma.$$

The claim holds.

Claim 7: $\varphi(G_r, I_m) = \lambda[G_r, I_m]$ for all $m \in \mathbb{Z}$, $r \in \Gamma$.

Note that $|\varphi(G_r, I_m)| = |G_r| + |I_m| = \bar{1}$. For any $m \in \mathbb{Z}$, $r \in \Gamma$, we can suppose that

$$\varphi(G_r, I_m) = \sum_{\alpha\in\Gamma} t_{m,r,\alpha}G_\alpha + \sum_{\beta\in\Gamma} w_{m,r,\beta}J_\beta,$$

where $t_{m,r,\alpha}, w_{m,r,\beta} \in \mathbb{C}$.

For the case $\epsilon = 0$**,** by Lemma 2(1), we have

$$[\varphi(G_r, I_m), [L_2, L_0]] = [[G_r, I_m], \varphi(L_2, L_0)],$$

we have that

$$\sum_{\alpha\in\mathbb{Z}}(\alpha-1)t_{m,r,\alpha}G_{\alpha+2} + \sum_{\beta\in\mathbb{Z}}(\beta+1)w_{m,r,\beta}J_{\beta+2} = m(m+r+1)a_{1,0,1}J_{m+r+2},$$

then we get $t_{m,r,\alpha} = 0$ if $\alpha \neq 1$; $w_{m,r,\beta} = 0$ if $\beta \neq m+r$ and $\beta \neq -1$; $w_{m,r,m+r} = ma_{1,0,1}$ if $\beta = m+r \neq -1$.

From

$$[\varphi(G_r, I_m), [L_4, L_0]] = [[G_r, I_m], \varphi(L_4, L_0)],$$

we have that

$$\sum_{\alpha\in\mathbb{Z}}(\alpha-2)t_{m,r,\alpha}G_{\alpha+4} + \sum_{\beta\in\mathbb{Z}}(\beta+2)w_{m,r,\beta}J_{\beta+4} = m(m+r+2)a_{1,0,1}J_{m+r+4},$$

then we get $t_{m,r,\alpha}=0$ if $\alpha\neq 2$; $w_{m,r,\beta}=0$ if $\beta\neq m+r$ and $\beta\neq -2$; $w_{m,r,m+r}=ma_{1,0,1}$ if $\beta=m+r\neq -2$.

Hence, we obtain that $t_{m,r,\alpha}=0$ for all $\alpha\in\mathbb{Z}$ and $w_{m,r,m+r}=ma_{1,0,1}$ if $m+r=\alpha$ with $\alpha\in\mathbb{Z}$, then $\varphi(G_r,I_m)=w_{m,r,m+r}G_{m+r}$ for all $m,r\in\mathbb{Z}$.

For the case $\epsilon=\frac{1}{2}$**,** by Lemma 2(1), we have

$$[\varphi(G_r,I_m),[L_1,L_0]]=[[G_r,I_m],\varphi(L_1,L_0)],$$

we have that

$$\sum_{\alpha\in\frac{1}{2}+\mathbb{Z}}(\alpha-\frac{1}{2})t_{m,r,\alpha}G_{\alpha+1}+\sum_{\beta\in\frac{1}{2}+\mathbb{Z}}(\beta+\frac{1}{2})w_{m,r,\beta}J_{\beta+1}=m(m+r+\frac{1}{2})a_{1,0,1}J_{m+r+1},$$

then we get $t_{m,r,\alpha}=0$ if $\alpha\neq\frac{1}{2}$; $w_{m,r,\beta}=0$ if $\beta\neq m+r$ and $\beta\neq-\frac{1}{2}$; $w_{m,r,m+r}=ma_{1,0,1}$ if $\beta=m+r\neq-\frac{1}{2}$.

From

$$[\varphi(G_r,I_m),[L_3,L_0]]=[[G_r,I_m],\varphi(L_3,L_0)],$$

we have that

$$\sum_{\alpha\in\frac{1}{2}+\mathbb{Z}}(\alpha-\frac{3}{2})t_{m,r,\alpha}G_{\alpha+3}+\sum_{\beta\in\frac{1}{2}+\mathbb{Z}}(\beta+\frac{3}{2})w_{m,r,\beta}J_{\beta+3}=m(m+r+\frac{3}{2})a_{1,0,1}J_{m+r+3},$$

then we get $t_{m,r,\alpha}=0$ if $\alpha\neq\frac{3}{2}$; $w_{m,r,\beta}=0$ if $\beta\neq m+r$ and $\beta\neq-\frac{3}{2}$; $w_{m,r,m+r}=ma_{1,0,1}$ if $\beta=m+r\neq-\frac{3}{2}$.

Hence, we obtain that $t_{m,r,\alpha}=0$ for all $\alpha\in\frac{1}{2}+\mathbb{Z}$ and $w_{m,r,m+r}=ma_{1,0,1}$ if $m+r=\alpha$ with $\alpha\in\frac{1}{2}+\mathbb{Z}$, then $\varphi(G_r,I_m)=w_{m,r,m+r}G_{m+r}$ for all $m,r\in\mathbb{Z}$, $r\in\frac{1}{2}+\mathbb{Z}$. Then we get that

$$\varphi(G_r,I_m)=\lambda\,[G_r,I_m]\quad\text{for all}\quad m\in\mathbb{Z},r\in\Gamma.$$

The claim holds.

Claim 8: $\varphi(I_m,I_n)=\kappa_{m,n,1}I_0$ for all $m,n\in\mathbb{Z}$.

For fixed $m,n\in\mathbb{Z}$, since $[I_m,I_n]=0$, by Lemma 2(1), $\varphi(I_m,I_n)\in Z([\mathcal{G}_\epsilon,\mathcal{G}_\epsilon])$. Since $[\mathcal{G}_\epsilon,\mathcal{G}_\epsilon]=\mathcal{G}_\epsilon$, then $\varphi(I_m,I_n)\in Z(\mathcal{G}_\epsilon)=\mathbb{C}I_0$. Then this claim holds.

Claim 9: $\varphi(I_m,J_r)=0$ for all $m\in\mathbb{Z}$, $r\in\Gamma$.

For fixed $m\in\mathbb{Z}$, $r\in\Gamma$, since $[I_m,J_r]=0$, by Lemma 2(1), $\varphi(I_m,J_r)\in Z([\mathcal{G}_\epsilon,\mathcal{G}_\epsilon])$. Since $[\mathcal{G}_\epsilon,\mathcal{G}_\epsilon]=\mathcal{G}_\epsilon$, then $\varphi(I_m,J_r)\in Z(\mathcal{G}_\epsilon)=\mathbb{C}I_0$. Meanwhile $|\varphi(I_m,J_r)|=|I_m|+|J_r|=\overline{1}$, that is, $\varphi(I_m,J_r)=\sum_{\alpha\in\Gamma}\nu_{m,r,\alpha}G_\alpha+\sum_{\beta\in\Gamma}\upsilon_{m,r,\beta}J_\beta$, so $\varphi(I_m,J_r)=\mathbb{C}I_0=0$. Then this claim holds.

Claim 10: $\varphi(J_r,J_s)=\kappa_{r,s,2}I_0$ for all $r,s\in\Gamma$.

For fixed $r,s\in\Gamma$, since $[J_r,J_s]=0$, by Lemma 2(1), $\varphi(J_r,J_s)\in Z([\mathcal{G}_\epsilon,\mathcal{G}_\epsilon])$. Since $[\mathcal{G}_\epsilon,\mathcal{G}_\epsilon]=\mathcal{G}_\epsilon$, then $\varphi(J_r,J_s)\in Z(\mathcal{G}_\epsilon)=\mathbb{C}I_0$. Then this claim holds.

Claim 11: There exists $\lambda \in \mathbb{C}$ such that

$$\begin{aligned}
\varphi : \mathcal{G}_\epsilon \times \mathcal{G}_\epsilon &\to \mathcal{G}_\epsilon \\
(L_m, L_n) &\mapsto \lambda [L_m, L_n] + \varphi_0(L_m, L_n), \\
(L_m, I_n) &\mapsto (1-\delta_{m+n,0})\lambda [L_m, I_n] + \varphi_0(L_m, I_n), \\
(L_m, G_r) &\mapsto \lambda [L_m, G_r], \\
(L_m, J_r) &\mapsto \lambda [L_m, J_r], \\
(G_r, G_s) &\mapsto \lambda [G_r, G_s] + \varphi_0(G_r, G_s), \\
(G_r, J_s) &\mapsto (1-\delta_{r+s,0})\lambda [G_r, J_s] + \varphi_0(G_r, J_s), \\
(G_r, I_m) &\mapsto \lambda [G_r, I_m], \\
(I_m, I_n) &\mapsto \varphi_0(I_m, I_n), \\
(I_m, J_r) &\mapsto 0, \\
(J_r, J_s) &\mapsto \varphi_0(J_r, J_s),
\end{aligned}$$

where $m, n \in \mathbb{Z}$, $r, s \in \Gamma$, and φ_0 is given by Definition 1.

It is straightforward by Claims 1-10.

According to Claim 1, we consider the case $\varphi(L_m, L_n) = \lambda [L_m, L_n] + \varphi_0(L_m, L_n)$. Replacing L_m, L_n by x, y and $b_{m,n,0}$ by the bilinear function $f : \mathcal{G}_\epsilon \times \mathcal{G}_\epsilon \to \mathbb{C}$, $(x, y) \longmapsto f(x, y)$, we note that

$$\varphi(x, y) = \lambda [x, y] + f(x, y) I_0,$$

which together with

$$\varphi([x, y], z) = [x, \varphi(y, z)] + (-1)^{|y||z|} [\varphi(x, z), y],$$

gives

$$\begin{aligned}
\varphi([x, y], z) &= \lambda [[x, y], z] + f([x, y], z) I_0 \\
&= [x, \lambda [y, z] + f(y, z) I_0] + (-1)^{|y||z|} [\lambda [x, z] + f(x, z) I_0, y] \\
&= \lambda \left([x, [y, z]] + (-1)^{|y||z|} [[x, z], y]\right) + f(y, z) [x, I_0] + (-1)^{|y||z|} f(x, z) [I_0, y] \\
&= \lambda [[x, y], z].
\end{aligned}$$

Then we claim $f([x, y], z) = 0$ for all $x, y, z \in \mathcal{G}_\epsilon$. Since the complete spectrum-generating superalgebra coincides with its derived subalgebra, one sees that f is exactly the zero function. Finally, we have that $\varphi(x, y) = \lambda [x, y]$ for all $x, y \in \mathcal{G}_\epsilon$. Thus, $\varphi(L_m, L_n) = \lambda [L_m, L_n]$. The proof of other occasions are similar with that of (L_m, L_n), so we omit them. This completes the proof. □

4 Super-skewsymmetric super-biderivations of the centerless super Virasoro algebra $\mathcal{W}_\epsilon$

In this section, we present the main result concerning super-skewsymmetric super-biderivation on the centerless super Virasoro algebra $\mathcal{W}_\epsilon$.

The definition of the centerless super Virasoro algebra $\mathcal{W}_\epsilon$ is given in the following.

Definition 2. *[25] For $\epsilon \in \{0, \frac{1}{2}\}$, the centerless super Virasoro algebra $\mathcal{W}_\epsilon$ is an infinite-dimensional Lie superalgebra over the complex field $\mathbb{C}$ with the basis $\{L_m, G_r \mid m \in \mathbb{Z}, r \in \Gamma\}$, admitting the following super-brackets:*

$$\begin{aligned}[L_m, L_n] &= (m-n)L_{m+n},\\ [L_m, G_r] &= \left(\tfrac{m}{2} - r\right) G_{r+m},\\ [G_r, G_s] &= 2L_{r+s},\end{aligned}$$

for $m, n \in \mathbb{Z}$, $r, s \in \Gamma$. Obviously, $\mathcal{W}_\epsilon$ is $\mathbb{Z}_2$-graded: $\mathcal{W}_\epsilon = \mathcal{W}_{\epsilon\bar{0}} \oplus \mathcal{W}_{\epsilon\bar{1}}$, with

$$\mathcal{W}_{\epsilon\bar{0}} = span\{L_m \mid m \in \mathbb{Z}\}, \tag{12}$$

$$\mathcal{W}_{\epsilon\bar{1}} = span\{G_r \mid r \in \Gamma\}. \tag{13}$$

By the above definition we can easily check that the centerless super Virasoro algebra $\mathcal{W}_\epsilon$ is perfect. Moreover, the centre of this Lie superalgebra is

$$Z(\mathcal{W}_\epsilon) = \text{span}\{0\}.$$

The main result in this section is as follows.

Theorem 2. *Every super-skewsymmetric super-biderivation ϕ of $\mathcal{W}_\epsilon$ is inner.*

Proof. Suppose ϕ is a super-biderivation of the centerless super Virasoro algebra $\mathcal{W}_\epsilon$. Assume that $\phi(L_0, L_n) = \sum_{m\in\mathbb{Z}} a_m^n L_m + \sum_{r\in\Gamma} b_r^n G_r$ and $\phi(L_0, G_s) = \sum_{m\in\mathbb{Z}} c_m^s L_m + \sum_{r\in\Gamma} d_r^s G_r$, where $a_m^n, b_r^n, c_m^s, d_r^s \in \mathbb{C}$ for any $m, n \in \mathbb{Z}$, $r, s \in \Gamma$.

If $n = s = 0$, then $[L_0, L_0] = [L_0, G_0] = 0$, by Lemma 2(3), $\phi(L_0, L_0), \phi(L_0, G_0) \in Z(\mathcal{W}_\epsilon) = \{0\}$. Hence, $\phi(L_0, L_0) = \phi(L_0, G_0) = 0$.

Next, assume that $n \neq 0$ and $s \neq 0$. Due to $L_m \in \mathcal{W}_{\epsilon\bar{0}}$, then $|L_m| + |L_n| = \bar{0}$ for any $m, n \in \mathbb{Z}$. By Lemma 2(2), we have

$$[[L_0, L_n], \phi(L_0, L_n)] = 0.$$

Then, we get

$$n \sum_{m\in\mathbb{Z}} (m-n) a_m^n L_{m+n} + \sum_{r\in\Gamma} \left(r - \frac{n}{2}\right) b_r^n G_{n+r} = 0.$$

One has

$$(m-n)a_m^n = \left(r - \frac{n}{2}\right) b_r^n = 0.$$

Thus, $a_m^n = 0$ for $m \neq n$ and $b_r^n = 0$ for $r \neq \frac{n}{2}$. So, for the case $\epsilon = 0$, we get

$$\phi(L_0, L_n) = \begin{cases} a_n^n L_n + b_{\frac{n}{2}}^n G_{\frac{n}{2}} & \text{if } n \text{ is even,} \\ a_n^n L_n & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, for the case when n is even, by Lemma 2(1), we have

$$\begin{aligned} 0 &= [\phi(L_0, L_n), [L_0, L_1]] - [[L_0, L_n], \phi(L_0, L_1)] \\ &= [a_n^n L_n + b_{\frac{n}{2}}^n G_{\frac{n}{2}}, L_1] - [nL_n, a_1^n L_1] \\ &= (n-1)a_n^n L_{n+1} + \tfrac{1}{2}(n-1)b_{\frac{n}{2}}^n G_{\frac{n}{2}+1} - n(n-1)a_1^n L_{n+1} \\ &= (n-1)(a_n^n - na_1^n)L_{n+1} + \tfrac{1}{2}(n-1)b_{\frac{n}{2}}^n G_{\frac{n}{2}+1}, \end{aligned}$$

hence, we deduce that $a_n^n = na_1^n$ and $b_{\frac{n}{2}}^n = 0$ for all $n \in \mathbb{Z}$. Taking $\lambda = a_1^n$, then we have

$$\phi(L_0, L_n) = \lambda n L_n.$$

By Lemma 2(1), we have

$$\begin{aligned} 0 &= [\phi(L_0, G_s), [L_0, G_s]] - [[L_0, G_s], \phi(L_0, G_s)] \\ &= \left[\sum_{m\in\mathbb{Z}} c_m^s L_m + \sum_{r\in\mathbb{Z}} d_r^s G_r, -sG_s\right] - \left[-sG_s, \sum_{m\in\mathbb{Z}} c_m^s L_m + \sum_{r\in\mathbb{Z}} d_r^s G_r\right] \\ &= \sum_{m\in\mathbb{Z}} s\left(s - \frac{m}{2}\right) c_m^s G_{m+s}, \end{aligned}$$

hence, we deduce that

$$c_m^s = 0 \text{ if } m \neq 2s. \tag{14}$$

So we get

$$\phi(L_0, G_s) = c_{2s}^s L_{2s} + \sum_{r\in\mathbb{Z}} d_r^s G_r.$$

Furthermore, by Lemma 2(1), we have

$$\begin{aligned} 0 &= [\phi(L_0, G_s), [L_0, L_1]] - [[L_0, G_s], \phi(L_0, L_1)] \\ &= \left[c_{2s}^s L_{2s} + \sum_{r\in\mathbb{Z}} d_r^s G_r, -L_1\right] - [-sG_s, \lambda L_1] \\ &= (1-2s)c_{2s}^s L_{2s+1} + \lambda s\left(s - \tfrac{1}{2}\right) G_{s+1} - \sum_{r\in\mathbb{Z}} \left(r - \frac{1}{2}\right) d_r^s G_{r+1}, \end{aligned}$$

which implies that

$$c_{2s}^s = 0, \quad d_r^s = 0 \text{ if } r \neq s \text{ and } d_s^s = \lambda s. \tag{15}$$

By Eqs. (14) and (15), we obtain

$$\phi(L_0, G_s) = \lambda s G_s.$$

For the case $\epsilon = \frac{1}{2}$, we get

$$\phi(L_0, L_n) = \begin{cases} a_n^n L_n + b_{\frac{n}{2}}^n G_{\frac{n}{2}} & \text{if } n \text{ is odd,} \\ a_n^n L_n & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, for the case when n is odd, by Lemma 2(1), we have

$$\begin{aligned} 0 &= [\phi(L_0, L_n), [L_0, L_2]] - [[L_0, L_n], \phi(L_0, L_2)] \\ &= \left[a_n^n L_n + b_{\frac{n}{2}}^n G_{\frac{n}{2}}, 2L_2\right] - [nL_n, a_2^n L_2] \\ &= 2(n-2)a_n^n L_{n+2} + \tfrac{1}{2}(n-2)b_{\frac{n}{2}}^n G_{\frac{n}{2}+2} - n(n-2)a_2^n L_{n+2} \\ &= (n-2)(2a_n^n - na_2^n)L_{n+2} + \tfrac{1}{2}(n-2)b_{\frac{n}{2}}^n G_{\frac{n}{2}+2}, \end{aligned}$$

hence, we deduce that $a_n^n = \frac{n}{2}a_2^n$ and $b_{\frac{n}{2}}^n = 0$ for all $n \in \mathbb{Z}$. Taking $\lambda = \frac{1}{2}a_2^n$, then we have

$$\phi(L_0, L_n) = \lambda n L_n.$$

Similar to Eq. (14), we still have

$$c_m^s = 0 \text{ if } m \neq 2s, \tag{16}$$

while Eq. (15) should be modified as

$$c_{2s}^s = 0 \text{ if } s \neq \frac{1}{2}, \quad d_r^s = 0 \text{ if } r \neq s \text{ and } d_s^s = \lambda s. \tag{17}$$

So by Eqs. (16) and (17) we get

$$\phi(L_0, G_s) = c_1^1 L_1 + \lambda s G_s.$$

Furthermore, by Lemma 2(1), we have

$$\begin{aligned} 0 &= [\phi(L_0, G_s), [L_0, L_2]] - [[L_0, G_s], \phi(L_0, L_2)] \\ &= [c_1^1 L_1 + \lambda s G_s, -2L_2] - [-sG_s, 2\lambda L_2] \\ &= -2c_1^1 L_3, \end{aligned}$$

which implies that $c_1^1 = 0$. Hence, we have

$$\phi(L_0, G_s) = \lambda s G_s.$$

Finally, we have proved the following equations

$$\begin{aligned} \phi(L_0, L_n) &= \lambda[L_0, L_n] \text{ for all } n \in \mathbb{Z}, \\ \phi(L_0, G_s) &= \lambda[L_0, G_s] \text{ for all } s \in \Gamma. \end{aligned}$$

For any $w \in \mathcal{W}_\epsilon$, we get

$$\phi(L_0, w) = \lambda[L_0, w].$$

By Lemma 2(1), we obtain

$$\begin{aligned} 0 &= [\phi(x, y), [L_0, w]] - [[x, y], \phi(L_0, w)] \\ &= [\phi(x, y), [L_0, w]] - [[x, y], \lambda[L_0, w]] \\ &= [\phi(x, y) - \lambda[x, y], [L_0, w]]. \end{aligned}$$

According to the arbitrariness of w, then $\phi(x, y) - \lambda[x, y] = 0$. Thus, $\phi(x, y) = \lambda[x, y]$ for all $x, y \in \mathcal{W}_\epsilon$. This completes the proof. □

5 Linear super-commuting maps on $\mathcal{G}_\epsilon$ and $\mathcal{W}_\epsilon$

In this section, we shall describe the linear super-commuting maps on $\mathcal{G}_\epsilon$ (resp. $\mathcal{W}_\epsilon$) based on Theorem 1 (resp. Theorem 2). We have the following theorems.

Theorem 3. *All linear super-commuting maps on the complete spectrum-generating superalgebra $\mathcal{G}_\epsilon$ are standard. Namely, each linear super-commuting map ψ on $\mathcal{G}_\epsilon$ has the following form*

$$\psi(x) = f(x)I_0 \text{ for all } x \in \mathcal{G}_\epsilon,$$

where f is a linear function from $\mathcal{G}_\epsilon$ to $\mathbb{C}$ mapping the odd part $\mathcal{W}_{\epsilon\bar{1}}$ of $\mathcal{G}_\epsilon$ to zero.

Proof. Let ψ be a linear super-commuting map on $\mathcal{G}_\epsilon$. Define

$$\varphi: \quad \mathcal{G}_\epsilon \times \mathcal{G}_\epsilon \to \mathcal{G}_\epsilon$$
$$(x, y) \mapsto [\psi(x), y]$$

for $x, y \in \mathcal{G}_\epsilon$. Note that ψ preserves the $\mathbb{Z}_2$-grading of $\mathcal{G}_\epsilon$. By the definition of φ, one can easily verify that

$$\varphi(x, [y, z]) = [\varphi(x, y), z] + (-1)^{|x||y|}[y, \varphi(x, z)] \text{ for } x, y, z \in \mathcal{G}_\epsilon.$$

Namely, φ satisfies the equation (6). Recalling $[\psi(x), y] = (-1)^{|x||y|}[x, \psi(y)]$ (ψ is a linear super-commuting map), one can easily check the other equation (5). In addition, φ is super-skewsymmetric by its definition. Thus, φ is a super-skewsymmetric super-biderivation of $\mathcal{G}_\epsilon$. By Theorem 1, there exists $\lambda \in \mathbb{C}$ such that

$$\varphi(x, y) = \lambda[x, y] \text{ for } x, y \in \mathcal{G}_\epsilon.$$

Considering the definition of φ, we have

$$[\psi(x) - \lambda x, y] \equiv 0 \text{ for } x, y \in \mathcal{G}_\epsilon. \tag{18}$$

This implies that

$$\psi(x) - \lambda x \in Z(\mathcal{G}_\epsilon) = \{I_0\} \text{ for } x \in \mathcal{G}_\epsilon.$$

Thus, we may assume that

$$\psi(x) - \lambda x = f(x)I_0 \text{ for } x \in \mathcal{G}_\epsilon,$$

where f is a linear function from $\mathcal{G}_\epsilon$ to $\mathbb{C}$. Furthermore, by choosing $x = G_r$ ($r \in \Gamma$) in the above formula and using the relation $[\psi(G_r), G_r] = 0$, one can immediately obtain that $\lambda = 0$, and f maps the odd part of $\mathcal{G}_\epsilon$ to zero. Hence, $\psi(x) = f(x)I_0$. Conversely, if ψ is of such form, it is indeed a linear super-commuting map. This completes the proof. □

Theorem 4. *Every linear super-commuting map on the centerless super Virasoro algebra $\mathcal{W}_\epsilon$ is the zero mapping. Namely, each linear super-commuting map Ψ on $\mathcal{W}_\epsilon$ has the following form*

$$\Psi(x) \equiv 0 \text{ for all } x \in \mathcal{W}_\epsilon.$$

Proof. Let Ψ be a linear super-commuting map on $\mathcal{W}_\epsilon$. Similarly to the proof of Theorem 3, we have

$$\Psi(x) - \lambda x = 0 \text{ for } x \in \mathcal{W}_\epsilon.$$

Furthermore, by choosing $x = G_r$ ($r \in \Gamma$) in the above formula and using the relation $[\Psi(G_r), G_r] = 0$, one can immediately obtain that $\lambda = 0$. Hence, $\Psi(x) \equiv 0$ for all $x \in \mathcal{W}_\epsilon$. This completes the proof. □

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