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Zero-divisor graphs of semirings with no S-vertices

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Abstract. Let R be a commutative semiring (ring) with identity $1 \neq 0$. A vertex a in a simple graph G is said to be a Smarandache vertex (or S-vertex for short) provided that there exist three distinct vertices x, y, and b (all different from a) in G such that x—a, a—b, and b—y are edges in G, but there is no edge between x and y. In this interdisciplinary subject, we investigate the interplay between the algebraic properties of the commutative semirings and their associated zero-divisor graphs, denoted by $\Gamma(R)$, using the notion of the S-vertices in connection with the nonexistence of S-vertices in $\Gamma(R)$. We discuss when $\Gamma(R)$ is a complete bipartite graph together with some of its other graph-theoretic properties and their relation to the nonexistence of S-vertices of $\Gamma(R)$.

Keywords: Complete bipartite graph, Weakly perfect graph, r-partite graph, Smarandache vertex (S-vertex) of a graph, Smarandache zero-divisor.
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1 Introduction

In this work, we will extend the notion of the *Smarandache vertex* (or S-vertex for short) of a simple graph (Definition 2), which was discussed for the *zero-divisor* and *comaximal graphs* of a *commutative ring* in [23] and [20], respectively, to the *zero-divisor graph* of a *commutative semiring* R, denoted $\Gamma(R)$ (Definition 1). Also, Mehdi-Nezhad and Siame studied the notion of S-vertices of the annihilation graphs of commutator posets and lattices with respect to an element in [21]. The motivation of this study is mainly based on providing a bridge between the structures of commutative semirings and their associated zero-divisor graphs using the notion of the Smarandache vertices in a natural way by comparing some results similar to the cases in commutative rings.

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In this introductory section, we define the notion of the zero-divisor graph of a commutative semiring (Definition 1), denoted by $\Gamma(R)$, which is similar to the definition of the zero-divisor graph of a commutative ring ([6]); and recall the notion of a Smarandache vertex in a simple graph (Definition 2) from [23] and we end the section with a brief description related to the *organization* of this paper. Note that for the main results of the paper, see Section 4, which are related to the *nonexistence of S-vertices* of $\Gamma(R)$ (see also Section 2 for some results and examples related to the *existence or nonexistence* of S-vertices of some simple graphs (Lemmas 1, 2, 3, and Proposition 1)). In addition, we assume that the reader is familiar with the basic notion and definitions of graph theory, commutative ring and semiring theory. For the notations and definitions regarding graph theory, the reader is referred to [11], [13], and [16] or any standard text of graph theory. For commutative ring; and semiring theory, see [8], [17], and [15], respectively.

Throughout this paper (unless otherwise indicated), all semirings (rings) R are commutative with identity $1 \neq 0$, $0x = 0$ for each x in R, $Z(R)$ denotes the set of zero-divisors of R, $U(R)$ denotes the set of units of R, and for $A \subseteq R$, we let $A^* = A \setminus \{0\}$. Also, $\mathrm{Ass}(R)$ and $\mathrm{Nil}(R)$ denote the set of associated prime ideals and the ideal of nilpotent elements of R, respectively. We will also emphasize whether R is a semiring or a ring whenever there is confusion in the context. As usual, the rings of integers, rationals, and integers modulo n will be denoted by $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{Z}_n$, respectively.

We now define the notion of the zero-divisor graph of a commutative semiring which is similar to the definition of the zero-divisor graph of a commutative ring as stated in [6].

Definition 1. *The zero-divisor graph of a commutative semiring (ring) R, denoted $\Gamma(R)$, is an undirected graph whose vertices are the nonzero zero-divisors of R and two distinct vertices x and y are adjacent if and only if $xy = 0$.*

$\Gamma(R)$ is the empty graph if and only if R is a semidomain (resp. an integral domain). To avoid trivialities when $\Gamma(R)$ is empty, we will implicitly assume when necessary that R is not a semidomain (an integral domain). Moreover, for a ring R, a nonempty graph $\Gamma(R)$ is finite if and only if R is finite and not a field [6, Theorem 2.2].

In [9] (1988), Beck introduced the concept of a *zero-divisor graph* of a commutative ring, but this work was mostly concerned with the *colorings* of rings. The above definition first appeared in the work of Anderson and Livingston [6] (1999), which contains several fundamental results concerning $\Gamma(R)$. This definition, unlike the earlier work of Anderson and Naseer [7] and Beck [9], does not take zero to be a vertex of $\Gamma(R)$.
The area of research on assigning a graph to an algebra (algebraic structure) has been very active (especially) since the last three decades and there are many papers that apply combinatorial methods (using graph-theoretic properties and parameters such as *connectedness, planarity, clique number, chromatic number, independence number, domination number*, and so on) to obtain algebraic results and vice versa, for instance, there are many papers on this interdisciplinary subject and for a short list of them, see [3–5, 10, 12, 22]. Moreover, the concept of a zero-divisor graph in a commutative ring has been generalized to a k-zero-divisor hypergraph by Eslahchi

and Rahimi in [14] (see also [18] for some improved results related to this subject).

The concept of a Smarandache vertex (or S-vertex for short) in a *simple graph* (Definition 2), which is *independent* of any *algebraic structure* was first introduced by Rahimi [23] to study the *Smarandache zero-divisors* of a commutative ring which Vasantha Kandasamy introduced in [25] for semigroups and rings (not necessarily commutative). Let R be a commutative semiring (ring) with identity $1 \neq 0$. We say that a nonzero element a in R is a *Smarandache zero-divisor* if there exist three different nonzero elements x, y, and b (all different from a) in R such that $ax = ab = by = 0$, but $xy \neq 0$. A Smarandache zero-divisor in a semiring (ring) is a zero-divisor and any semiring (ring) with a Smarandache zero-divisor must have at least four nontrivial zero-divisors. This definition of a Smarandache zero-divisor (which was given in [23]) is slightly different from the definition of Vasantha Kandasamy in [25], where in her definition b could also be equal to a.

We now recall the definition of a Smarandache vertex for a simple graph (i.e., an undirected graph with no loops and no multiple edges) from [23].

Definition 2. *A vertex a in a simple graph G is said to be a Smarandache vertex (or S-vertex for short) provided that there exist three distinct vertices x, y, and b (all different from a) in G such that x—a, a—b, and b—y are edges in G but there is no edge between x and y.*

Consequently, by this generalization, the study of S-vertices of any simple graph can be done directly in a pure graph-theoretic sense, and especially, discussing the S-vertices of any simple graph associated to an algebra (algebraic structure) is possible and can lead to the study of the interplay between some graph-theoretic properties and algebraic properties of the related algebra. For instance, S. Visweswaran and Hiren D. Patel [26] studied the S-vertices of *the complement of the annihilating-ideal graph* in connection to some ring-theoretic properties in Sections 2 (Lemma 2.5), 4 (Lemma 4.2(v)), and 5 (Proposition 5.1(iv)) of their paper.

Note that a graph containing a Smarandache vertex should have at least four vertices and three edges, and also the *degree* of each S-vertex must be at least 2.

The organization of this paper is as follows: In the introductory section (Section 1), besides recalling the notion of the Smarandache zero-divisor of a ring [23] and extending it to commutative semirings, we defined the notion of the zero-divisor graph of a commutative semiring R (Definition 1), denoted by $\Gamma(R)$, and the S-vertex of a simple graph (Definition 2).

In Section 2, we give several examples and study the S-vertices of a simple graph in general (Example 1, Lemmas 1, 2, 3, and Proposition 1 which provides a relation between a weakly perfect graph and its S-vertices). In Lemma 1, we show the existence or nonexistence of S-vertices of some known graphs, and in Lemmas 2 and 3 show how to construct the S-vertices from the cliques of a graph. We end this section with an example that shows the converse of Proposition 1 need not be true in general (Example 2), see also Remark 1. In Section 3, similar to the case in commutative rings, we show that $\Gamma(R)$ is connected with a diameter less than or equal to 3 (Theorem 1); and also show that the girth of $\Gamma(R)$ is less than or equal to 4 provided that it contains a cycle (Theorem 2). Section 4 is related to the girth of $\Gamma(R)$ and cases when $\Gamma(R)$ has no S-vertices regarding a relationship between the complete bipartiteness of $\Gamma(R)$ and prime ideals of the commutative semiring (ring) R with their nonexistence of the S-vertices (Theorems 3, 4, 5, and 6). We also characterize all regular graphs which can be realized as the zero-divisor graph of decomposable commutative semirings and their relation to the nonexistence of the S-vertices (Theorem 7). Finally, we end the paper with a result (Theorem 8) following a conjecture related to a universal vertex (i.e., a vertex which is adjacent to all other vertices of the graph) of $\Gamma(R)$, which implies nonexistence of S-vertices in $\Gamma(R)$.

2 Some examples and results related to S-vertices of some known simple graphs

We provide several (in particular, graph-theoretic) examples (Lemmas 1, 2, 3, and Proposition 1 which provides a relation between a weakly perfect graph and its S-vertices). In Lemma 1, we show the existence or nonexistence of S-vertices of some known graphs and in Lemmas 2 and 3 show how to construct the S-vertices from the cliques of a graph. Finally, we end the section with an example that shows the converse of Proposition 1 need not be true in general (Example 2).

We begin this section by defining the join of two graphs and providing an example (Example 1), respectively, to show the existence of the S-vertices in the join of two special graphs (see also Lemma 3).

Definition 3. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertex sets V_i and edge sets E_i $(i = 1, 2)$. The join of G_1 and G_2 is denoted by $G = G_1 \vee G_2$ with the vertex set $V_1 \cup V_2$ and the set of edges is*

$$E_1 \cup E_2 \cup \{xy | x \in V_1 \textit{and } y \in V_2\},$$

where xy is the edge between x and y.

Example 1. Let for any graph G, $\overline{G}$ denote the complement of G. The graph $K_n \vee \overline{K_m}$ contains exactly n S-vertices provided that $n \geq 2$ and $m \geq 2$ (see also Lemma 3 below). Moreover, for any $n \geq 2$, $K_n \vee \overline{K_2}$ has exactly n S-vertices.

The proof of the next three lemmas is not difficult and can be followed directly from the definition of the S-vertex and we leave them to the reader. In these lemmas, we show the existence (or nonexistence) of S-vertices of some known graphs (Lemma 1) and in Lemmas 2 and 3 we show how to construct the S-vertices from the cliques of a graph.

Lemma 1. *For any simple graph, the following results are true:*

(1) *A complete graph does not have any S-vertices.*

(2) *A star graph does not have any S-vertices.*

(3) *A complete bipartite graph has no S-vertices.*

(4) *Let G be a complete r-partite graph ($r \geq 3$) with parts $V_1, V_2, \ldots, V_r$. If at least one part, say V_1, has at least two elements, then every element not in V_1 is an S-vertex. Further, if there exist at least two parts of G such that each of which has at least two elements, then every element of g is an S-vertex.*

(5) *A bistar graph has two S-vertices; namely the center of each star is an S-vertex. Recall that a bistar graph is a graph generated by two star graphs when their centers are joined.*

(6) *Every vertex in a cycle of size greater than or equal to five in a graph is an S-vertex provided that there is no edge between the nonneighboring vertices. In particular, every vertex in the cycle C_n of size larger than or equal to 5 is an S-vertex. Note that for odd integers $n \geq 5$, $\chi(C_n) = 3$ and $\omega(C_n) = 2$. Moreover, for even integers $n \geq 5$, $\chi(C_n) = \omega(C_n) = 2$.*

(7) *Let G be a simple graph containing two distinct vertices x and y such that $d(x, y) = 3$. Then G has an S-vertex. But the converse is not true in general. Suppose G is the graph x—a, a—b, b—y, and a—y; where obviously, a is an S-vertex and $d(x, y) = 2$. Note that if the diameter of G is 3, then it has an S-vertex since there exist two distinct vertices x and y in G such that $d(x, y) = 3$.*

Lemma 2. *Let C be a clique in a graph G such that $|C| \geq 3$. Suppose that x is a vertex in $G \setminus C$ and x makes a link with at least one vertex or at most $|C| - 2$ vertices of C. Then every vertex of C is an S-vertex. In other case, if x makes links with $|C| - 1$ vertices of C, then all those $|C| - 1$ vertices are S-vertices.*

Lemma 3. *Let C be a clique in a graph G such that $|C| = n \geq 3$. Then by removing any edge e from C, the remaining subgraph $C \setminus \{e\}$ is isomorphic to $K_{n-2} \vee \overline{K_2}$. In this case, $C \setminus \{e\}$ has exactly $n - 2$ S-vertices provided that $n \geq 4$ (see Example 1).*

The following proposition provides a relation between a weakly perfect graph and its S-vertices.

Proposition 1. *Let G be a connected simple graph whose clique number is strictly larger than 2. If $\omega(G) \neq \chi(G)$, then G has an S-vertex. In other words, for any connected simple graph G with $\omega(G) \geq 3$ and no S-vertices, then $\omega(G) = \chi(G)$ (i.e., G is weakly perfect).*

Proof. Let C be a (largest) clique in G with $|C| \geq 3$. Since $\omega(G) \neq \chi(G)$, then G is not a complete graph. Thus, there exists a vertex x in $G \setminus C$ which makes edge(s) with at least one or at most $\omega(G) - 1$ element(s) of C. Now the proof is immediate from Lemma 2. □

Remark 1. *In the next example, we show that the converse of the above proposition need not be true in general (Example 2). Also none of the graphs in Parts (1), (2), and (3) of Lemma 1, has an S-vertex where $\omega(G) = \chi(G)$. Note that each of the graphs in Parts (2) and (3) has $\omega(G) = \chi(G) = 2$. The graph in Part (5) has two S-vertices and $\omega(G) = \chi(G) = 2$. See also Part (6) of Lemma 1.*

We end this section with an example that shows the converse of Proposition 1 is not true in general.

Example 2. Suppose G is the graph x—a, a—b, b—y, and a—y, where obviously, a is an S-vertex of G. Clearly, this graph shows that the converse of Proposition 1 is not true in general since it has an S-vertex and $\omega(G) = \chi(G) = 3$.

3 Connectivity and girth of $\Gamma(R)$

In this section, we show that $\Gamma(R)$ is connected with diameter less than or equal to 3 (Theorem 1); and also show that the girth of $\Gamma(R)$ is less than or equal to 4 provided that it contains a cycle (Theorem 2).

Theorem 1. (cf. [6, Theorem 2.3] or [24, Theorem 2.4]) *Let R be a commutative semiring (ring). Then $\Gamma(R)$ is connected and* $\operatorname{diam}(\Gamma(R)) \leq 3$.

Proof. Let $x, y \in Z(R)^*$ be distinct. If $xy = 0$, then $d(x, y) = 1$. So suppose that xy is nonzero. If $x^2 = y^2 = 0$, then x— xy— y is a path of length 2; thus $d(x, y) = 2$. If $x^2 = 0$ and $y^2 \neq 0$, then there is a $b \in Z(R)^* \setminus \{x, y\}$ with $by = 0$. If $bx = 0$, then x— b — y is a path of length 2. If $bx \neq 0$, then x —bx — y is a path of length 2. In either case, $d(x, y) = 2$. A similar argument holds if $y^2 = 0$ and $x^2 \neq 0$. Thus we may assume that xy, x^2, and y^2 are all nonzero. Hence there are $a, b \in Z(R)^* \setminus \{x, y\}$ with $ax = by = 0$. If $a = b$, then x— a — y is a path of length 2. Thus we may assume that $a \neq b$. If $ab = 0$, then x — a — b — y is a path of length 3, and hence $d(x, y) \leq 3$. If $ab \neq 0$, then x — ab — y is a path of length 2; thus $d(x, y) = 2$. Hence $d(x, y) \leq 3$, and thus $\operatorname{diam}(\Gamma(R)) \leq 3$. □

Theorem 2. (cf. [19, Theorem 3.4]) *Let R be a commutative semiring (ring). Then* $\operatorname{gr}(\Gamma(R)) \leq 4$ *provided $\Gamma(R)$ contains a cycle.*

Proof. Suppose that $\Gamma(R)$ contains a cycle C of size n and let $C = a_1$— $\cdots$ —a_n—a_1 be a cycle with the least length. If $n \leq 4$, we are done. Otherwise, we have $a_1a_4 \neq 0$. Thus, we need only consider 3 cases.

Case 1: $a_1a_4 = a_1$. Then $a_1 = a_1a_4$ implies a_1—a_2—a_3—a_1 is a cycle, a contradiction. Similarly, the case $a_1a_4 = a_4$ yielding a contradiction since a_2, a_3, and a_4 form a cycle of length three.

Case 2: $a_1a_4 = a_2$. Then $a_na_2 = 0$ implies a_2—$\cdots$— a_n—a_2 is a cycle with length $n - 1$, a contradiction. The case $a_1a_4 = a_3$ yielding a contradiction similarly.

Case 3: $a_1a_4 \neq a_1, a_2, a_3, a_4$. Then $a_2(a_1a_4) = 0$, $a_3(a_1a_4) = 0$, and a_2—a_1a_4—a_3—a_2 is a cycle, yielding a contradiction. Thus $n \leq 4$, i.e., $\operatorname{gr}(\Gamma(R)) \leq 4$. □

4 No S-vertices in $\Gamma(R)$ as a complete bipartite graph

In this section, we mainly discuss a relationship between the girth, complete bipartiteness of $\Gamma(R)$ and prime ideals of the semiring (ring) R and their relation to the nonexistence of the S-vertices(Theorems 3, 4, 5, and 6). We also characterize all regular graphs which can be realized as the zero-divisor graph of decomposable commutative semirings and their relation to the nonexistence of the S-vertices (Theorem 7). Finally, we end the paper with a result

(Theorem 8) and a conjecture related to a universal vertex of $\Gamma(R)$, which implies nonexistence of S-vertices in $\Gamma(R)$.

Example 3. Let $R \cong R_1 \times R_2$ be the direct product of two commutative semidomains (integral domains). Then it is not difficult to show that $\Gamma(R)$ is a complete bipartite graph with parts

$$\{(a, 0) \mid a \neq 0\}$$

and

$$\{(0, b) \mid b \neq 0\}$$

and hence has no S-vertices by Lemma 1(3). See also Example 2.1(d) of [6] when R is the direct product of two integral domains.

Theorem 3. (cf. [1, Theorem 2.4]) *Let P_1 and P_2 be two distinct prime ideals of the semiring (ring) R. If $P_1 \cap P_2 = \{0\}$, then $\Gamma(R)$ is a complete bipartite graph and consequently has no S-vertices.*

Proof. First, observe that $P_1, P_2 \in \mathrm{Ass}(R)$ since $P_1 \cap P_2 = \{0\}$. That is, $P_1 = \mathrm{Ann}(x)$ and $P_2 = \mathrm{Ann}(y)$ for some $x, y \in R$. For instance, let $x \in P_2 \setminus P_1$. Then it is clear that $P_1 \subseteq \mathrm{Ann}(x)$. On the other hand, $r \in \mathrm{Ann}(x)$ implies $rx = 0 \in P_1$ and hence $r \in P_1$ which implies $\mathrm{Ann}(x) \subseteq P_1$. We claim that $Z(R) = P_1 \cup P_2$. It is clear that $P_1 \cup P_2 \subseteq Z(R)$. On the other hand, if $x \in Z(R) \setminus P_1 \cup P_2$, then there exists $0 \neq y \in R$ such that $xy = 0$, and hence $y \in P_1 \cap P_2 = \{0\}$, yielding a contradiction.

Set $V_1 = P_1 \setminus \{0\}$ and $V_2 = P_2 \setminus \{0\}$. We claim that $\Gamma(R)$ is bipartite with two parts V_1 and V_2. It is enough to show that there is no edge between two vertices in V_1. If $a, b \in V_1$ and $ab = 0$, then $ab \in P_2$,

and hence $a \in P_2$ or $b \in P_2$, which is a contradiction. Therefore $\Gamma(R)$ is a bipartite graph.

Now, we show that $\Gamma(R)$ is complete bipartite. If, $a \in V_1$ and $b \in V_2$, then $ab \in P_1 \cap P_2$, and hence $ab = 0$. Thus $\Gamma(R)$ is a complete bipartite graph and hence has no S-vertices by Lemma 1(3). □

Theorem 4. *The following statements are true for a commutative semiring (ring) R.*

(a) *If $\mathrm{gr}(\Gamma(R)) = 4$ and $a^2 \neq 0$ for all $a \in Z(R)^*$ (in particular, R could be a reduced semiring (ring)), then $\Gamma(R)$ is a complete bipartite graph and consequently has no S-vertices.*

(b) *If $\Gamma(R)$ is a complete bipartite graph, then $\mathrm{gr}(\Gamma(R)) = 4$ or ∞.*

Proof. First, we show that $\mathrm{diam}(\Gamma(R)) = 2$. If $\mathrm{diam}(\Gamma(R)) = 0$ or 1, then $\Gamma(R)$ is a complete graph and so $\mathrm{gr}(\Gamma(R))$ is either ∞ or 3, yielding a contradiction. If $\mathrm{diam}(\Gamma(R)) = 3$, then there exist $a_1, a_2, a_3, a_4 \in \Gamma(R)$ such that a_1—a_2—a_3—a_4 is a path in $\Gamma(R)$. Thus, $a_1a_3 \neq 0$, $a_2a_4 \neq 0$, and $a_1a_4 \neq 0$ since $d(a_1, a_4) = 3$. If $a_1a_4 = a_2$, then since $(a_1a_4)a_2 = 0$, $(a_2)^2 = 0$, yielding a contradiction by hypothesis. Similarly $a_1a_4 \neq a_3$. Thus a_2—a_3—a_1a_4—a_2 is a cycle and so $\mathrm{gr}(\Gamma(R)) = 3$, yielding a contradiction. Therefore, $\mathrm{diam}(\Gamma(R)) = 2$. We now show that $\Gamma(R)$ is a complete bipartite graph. Since $\mathrm{gr}(\Gamma(R)) = 4$, there exist $a, b, c, d \in \Gamma(R)$ such that a—b—c—d—a. We show that $\Gamma(R) \cong K_{|V_1|,|V_2|}$, where,

$$V_1 = \{t \in Z(R)^* \mid ta = 0\}$$

and

$$V_2 = \{s \in Z(R)^* \mid sa \neq 0\}.$$

Let $t, t_1 \in V_1$ and $s, s_1 \in V_2$. Then $ta = 0$ and $sa \neq 0$. Assume that $ts \neq 0$. Since $\mathrm{diam}(\Gamma(R)) = 2$, there exists $x \in R$ such that a—x—s. If $ts = x$ or $ts = a$, then $(ts)^2 = 0$, yielding a contradiction by hypothesis. Therefore, a—ts—x—a is a cycle, yielding a contradiction since $\mathrm{gr}(\Gamma(R)) = 4$. Thus $ts = 0$. If $tt_1 = 0$, then a—t—t_1—a is a cycle, yielding a contradiction. So, $tt_1 \neq 0$. Similarly, $ss_1 \neq 0$. Also $V_1 \cap V_2 = \emptyset$. Therefore, $\Gamma(R) \cong K_{|V_1|,|V_2|}$ and so $\Gamma(R)$ is a complete bipartite graph and consequently has no S-vertices by Lemma 1(3). The proof of the other part is clear. □

We now, somewhat similar to the above theorem, provide a relation between the girth and nonexistence of S-vertices of $\Gamma(R)$.

Theorem 5. *The following statements are true for a semiring (ring) R.*

(1) *Let $\mathrm{gr}(\Gamma(R)) = 4$ and $a - b - c - d - a$ be a cycle in $\Gamma(R)$ such that $a^2 \neq 0$. Then $\Gamma(R)$ is complete bipartite when it has no S-vertices.*

(2) *If $\Gamma(R)$ is complete bipartite, then $\Gamma(R)$ has no S-vertices with $\mathrm{gr}(\Gamma(R)) = 4$ or ∞.*

Proof. We just give a proof for Part (1) since the other part is obvious (see also Lemma 1(3)). Clearly, by Lemma 1(7), $\mathrm{diam}(\Gamma(R)) \neq 3$ since $\Gamma(R)$ has no S-vertices. If $\mathrm{diam}(\Gamma(R)) = 0$ or 1, then $\Gamma(R)$ is a complete graph and so $\mathrm{gr}(\Gamma(R))$ is ∞ or 3, yielding a contradiction. Therefore, $\mathrm{diam}(\Gamma(R)) = 2$. We now show that $\Gamma(R)$ is a complete bipartite graph. Since $\mathrm{gr}(\Gamma(R)) = 4$, there exist $a, b, c, d \in \Gamma(R)$ such that $a - b - c - d - a$ is a cycle with $a^2 \neq 0$ by hypothesis. We show that $\Gamma(R) \cong K_{|V_1|,|V_2|}$, where

$$V_1 = \{t \in Z(R)^* : ta = 0\}$$

and

$$V_2 = \{s \in Z(R)^* : sa \neq 0\}.$$

Let $t, t_1 \in V_1$ and $s, s_1 \in V_2$. Then $ta = 0$ and $sa \neq 0$. Assume that $ts \neq 0$. Since $\mathrm{diam}(\Gamma(R)) = 2$, there exists $h \in Z(R)^*$ such that $a - h - s$. Clearly, $ts \neq 0$ implies that $t \neq h$. If $th = 0$, then $t - a - h - t$ implies $\mathrm{gr}(\Gamma(R)) = 3$, yielding a contradiction. Also $t \neq s$ since $th \neq 0$. Actually, $t \neq s$ since $t = s$ implies $0 = ta = sa \neq 0$, yielding a contradiction by the construction of V_1 and V_2. Thus a is an S-vertex in $\Gamma(R)$, which is a contradiction by hypothesis. Therefore $ts = 0$. If $tt_1 = 0$, then $a - t - t_1 - a$ is a cycle, yielding a contradiction. So, $tt_1 \neq 0$. Similarly $ss_1 \neq 0$, otherwise, $t - s - s_1 - t$ is a cycle of length 3, yielding a contradiction. Also $V_1 \cap V_2 = \emptyset$. Therefore, $\Gamma(R) \cong K_{|V_1|,|V_2|}$ and so $\Gamma(R)$ is a complete bipartite graph. □

Lemma 4. *Let $n \geq 3$ be a fixed integer and $R \cong R_1 \times R_2 \times \cdots \times R_n$ the direct product of n commutative semirings (rings). Then the girth of $\Gamma(R)$ is three.*

Proof. Without loss of generality, assume $n = 3$. Now the proof follows from the fact that the set consisting of $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ is a clique of size 3 in $\Gamma(R)$. □

Theorem 6. *Let $n \geq 2$ be a fixed integer and $R \cong R_1 \times R_2 \times \cdots \times R_n$ the direct product of n commutative semirings (rings). Then $n = 2$ and $\Gamma(R)$ is a complete bipartite graph with each part of size greater than or equal to 2 if and only if it contains no S-vertices and its girth is 4.*

Proof. The necessary part is clear. For the sufficient part, by the above lemma, $n = 2$ since $n \geq 3$ implies that the girth of the graph is equal to three, which is a contradiction by hypothesis. That is, R must be isomorphic to the product of two commutative semirings (rings). Now, without loss of generality, assume that $R = R_1 \times R_2$ and R_2 has two nontrivial zero divisors. Consequently, there exist two nontrivial zero divisors $a, b \in R_2$ (not necessarily distinct) such that $ab = 0$. Therefore,

$$(0,1) - (1,0) - (0,a) - (1,b)$$

is a path in $\Gamma(R)$, yielding a contradiction since $\Gamma(R)$ contains no S-vertices by hypothesis. Thus, each of R_1 and R_2 has no nontrivial zero divisors, respectively, which implies $\Gamma(R)$ is complete bipartite by Example 3 and the cardinality of each is greater than or equal to 3 since $\mathrm{gr}(\Gamma(R)) = 4$. □

Next, we show the nonexistence of S-vertices in $\Gamma(R)$ by characterizing all regular graphs which can be realized as the zero-divisor graph of decomposable commutative semirings. A semiring R is said to be *indecomposable* if R can not be represented as $R_1 \times R_2$, where R_1 and R_2 are two semirings, otherwise, R is said to be *decomposable*. A graph G is *regular* if the degrees of all vertices of G are the same.

Theorem 7. (cf. [2, Theorem 7]) *Let R be a finite commutative semiring such that $\Gamma(R)$ is a regular graph. If R is a decomposable semiring, then $\Gamma(R)$ is a complete bipartite graph and consequently has no S-vertices.*

Proof. Assume that $\Gamma(R)$ is a regular graph of degree r. Suppose that $R = R_1 \times R_2$, is a decomposable semiring. Since the degree of $(1,0)$ is $|R_2| - 1$ and the degree of $(0,1)$ is $|R_1| - 1$, we have $|R_1| = |R_2| = r + 1$. We show that R_1 is a semidomain. If not, then there exist two nonzero elements a and b in R_1 such that $ab = 0$. But

$$(\{0\} \times R_2) \cup \{(b,1)\} \subseteq \mathrm{Ann}((a,0))$$

and this follows that $\deg((a,0)) \geq r + 1$, a contradiction. Similarly, R_2 must be a semidomain. So in this case, $\Gamma(R) \cong K_{r,r}$. Thus, $\Gamma(R)$ is a complete bipartite graph and hence has no S-vertices by Lemma 1(3). □

Finally, we end the paper with a result and a conjecture related to a universal vertex of $\Gamma(R)$, which implies nonexistence of S-vertices in $\Gamma(R)$.

Theorem 8. *Let R be a reduced semiring (ring) such that $\Gamma(R)$ is nonempty with $\mathrm{gr}(\Gamma(R)) = \infty$. Then there is a vertex in $\Gamma(R)$ which is adjacent to every vertex of $\Gamma(R)$.*

Proof. Note that $\Gamma(R)$ has more than one vertex since it is nonempty and R is reduced by hypothesis and hence has no loops. Now, suppose to the contrary that there is not a vertex in $\Gamma(R)$ which is adjacent to every vertex of $\Gamma(R)$. Therefore, there exist distinct vertices

a_1, a_2, a_3, a_4 in $\Gamma(R)$ such that $a_1 - a_2 - a_3 - a_4$ is a path in $\Gamma(R)$ with $a_1a_3 \neq 0$ and $a_2a_4 \neq 0$. If $a_1a_4 = 0$, then $a_1 - a_2 - a_3 - a_4 - a_1$ is a cycle in $\Gamma(R)$, contrary to $\mathrm{gr}(\Gamma(R)) = \infty$. So we may assume that $a_1a_4 \neq 0$. Therefore, $(a_1a_4)a_2 = 0$ and $(a_1a_4)a_3 = 0$ implies $a_2 - a_3 - a_1a_4 - a_2$ is a cycle, contrary to $\mathrm{gr}(\Gamma(R)) = \infty$ (note that $a_1a_4 \neq a_2, a_3$ since R is reduced). Thus, there exists a vertex in $\Gamma(R)$ which is adjacent to every vertex of $\Gamma(R)$. □

Conjecture: Let R be a reduced semiring such that there exists a vertex in $\Gamma(R)$ which is adjacent to every vertex of $\Gamma(R)$. Then $\Gamma(R)$ is a star graph and hence has no S-vertices.

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