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Stability of the depth function of good filtrations

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Abstract. Let A be a Noetherian local ring, and let $I \subset J$ be two ideals of A. Let M be a finitely generated A-module. Brodmann proved that the function $n \mapsto \text{depth}_J(\frac{M}{I^nM})$ is constant for large n. In this paper, we consider a filtration $\phi = (M_n)_{n\in\mathbb{N}}$ of M and a filtration $f = (I_n)_{n\in\mathbb{N}}$ of A. Generalizing Brodmann's result, we first show that the function $n \mapsto \text{depth}_J(\frac{M}{M_n})$ is constant for large n of value $\text{depth}_J(f, M)$, provided that ϕ is f-good and f is strongly Noetherian. Secondly, we establish the inequality $\gamma_J(f, M) \leq \dim_A(M) - \text{depth}_J(f, M)$, where $\gamma_J(f, M)$ denotes the analytic spread of f at J with respect to M.

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1 Introduction

One of the first authors to have studied the asymptotic stability of the depth of the homogeneous pieces of a graded module is Burch [6], he proved that for an ideal I in a Noetherian local ring $(A, \mathfrak{m})$ the following inequality holds,

$$\ell(I) \leq \dim A - \min_{n\geq 1} \text{depth}\left(\frac{A}{I^n}\right),$$

where $\ell(I)$ is the analytic spread of I. Years laters, Brodmann [3] replaced the minimum of these depths by the limit value of the depth function $n \mapsto \text{depth}(\frac{A}{I^n})$, a limit whose existense he proved. In fact he studied the application from $\mathbb{N}$ into the subsets of $\text{Spec}(A)$ defined by

$$n \mapsto \text{Ass}_A(\frac{M}{I^nM}),$$

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and he showed that it is stationnary [4]. Analogous investigations on the stability of associated primes, but in the context of semi-prime operations, where later carried out by Kamano, Essan, Abdoulaye and Akeke [8]. As consequence, depth$(\frac{M}{I^nM})$ is constant for large n and

$$\ell(I,M) \leq \dim M - \lim_{n\to+\infty} \operatorname{depth}\left(\frac{M}{I^nM}\right)$$

being $\ell(I,M) = \dim \bigoplus_{n\geq 0} \frac{I^nM}{\mathfrak{m}I^nM}$. Other authors such as Bandari, Herzog and Hibi [1], Nam and Verbao [10] have also studied the depth function. The aim of this article is to extend Brodmann's results to a wider family of filtrations. Let $\phi = (M_n)_{n\in\mathbb{N}}$ be an $I-$good filtration of M. We will show that, for $n \gg 0$, Ass$(\frac{M}{M_n})$ is stable. Which makes it easy to prove that depth$(\frac{M}{M_n})$ is constant for large n. We also prove that, if $\phi = (M_n)_{n\in\mathbb{N}}$ is $f-$good, where $f = (I_n)_{n\in\mathbb{N}}$ is a strongly noetherian filtration of A, then depth$(\frac{M}{M_n})$ remains constant for $n \gg 0$. The existence of this asymptotic value allows us to obtain the following inequality.

$$\gamma_J(f,M) \leq \dim M - \lim_{n\to+\infty} \operatorname{depth}\left(\frac{M}{I_nM}\right),$$

with $\gamma_J(f,M) = \dim_{R(A,f)}\left(\bigoplus_{n\geq 0} \frac{I_nM}{JI_nM}\right)$.

2 Generalities

In this section we give basic definitions and notations.

Definition 1 ([2]). *(1) A filtration on a ring A is a family $f = (I_n)_{n\in\mathbb{N}}$ of ideals of A verifying*

$$I_0 = A, I_{n+1} \subseteq I_n \text{ and } I_nI_m \subseteq I_{n+m}, \text{ for all } n,m \in \mathbb{N}.$$

The set of filtrations of A is denoted by $\mathbb{F}(A)$. If I is an ideal of A then the family $f_I = (I^n)_{n\in\mathbb{N}}$ is a filtration of A called I-adic filtration.
Let M be a A-module. A filtration of M is a family $\Phi = (M_n)_{n\in\mathbb{N}}$ of sub-modules of M such as

$$M_0 = M, M_{n+1} \subseteq M_n, \text{ for all } n \in \mathbb{N}.$$

Φ is said to be f-compatible if $I_nM_p \subseteq M_{n+p}$ for all $n,p \in \mathbb{N}$.

(2) A filtration $f = (I_n)_{n\in\mathbb{N}}$ of A is said to be strongly Noetherian if A is a Noetherian ring and there exists an integer $k > 1$ such that $I_{n+m} = I_nI_m$ for all m,n $\geq k$.

(3) Let I be an ideal of A, a filtration $\Phi = (M_n)_{n\in\mathbb{N}}$ of M is said to be $I-$good if $IM_n \subseteq M_{n+1}$ and there exists $n_0 \in \mathbb{N}$ such that $IM_n = M_{n+1}$ for all $n \geq n_0$.

(4) Let M be an $A-$module, $f = (I_n)_{n\in\mathbb{N}}$ be a filtration of A and $\Phi = (M_n)_{n\in\mathbb{N}}$ be a filtration of M compatible with f.
Φ is said $f-$ good if there exists $N > 1$ such that $M_n = \sum_{p=1}^{N} I_{n-p}M_p$ for all $n \geq N$.

Definition 2 ([9])**.** *Let A be a noetherian ring and M be a $A-$module. A prime ideal P of A is said to be prime ideal associated with M if one of the following conditions is verified.*

(i) $\exists x \in M$ *such that* $\operatorname{ann}(x) = P$.

(ii) M *contains a submodule isomorphic to* $\frac{A}{P}$.

The set of prime ideals of A associated with M is denoted $\operatorname{Ass}_A(M)$.

Definition 3 ([5])**.** *(1) A sequence $a_1, a_2, \ldots, a_n$ of elements of A is called a regular sequence on a $A-$module M, or an $M-$regular sequence, if the following conditions hold.*

- *(i)* a_1 *is an $M-$regular element.*
- *(ii)* a_i *is an* $\frac{M}{(a_1,\ldots,a_{i-1})M}-$*regular element, for* $i = 2, \ldots, n$.
- *(iii)* $\frac{M}{(a_1,\ldots,a_n)M} \neq 0$.

(2) Let A be a Noetherian ring, I be an ideal of A, and M be a finite $A-$module such that $IM \neq M$. We define the depth of M on I, $\operatorname{depth}_I(M)$*, as the length of all maximal $M-$sequence contained in I.*
It is also called grade of M on I and is denoted $\operatorname{grade}(I, M)$.

3 Stability of the depth function $n \mapsto \operatorname{depth}_J(\frac{M}{M_n})$ with (M_n) an $I-$good filtration

We first recall this result of Brodmann.

Theorem 1 ([4])**.** *Let A be a Noetherian ring, I be an ideal of A, and M be a finitely generated $A-$module. The map from $\mathbb{N}$ into the subsets of* $\operatorname{Spec}(A)$ *defined by* $n \mapsto \operatorname{Ass}_A(\frac{M}{I^n M})$ *is independent of n, for n sufficiently large.*

We establish the following result which generalizes Brodmann's theorem to $I-$good filtrations.

Theorem 2. *Let A be a Noetherian ring, I be an ideal of A, M be a finitely generated $A-$module and $\Phi = (M_n)_{n\in\mathbb{N}}$ be a $I-$good filtration of M. Then the map* $n \mapsto \operatorname{Ass}_A(\frac{M}{M_n})$ *is stationnary.*

Proof. $\Phi = (M_n)_{n\in\mathbb{N}}$ a $I-$good filtration thus there exists $r \in \mathbb{N}$ such that for all $n \geqslant r$, $IM_n = M_{n+1}$. Therefore for all $n \geqslant 0$, $M_{n+r} = I^n M_r$.
We have the following exact sequence,

$$(0) \to \frac{M_{k+r}}{M_{n+k+r}} \to \frac{M}{M_{n+k+r}} \to \frac{M}{M_{k+r}} \to (0).$$

Thus

$$\operatorname{Ass}_A\left(\frac{M}{M_{n+k+r}}\right) \subset \operatorname{Ass}_A\left(\frac{M_{k+r}}{M_{n+k+r}}\right) \cup \operatorname{Ass}_A\left(\frac{M}{M_{k+r}}\right)$$

$$\subset \operatorname{Ass}_A\left(\frac{M_r}{I^{n+k}M_r}\right) \cup \operatorname{Ass}_A\left(\frac{M}{I^k M_r}\right), \text{for all } k \gg 0.$$

As according Theorem 1, for all $k \gg 0, \operatorname{Ass}_A\left(\frac{M_r}{I^{n+k}M_r}\right) = \operatorname{Ass}_A\left(\frac{M_r}{I^k M_r}\right)$. Then

$$\begin{aligned}\operatorname{Ass}_A\left(\frac{M}{M_{n+k+r}}\right) &\subset \operatorname{Ass}_A\left(\frac{M_r}{I^k M_r}\right) \cup \operatorname{Ass}_A\left(\frac{M}{I^k M_r}\right) \\ &\subset \operatorname{Ass}_A\left(\frac{M}{I^k M_r}\right) = \operatorname{Ass}_A\left(\frac{M}{M_{k+r}}\right).\end{aligned}$$

For $n = 1$, we have $\operatorname{Ass}_A\left(\frac{M}{M_{k+r+1}}\right) \subset \operatorname{Ass}_A\left(\frac{M}{M_{k+r}}\right)$, which shows that $\left(\operatorname{Ass}_A\left(\frac{M}{M_{n+k+r}}\right)\right)_n$ is decreasing. Therefore $\left(\operatorname{Card}\left(\frac{M}{M_{n+k+r}}\right)\right)_n$ is decreasing. As $\operatorname{Ass}_A\left(\frac{M}{M_{n+k+r}}\right)$ is a finite set, then $\left(\operatorname{Card}\left(\frac{M}{M_{n+k+r}}\right)\right)_n$ is a decreasing sequence of integers and therefore convergent. Hence $\left(\operatorname{Ass}_A\left(\frac{M}{M_{n+k+r}}\right)\right)_n$ is stationary. □

Theorem 3. *Let A be a Noetherian ring, I and J be ideals of A such that $I \subset J$, M be a finitely generated $A-$module and $\Phi = (M_n)_{n\in\mathbb{N}}$ be a $I-$good filtration of M. Then the function $n \mapsto depth_J(\frac{M}{M_n})$ is stationnary.*

Proof. Put $h_M = \liminf\limits_{n\longrightarrow+\infty} \operatorname{depth}_J(\frac{M}{M_n})$. The proof is done by induction on $p = h_M$.

- Suppose $p = 0$.
  $h_M = 0$ then, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $\inf\limits_{k\geqslant n} depth_J(\frac{M}{M_k}) = 0$. Therefore, there exists $N \geqslant n_0$ such that $\operatorname{depth}_J(\frac{M}{M_N}) = 0$. For all $a \in J$, a is not regular in $\frac{M}{M_N}$, then $a \in Z\left(\frac{M}{M_N}\right)$ where $Z\left(\frac{M}{M_N}\right)$ denote the set of zero-divisor of a module $\left(\frac{M}{M_N}\right)$; thus $J \subseteq \bigcup\limits_{P\in Ass\frac{M}{M_N}} P = Z\left(\frac{M}{M_N}\right)$. As $\left(Ass_A(\frac{M}{M_n})\right)_n$ is stationnary by Theorem 2, then we can assume that for all $n > N$, $Ass_A(\frac{M}{M_N}) = Ass_A(\frac{M}{M_n})$. Thus $J \subseteq \bigcup\limits_{P\in Ass(\frac{M}{M_n})} P$, hence $\operatorname{depth}_J(\frac{M}{M_n}) = 0$ for all $n \gg 0$.

- Let $p > 0$.
  Suppose the property is true at order p and $h_M = p+1 = \liminf\limits_{n\longrightarrow+\infty} \operatorname{depth}_J(\frac{M}{M_n})$, then for all n_0, there exists $N > n_0$ such that $\inf\limits_{k\geqslant N} \operatorname{depth}_J(\frac{M}{M_k}) \neq 0$. For all $k > N$, $\operatorname{depth}_J(\frac{M}{M_k}) \neq 0$ thus there exists $a \in J$ such that $a \notin \bigcup\limits_{P\in Ass\frac{M}{M_k}} P$. Put $\overline{M} = \frac{M}{aM}$ and $\overline{M_k} = \frac{M_k+aM}{aM}$. $(\overline{M_k})_k$ is then a $I-$good filtration on $\overline{M}$.
  Let $a_1, \ldots, a_r$ be an $\frac{\overline{M}}{\overline{M_k}}-$regular sequence in J. Let us show that $a, a_1, \ldots, a_r$ is an $\frac{M}{M_k}-$regular sequence in J. Since $a \notin \bigcup\limits_{P\in Ass\frac{M}{M_k}} P$ then a is an $\frac{M}{M_k}-$regular element. By

hypothesis a_1 is regular on $\frac{\overline{M}}{\overline{M}_k}$ which is isomorphic to $\dfrac{M}{aM+M_k}$. However, $\dfrac{\frac{M}{M_k}}{a\frac{M}{M_k}}$ is also isomorphic to $\dfrac{M}{aM+M_k}$. Thus, a_1 is regular on $\dfrac{\frac{M}{M_k}}{a\frac{M}{M_k}}$.
Therefore, a, a_1 is an $\frac{M}{M_k}-$regular sequence in J. We have

$$\begin{aligned}
\frac{\frac{\overline{M}}{\overline{M}_k}}{(a_1,\ldots,a_{i-1})\frac{\overline{M}}{\overline{M}_k}} &\cong \frac{\frac{M}{M_k+aM}}{(a_1,\ldots,a_{i-1})\frac{M}{M_k+aM}} = \frac{\frac{M}{M_k+aM}}{\frac{(a_1,\ldots,a_{i-1})M+M_k+aM}{M_k+aM}} \\
&\cong \frac{M}{(a,a_1,\ldots,a_{i-1})M+M_k} \cong \frac{\frac{M}{M_k}}{(a,a_1,\ldots,a_{i-1})\frac{M}{M_k}}.
\end{aligned}$$

Given that $a_1,\ldots,a_r$ is an $\frac{\overline{M}}{\overline{M}_k}-$regular sequence in J, we have that a_i is regular on $\frac{\frac{M}{M_k}}{(a,a_1,\ldots,a_{i-1})\frac{M}{M_k}}$. Therefore $a, a_1,\ldots,a_r$ is an $\frac{M}{M_k}-$regular sequence in J. We deduce that $\operatorname{depth}_J(\frac{M}{M_k}) \geq \operatorname{depth}_J(\frac{\overline{M}}{\overline{M}_k})+1$.
Let $a, a_1,\ldots,a_r$ be an $\frac{M}{M_k}-$regular sequence in J. Then, $a_1,\ldots,a_r$ is also an $\frac{\overline{M}}{\overline{M}_k}-$regular sequence in J. We obtain $\operatorname{depth}_J(\frac{\overline{M}}{\overline{M}_k}) \geq \operatorname{depth}_J(\frac{M}{M_k})-1$. Thus $\operatorname{depth}_J(\frac{\overline{M}}{\overline{M}_k}) = \operatorname{depth}_J(\frac{M}{M_k})-1$. It follows that $h_{\overline{M}} < h_M$. As $h_M = p+1$ then $h_{\overline{M}} \leq p$. According to the recurrence hypothesis $\left(\operatorname{depth}_J(\frac{\overline{M}}{\overline{M}_k})\right)_k$ is stable. Therefore the sentence $\left(\operatorname{depth}_J(\frac{M}{M_k})\right)_k$ is stationary.

□

4 Stability of the depth function $n \mapsto \operatorname{depth}_J\left(\frac{M}{M_n}\right)$ with (M_n) a $f-$good filtration

Let $f = (I_n)_{n\in\mathbb{N}}$ be a strongly Noetherian filtration of A and $\Phi = (M_n)_{n\in\mathbb{N}}$ be a $f-$ *good* filtration of M. We have

$$M_{n+m} = I_n M_m = I_m M_n \text{ for all } m, n \gg 0.$$

This result will allow us to extend Theorem 3 to $f-$good filtrations where f is strongly Noetherian.

Lemma 1. *Let k_1 and k_2 be two coprime positive integers. Then for any sufficently large integers s there exists $n, m \in \mathbb{N}$ such that $s = nk_1 + mk_2$.*

Proof. Since $\gcd(k_1, k_2) = 1$, there exist two integers u, v such that $uk_1 + vk_2 = 1$. Put $N = |u|k_1 + |v|k_2$. Let s be an integer such that $s > N^2$. From the Euclidean division of s by N, we have $s = Nq + j$, where $0 \leqslant j < N$ and $[\frac{s}{N}] = q$. As $uk_1 + vk_2 = 1$, it follows that

$$s = (|u|k_1 + |v|k_2)q + (uk_1 + vk_2)j = k_1(|u|q + uj) + k_2(|v|q + vj).$$

We observe that $j < N < q$, therefore, $0 \leq uj + |u|j < |u|q + uj$. Similarly, we also show that $|v|q + vj \geq 0$. Therefore, s can be written in the form $s = nk_1 + mk_2$ where $n, m \in \mathbb{N}$. □

Theorem 4. *Let A be a Noetherian ring, I and J be ideals of A such that $I \subset J$, M be a finitely generated $A-$module, $f = (I_n)_{n\in\mathbb{N}}$ be a strongly Noetherian filtration and $\Phi = (M_n)_{n\in\mathbb{N}}$ be a $f-$good filtration of M. Then the function $n \mapsto \mathrm{depht}_J \left(\frac{M}{M_n}\right)$ is stationnary.*

Proof. Let $s \gg 0$, k_1 and k_2 sufficiently large integers such that $\gcd(k_1, k_2) = 1$. s can be written (see Lemma 1) as $s = nk_1 + mk_2$ with $n, m \in \mathbb{N}$. As $(k_1, k_2) = 1$, there exist two integers u and v such that $uk_1 + vk_2 = 1$. We have $(M_{nk_1+mk_2})_n$ is $I_{k_1}-$good. Similary, $(M_{nk_1+mk_2})_m$ is $I_{k_2}-$good. According to Theorem 3, $\left(\mathrm{depth}_J\left(\frac{M}{M_{nk_1+mk_2}}\right)\right)_n$ is constant for large n. For all $n \gg 0$, we have $\mathrm{depth}_J\left(\frac{M}{M_{nk_1+mk_2}}\right) = \mathrm{depth}_J\left(\frac{M}{M_{(n+u)k_1+mk_2}}\right)$. Hence $\left(\mathrm{depth}_J\left(\frac{M}{M_{(n+u)k_1+mk_2}}\right)\right)_m$ is stationary. Then $\mathrm{depth}_J\left(\frac{M}{M_{(n+u)k_1+mk_2}}\right) = \mathrm{depth}_J\left(\frac{M}{M_{(n+u)k_1+(m+v)k_2}}\right)$, for all $m \gg 0$.
Therefore,

$$\mathrm{depth}_J\left(\frac{M}{M_{nk_1+mk_2}}\right) = \mathrm{depth}_J\left(\frac{M}{M_{(n+u)k_1+(m+v)k_2}}\right) = \mathrm{depth}_J\left(\frac{M}{M_{k_1n+k_2m+k_1u+k_2v}}\right).$$

Thus, $\mathrm{depth}_J\left(\frac{M}{M_s}\right) = \mathrm{depth}_J\left(\frac{M}{M_{s+1}}\right)$. Hence, $\left(\mathrm{depth}_J\left(\frac{M}{M_n}\right)\right)_n$ is stationary. □

5 Depth and analytic spread

Following [7] we now introduce the following definition.

Definition 4. *Let A be a ring, M be a finitely generated $A-$module, J be an ideal of A, and $f = (I_n)_{n\in\mathbb{N}}$ be a filtration of A. The analytic spread of f in J relative to M as the integer*

$$\gamma_J(f, M) = \dim_{\mathrm{R}(A,f)} \frac{\mathrm{R}(M, f)}{J\mathrm{R}(M, f)} = \dim_{\mathrm{R}(A,f)} \bigoplus_{n\geq 0} \frac{I_n M}{J I_n M},$$

where $\mathrm{R}(A, f) = \bigoplus_{n\geq 0} I_n$ and $\mathrm{R}(M, f) = \bigoplus_{n\geq 0} I_n M$.

Proposition 1. *Let A be a ring, J be an ideal of A, M be a finitely generated $A-$module, $f = (I_n)_{n\in\mathbb{N}}$ be a filtration of A. Then $\gamma_J(f^{(q)}, M) = \gamma_J(f, M)$, where $f^{(q)} = (I_{nq})_{n\geq 0}$ for all $q \geq 1$.*

Proof. We have

$$\frac{R(M, f^{(q)})}{JR(M, f^{(q)})} \cong \bigoplus_{n\geq 0} \frac{I_{qn} M}{J I_{qn} M} \subseteq \bigoplus_{n\geq 0} \frac{I_n M}{J I_n M}.$$

Therefore,

$$\mathrm{ann}_{R(A,f^{(q)})} \bigoplus_{n\geq 0} \frac{I_n M}{J I_n M} \subseteq \mathrm{ann}_{R(A,f^{(q)})} \bigoplus_{n\geq 0} \frac{I_{qn} M}{J I_{qn} M}.$$

Let a be an homogeneous element of degree r in $R(A,f^{(q)})$ such that $a \in \mathrm{ann}_{R(A,f^{(q)})}\bigoplus_{n\geq 0}\dfrac{I_{qn}M}{JI_{qn}M}$.
We have $aI_{qn}M \subset JI_{rq+nq}M$. So,

$$a^2I_nM = aaI_nM \subseteq aI_{rq}I_nM \subseteq I_n(aI_{rq}M) \subseteq I_n(JI_{2qr}M) \subseteq JI_{2rq+n},$$

thus, $a^2 \in \mathrm{ann}_{R(A,f^{(q)})}\bigoplus_{n\geq 0}\dfrac{I_nM}{JI_nM}$. Then,

$$\mathrm{ann}_{R(A,f^{(q)})}\bigoplus_{n\geq 0}\frac{I_{qn}M}{JI_{qn}M} \subseteq \sqrt{\mathrm{ann}_{R(A,f^{(q)})}\bigoplus_{n\geq 0}\frac{I_nM}{JI_nM}}.$$

Hence,

$$\sqrt{\mathrm{ann}_{R(A,f^{(q)})}\bigoplus_{n\geq 0}\frac{I_{qn}M}{JI_{qn}M}} \subseteq \sqrt{\mathrm{ann}_{R(A,f^{(q)})}\bigoplus_{n\geq 0}\frac{I_nM}{JI_nM}}.$$

As $\mathrm{ann}_{R(A,f^{(q)})}\bigoplus_{n\geq 0}\dfrac{I_nM}{JI_nM} \subseteq \mathrm{ann}_{R(A,f^{(q)})}\bigoplus_{n\geq 0}\dfrac{I_{qn}M}{JI_{qn}M}$,
then,

$$\sqrt{\mathrm{ann}_{R(A,f^{(q)})}\bigoplus_{n\geq 0}\frac{I_nM}{JI_nM}} \subseteq \sqrt{\mathrm{ann}_{R(A,f^{(q)})}\bigoplus_{n\geq 0}\frac{I_{qn}M}{JI_{qn}M}}.$$

Hence,

$$\sqrt{\mathrm{ann}_{R(A,f^{(q)})}\bigoplus_{n\geq 0}\frac{I_{qn}M}{JI_{qn}M}} = \sqrt{\mathrm{ann}_{R(A,f^{(q)})}\bigoplus_{n\geq 0}\frac{I_nM}{JI_nM}}.$$

Therefore,

$$\dim \frac{R(A,f^{(q)})}{\mathrm{ann}_{R(A,f^{(q)})}\dfrac{R(M,f^{(q)})}{JR(M,f^{(q)})}} = \dim \frac{R(A,f^{(q)})}{\mathrm{ann}_{R(A,f^{(q)})}\dfrac{R(M,f)}{JR(M,f)}}.$$

We have $\dim_{R(A,f^{(q)})}\dfrac{R(M,f)}{JR(M,f)} = \dim_{R(A,f^{(q)})}\dfrac{R(M,f^{(q)})}{JR(M,f^{(q)})}$.
Moreover, let $x \in R(A,f)$ homogeneous of degree n, we have $x^q \in I_{nq} \subseteq R(A,f^{(q)})$ thus $R(A,f)$ is integral over $R(A,f^{(q)})$.
Then $\dfrac{R(A,f)}{\mathrm{ann}_{R(A,f)}\dfrac{R(M,f)}{JR(M,f)}}$ is integral over $\dfrac{R(A,f^{(q)})}{\mathrm{ann}_{R(A,f)}\dfrac{R(M,f)}{JR(M,f)}\bigcap R(A,f^{(q)})}$.
Consequently,

$$\dim_{R(A,f)} \frac{R(A,f)}{\mathrm{ann}_{R(A,f)}\dfrac{R(M,f)}{JR(M,f)}} = \dim_{R(A,f^{(q)})} \frac{R(A,f^{(q)})}{\mathrm{ann}_{R(A,f)}\dfrac{R(M,f)}{JR(M,f)}\bigcap R(A,f^{(q)})}.$$

As $\mathrm{ann}_{R(A,f)}\dfrac{R(M,f)}{JR(M,f)}\bigcap R(A,f^{(q)}) = \mathrm{ann}_{R(A,f^{(q)})}\dfrac{R(M,f)}{JR(M,f)}$, therefore

$$\dim_{R(A,f)} \frac{R(M,f)}{JR(M,f)} = \dim_{R(A,f^{(q)})} \frac{R(A,f^{(q)})}{\mathrm{ann}_{R(A,f^{(q)})}\dfrac{R(M,f)}{JR(M,f)}}$$

$$= \dim_{R(A,f^{(q)})} \frac{R(M,f)}{JR(M,f)}$$
$$= \dim_{R(A,f^{(q)})} \frac{R(M,f^{(q)})}{JR(M,f^{(q)})}.$$

Hence, $\gamma_J(f,M) = \gamma_J(f^{(q)},M)$.

□

Theorem 5. *Let M be a finitely generated $A-$module, $f = (I_n)_{n\in\mathbb{N}}$ be a strongly noetherian filtration of A and J be an ideal of A. If $I_1 \subseteq J$, we have the inequality*

$$\gamma_J(f,M) \leq \dim_A(M) - \operatorname{depth}_J(f,M),$$

where $\operatorname{depth}_J(f,M) = \lim\limits_{n\longrightarrow+\infty} \operatorname{depth}_J(\frac{M}{I_nM})$.

Proof. The filtration $(I_nM)_{n\geq 0}$ of M is $f-$good where f is strongly noetherian, hence $\operatorname{depth}_J(f,M)$ exists. There exists an integer N such that for all $n,m \geq N$, $I_{m+n} = I_mI_n$. In particular, for each $q \geq N$, $f^{(q)} = (I_{nq})_{n\geq 0} = (I_q^n)_{n\geq 0} = f_{I_q}$. We have $\bigoplus\limits_{n\geq 0}\frac{I_{qn}M}{JI_{qn}M} = \bigoplus\limits_{n\geq 0}\frac{I_q^nM}{JI_q^nM}$. Therefore, according to Brodmann [3], $\gamma_J(I_q,M) \leq \dim_A(M) - \operatorname{depth}_J(I_q,M)$. We know that $\gamma_J(I_q,M) = \gamma_J(f_{I_q},M) = \gamma_J(f,M)$. Also, $\operatorname{depth}_J(I_q,M) = \lim\limits_{n\longrightarrow+\infty}\operatorname{depth}_J(\frac{M}{I_q^nM}) = \lim\limits_{n\longrightarrow+\infty}\operatorname{depth}_J(\frac{M}{I_{qn}M}) = \operatorname{depth}_J(f,M)$. Hence, $\gamma_J(f,M) \leq \dim_A(M) - \operatorname{depth}_J(f,M)$. □

Example 1. Let $A = K[[X,Y,Z]]$ denote the ring of formal power series over a field K. Consider the ideals $J = (X,Y,Z)$, $I = (XY,XZ,X^2)$, and for each $n \in \mathbb{N}$, define

$$I_n = (XY,XZ,X^2,Y^nZ^n).$$

Each I_n is an ideal of A. Let $B = \frac{A}{I}$, $\mathfrak{J} = \frac{J}{I}$, and $\mathfrak{g} = (J_n)_{n\in\mathbb{N}} = \left(\frac{I_n}{I}\right)_{n\in\mathbb{N}}$. Then $\mathfrak{J}$ is an ideal of B, and $\mathfrak{g}$ defines a strongly Noetherian filtration on B. We consider B as a B-module and set $M = B$. We now verify the inequality given in Theorem 5.

Computation of $\operatorname{depth}_{\mathfrak{J}}(\mathfrak{g},M)$. By definition, $\operatorname{depth}_{\mathfrak{J}}(\mathfrak{g},M) = \lim_{n\to\infty}\operatorname{depth}_{\mathfrak{J}}\left(\frac{M}{J_nM}\right)$.
According to [5], we have $\operatorname{depth}_{\mathfrak{J}}\left(\frac{M}{J_nM}\right) \leq \dim(\frac{B}{\mathfrak{P}})$, for some $\mathfrak{P} \in \operatorname{Ass}_B\left(\frac{M}{J_nM}\right)$. Note that $\frac{M}{J_nM} \cong \frac{A}{I_n}$, hence $\operatorname{Ass}_B\left(\frac{M}{J_nM}\right) = \operatorname{Ass}_A\left(\frac{A}{I_n}\right)$. Let $P = \operatorname{Ann}_A(X + I_n)$. Since $X,Y,Z \in P$, we have $(X,Y,Z) \subseteq P$, and thus $P = (X,Y,Z) \in \operatorname{Ass}_A(\frac{A}{I_n})$. Setting $\mathfrak{P} = \frac{P}{I}$, we obtain $\mathfrak{P} \in \operatorname{Ass}_B(\frac{M}{J_nM})$, and $\dim(\frac{B}{\mathfrak{P}}) = \dim(\frac{A}{P}) = 0$. Therefore,

$$\operatorname{depth}_{\mathfrak{J}}\left(\frac{M}{J_nM}\right) = 0 \quad \text{for all } n,$$

and consequently, $\operatorname{depth}_{\mathfrak{J}}(\mathfrak{g},M) = 0$.

Computation of $\dim_B(M)$. We have $\dim_B(M) = \dim(M) = \dim(\frac{A}{I}) = \sup\left\{\dim \frac{A}{P} \mid P \in \min(I)\right\}$, where $\min(I)$ is the set of prime ideals minimal over I. Since $\min(I) = \{(X)\}$, we obtain

$$\dim(M) = \dim \frac{A}{(X)} = \dim K[[Y,Z]] = 2.$$

Computation of $\gamma_{\mathfrak{J}}(\mathfrak{g}, M)$. We compute the associated graded components.

$$\frac{J_n M}{\mathfrak{J} J_n M} \cong \frac{J_n}{J(Y^n Z^n) + I} \cong \frac{I + (Y^n Z^n)}{I + J(Y^n Z^n)} \cong \frac{(Y^n Z^n)}{J(Y^n Z^n)}.$$

Therefore,

$$\gamma_{\mathfrak{J}}(\mathfrak{g}, M) = \dim \bigoplus_{n \geq 0} \frac{J^n M}{\mathfrak{J} J^n M} = \dim \bigoplus_{n \geq 0} \frac{(Y^n Z^n)}{J(Y^n Z^n)} = 1.$$

Conclusion. We conclude that the inequality in Theorem 5 is satisfied

$$\gamma_{\mathfrak{J}}(\mathfrak{g}, M) \leq \dim_B(M) - \operatorname{depth}_{\mathfrak{J}}(\mathfrak{g}, M) = 2 - 0 = 2.$$

Acknowledgments

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