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On non-closure degree for finite groups

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Abstract. The non-closure degree for a finite group G has to do with obtaining a probability of selecting any pair of elements $x, y \in G$ such that $xy \notin H$, where H is normal in G. It is shown in the paper that, the probability lies in the interval $[0, \frac{|G|-1}{|G|})$, with the result as 0 if and only if $H = G$. Illustrations were done using both abelian and non-abelian groups relative to their normal subgroups.

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1 Introduction

A lot of researchers worked on probabilities defined for finite groups i.e $(P(G))$; the stage was set ready for such concept by the works of Miller [7] in 1944, Erdos et al. [5] in 1968 and Gustafson [6] in 1973 where they presented most of the first results in probability group theory. The notion was on the probability of two distinct elements of a group commuting, which was termed as commutativity degree of non-abelian groups. Since then there have been a lot of generalizations and forms of the commutativity degree of groups; just like the $n-th$ commutativity degree of a group and relative $n-th$ commutativity degree of a group by Erfanian et al. [4], $n-th$ power commutativity degree [2] and the work of Abdul Hamid et al. [1] on the productivity degree of subgroups. Other types of probabilities include the cyclic degree of a finite group [8] which has to do with the probability of selecting two distinct elements $x, y \in G$ such that the subgroup $\langle x, y\rangle$ is cyclic, the nilpotency degree of a finite group [3] that has to do with selecting $x, y \in G$ such that the subgroup $\langle x, y\rangle$ is nilpotent.

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The non-closure degree entails selecting a pair of elements (x, y) from a finite group G, where $xy \notin H$ with H a normal subgroup of G. The selection can be made easier by considering elements of the quotient group of G by H (G/H). Quotient group consists of cosets, some cosets exhibits some featrures as can be seen in the definition of lone coset [9]. A lone coset is a set $X \in G/H$ such that $X \neq H$ and for all $x \in X$ the inverse $x^{-1} \in X$. It is also shown in [9] that a quotient group of even order consists of a lone coset.

In this paper, the upper and lower bounds for the non-closure degree of finite groups are given; the work showed that the probability of selecting two distinct elements $x, y \in G$ relative to its normal subgroup H such that $xy \notin H$ lies in the interval $[0, \frac{|G|-1}{|G|})$. Illustrations were done using both abelian and non-abelian groups.

2 Results and discussion

The non-closure degree for a finite group G shows the probability of selecting any pair $x, y \in G$ with $xy \notin H$, where $H \lhd G$. It is denoted by $P_{xy \notin H}(G)$ as can be seen in the following definition.

Definition 1. *Let H be a normal subgroup of a finite group G, then the non-closure degree of G is defined as* $P_{xy \notin H}(G) = \frac{|\{(x,y) \in G \times G : xy \notin H\}|}{|G|^2}$.

Besides the lone coset we have cosets in G/H which are also useful in determining the non-closure degree for finite groups, as such we give a new definition below.

Definition 2. *Related Pair of Cosets: The quotient group of G by H has a pair of cosets say A and B which are related pair if for every $x \in A$ the inverse x^{-1} belongs to set B.*

Remark 1. *Note that:*

- *We have a lone coset when*
  - *i.* $xH = x^{-1}H$.
  - *ii.* $xH = yH$ *with x and y each a self inverse element.*
- *For related pair of cosets A and B, xH gives set A while $x^{-1}H$ gives the other set B.*

The following examples show how lone cosets and related pair of cosets look like.

Example 1. Let $G = D_6$ with a normal subgroup $H = \{e, a^3\}$, we have the following lone cosets and related pair of cosets.
The lone cosets are: $bH = \{b, a^3b\} = a^3bH$, $abH = \{ab, a^4b\} = a^4bH$, and $a^2bH = \{a^2b, a^5b\} = a^5bH$. We have the related pair of cosets as: $aH = \{a, a^4\}$ and $a^5H = \{a^2, a^5\}$, $a^2H = \{a^2, a^5\}$ and $a^4H = \{a, a^4\}$.

Example 2. The group $\mathbb{Z}_{15}$ with its subgroup $H = \{0, 5, 10\}$ has pairs of related cosets but no lone coset. $A = \{1, 6, 11\}$ and $D = \{4, 9, 14\}$, $B = \{2, 7, 12\}$ and $C = \{3, 8, 13\}$.

Example 3. Let $G = \mathbb{Z}_{16}$ and $H = \{0, 4, 8, 12\}$, then $G/H = \{H, A, B, C\}$, where $A = \{1, 5, 9, 13\}$, $B = \{2, 6, 10, 14\}$ and $C = \{3, 7, 11, 15\}$. Observe that B is a lone coset while A and C are related, B is obtained from $2H = 14H$ as well as $6H = 10H$.

Depending on the order of H there could be a number of lone cosets in a quotient group of G by H. Below are some results on cosets in G/H.

Proposition 1. *Let H be a normal subgroup of a finite group G, then each element of a coset in G/H generates the coset.*

Proof. For a coset aH in G/H, $aH = \{ah : h \in H\} = \{ah_1, ah_2, ah_3, ., ., ., ah_n\}$, so $(ah_i)H = \{(ah_i)h : h \in H\} = \{ah_ih_1, ah_ih_2, ah_ih_3, ., ., ., ah_ih_n\} = \{ah_ih_j\} = \{ah_k\} = aH$ since $h_k = h_ih_j \in H$. □

Lemma 1. *Let H be a normal subgroup of a finite group G and P a lone coset in the quotient group G/H, then for all $x, y \in P$, xy belongs to H.*

Proof. Let P be a lone coset in G/H, from Proposition 1 we can see that for every $x, y \in P$ we have $xH = yH = P$. And by the property of cosets if $xH = yH$ then $x^{-1}y \in H$. Since P is a lone coset, if $x \in P$ the inverse $x^{-1} \in P$. Which means that $x^{-1}H = yH$. Hence, $xy \in H$. □

Below are some corollaries to Lemma 1.

Corollary 1. *Let A and B be related pair of cosets in the quotient group of a finite group G by its normal subgroup H, then for every $x, y \in A$, $xy \notin H$.*

Proof. Let A and B be the related pair of cosets in the quotient group G/H for $H \lhd G$. Let $x, y \in A$, then from Proposition 1 above it is clear that $xH = yH$ and $x^{-1}y \in H$. But A has B as a related coset making $x^{-1} \in B$, which indicates that $x^{-1}H \neq xH = yH$ and $x^{-1}H \neq yH$, leading to $(x^{-1})^{-1}y \notin H$ as such $xy \notin H$. □

Note that for $x \in A$ and $y \in B$, xy results to an element of H.

Corollary 2. *For two different lone cosets say P_1 and P_2 found in the quotient group of G by H, if $x \in P_1$ and $y \in P_2$, then $xy \notin H$.*

Proof. Let $x \in P_1$ and $y \in P_2$, since $x^{-1} \in P_1$ it implies that $x^{-1}H \neq yH$ which makes $xy \notin H$. □

Corollary 3. *For two ordinary cosets say Q and R which are not pairwise related in the quotient group of a finite group G by its normal subgroup H, if $x \in Q$ and $y \in R$, then $xy \notin H$.*

Proof. Since Q and R are distinct and not related, for $x \in Q$ and $y \in R$ we have $x^{-1}H \neq yH$ (as x^{-1} can not be in R) leading to $xy \notin H$. □

Below are theorems and examples on the non-closure degree of G considering different forms of normal subgroups.

Theorem 1. *For a finite group G and $H \lhd G$ with $|H| = \frac{|G|}{2}$, $P_{xy \notin H}(G) = 1/2$.*

Proof. Since H satisfies the closure property and H is a normal subgroup of G, the set $P_1 = \{(x, y) \in H \times H : xy \in H\}$ is of order $|H \times H| = \frac{|G|^2}{4}$. Let $L = G - H$, then L in this case is a lone coset since $H \cup L = G$ and $|L| = |H|$.

Let $P_2 = \{(x, y) \in L \times L : xy \in H\}$, from Lemma 1 $x, y \in L$ implies $xy \in H$, then $|P_2| = \frac{|G|^2}{4}$. So, $P_{xy\notin H}(G) = \frac{|\{(x,y)\in G\times G\,:\,xy\notin H\}|}{|G|^2} = \frac{|G\times G| - (|P_1|+|P_2|)}{|G|^2} = \frac{|G|^2 - 2(\frac{|G|^2}{4})}{|G|^2} = 1/2.$ □

Theorem 1 above shows that the non-closure degree for a group of even order relative to its maximal normal subgroup is 0.5.

Example 4. Let $G = \mathbb{Z}_8$ and $H = \{0, 2, 4, 6\}$, $G/H = \{H, L\}$ with $L = \{1, 3, 5, 7\}$.The set $\{(x, y) \in G \times G : xy \notin H\}$ has the following elements; $\{(0, 1), (0, 3), (0, 5), (0, 7), (1, 0), (3, 0), (5, 0), (7, 0), (2, 1), (2, 3), (2, 5), (2, 7), (1, 2), (3, 2), (5, 2), (7, 2), (4, 1), (4, 3), (4, 5), (4, 7), (1, 4), (3, 4), (5, 4), (7, 4), (6, 1), (6, 3), (6, 5), (6, 7), (1, 6), (3, 6), (5, 6), (7, 6)\}$. So, $P_{xy\notin H}(G) = \frac{|\{(x,y)\in G\times G\,:\,xy\notin H\}|}{|G|^2} = \frac{32}{64} = 1/2.$

Example 5. For $G = D_6$ and $H = \{1, a, a^2\}$, we have $L = \{b, ab, a^2b\}$ as element of G/H. The set $\{(x, y) \in G\times G : xy \notin H\}$ contains the following elements; $\{(1, b), (1, ab), (1, a^2b), (b, 1), (ab, 1), (a^2b, 1), (a, b), (a, ab), (a, a^2b), (b, a), (ab, a), (a^2b, a), (a^2, b), (a^2, ab), (a^2, a^2b), (b, a^2), (ab, a^2), (a^2b, a^2)\}$. $P_{xy\notin H}(G) = \frac{|\{(x,y)\in G\times G\,:\,xy\notin H\}|}{|G|^2} = \frac{18}{36} = 1/2.$

Theorem 2. *Let H be the trivial subgroup of a finite group G, the non-closure degree for G is given by* $P_{xy\notin H}(G) = \frac{|G| - 1}{|G|}$.

Proof. Excluding the self inverse elements, for every pair $(x, y) \in G \times G : xy = e$ there exists $(y, x) \in G \times G : yx = e$. Let π be the number of self inverse elements in G and $P_1 = \{(x, y) \in G \times G : xy = e\}$.We can see that $|P_1| = \pi + 2(\frac{|G|-\pi}{2}) = |G|$. Then we obtain $P_{xy\notin H}(G) = \frac{|\{(x,y)\in G\times G\,:\,xy\notin H\}|}{|G|^2} = \frac{|G\times G| - |P_1|}{|G|^2} = \frac{|G|^2 - |G|}{|G|^2} = \frac{|G| - 1}{|G|}.$ □

The next examples are illustrated using the trivial subgroup.

Example 6. Let $G = \mathbb{Z}_7$ and $H = \{0\}$, we then extract the elements of the set $\{(x, y) \in \mathbb{Z}_7 \times \mathbb{Z}_7 : xy \notin H\}$. Thus the elements are; $\{(0, 1,), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (1, 0), (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 0), (2, 1), (2, 2), (2, 3), (2, 4), (2, 6), (3, 0), (3, 1), (3, 2), (3, 3), (3, 5)(3, 6), (4, 0), (4, 1), (4, 2), (4, 4), (4, 5), (4, 6), (5, 0), (5, 1), (5, 3), (5, 4), (5, 5), (5, 6), (6, 0), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$, therefore $P_{xy\notin H}(G) = \frac{|\{(x,y)\in G\times G\,:\,xy\notin H\}|}{|G|^2} = \frac{42}{49} = \frac{6}{7}$.

By using Theorem 2 we get $P_{xy\notin H}(G) = \frac{|G| - 1}{|G|} = \frac{7-1}{7} = \frac{6}{7}$.

Example 7. For a group $G = D_8 = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$ and its normal subgroup $H = \{1\}$. The set $\{(x, y) \in G\times G : xy \notin H\} = \{(1, a), (1, a^2), (1, a^3), (1, b), (1, ab), (1, a^2b), (1, a^3b), (a, 1), (a^2, 1), (a^3, 1), (b, 1), (ab, 1), (a^2b, 1), (a^3b, 1), (a, a), (a, a^2), (a, b), (a, ab), (a, a^2b), (a, a^3b), (a^2, a), (b, a), (ab, a), (a^2b, a), (a^3b, a), (a^2, a^3), (a^2, b), (a^2, ab), (a^2, a^2b), (a^2, a^3b), (a^3, a^2), (b, a^2), (ab, a^2), (a^2b, a^2), (a^3b, a^2), (a^3, a^3), (a^3, b), (a^3, ab), (a^3, a^2b), (a^3, a^3b), (b, a^3), (ab, a^3), (a^2b, a^3), (a^3b, a^3), (b, ab), (b, a^2b), (b, a^3b), (ab, b), (a^2b, b), (a^3b, b), (ab, a^2b), (ab, a^3b), (a^2b, ab), (a^3b, ab), (a^2b, a^3b), (a^3b, a^2b)\}$. Thus, $P_{xy\notin H}(G) = \frac{|\{(x,y)\in G\times G\,:\,xy\notin H\}|}{|G|^2} = \frac{56}{64} = \frac{7}{8}$.

In this case if $Z(D_8)$ is used as the normal subgroup we get same result.

The following theorem takes care of the non-closure degree for a finite group G relative to a non-trivial subgroup H when G/H consists of lone cosets and related pairs of cosets.

Theorem 3. *Let H be a non-trivial normal subgroup of a finite group G with the order of H as β and m the index of H in G, where ψ is the number of lone cosets and ω the number of pairs of related cosets. Then the non-closure degree of G is;*

$$P_{xy\ \notin H}(G) = \frac{2\beta^2 \displaystyle\sum_{i=1}^{\psi+1}(m-i) + 8\beta^2(\omega-1) + 2\omega(2\ {}^{\beta}C_2 + \beta)}{|G|^2}.$$

Proof. Let m be the index of H in G, if m is even the quotient group G/H contains lone coset(s) and related pair of cosets. Let ψ be the number of lone cosets and ω the number of pairs of related cosets.

In the set $\{(x,y)\ \in G\times G\ :\ xy\ \notin H\}$ the pairs (x,y) could be by pairing

i. x from H and y from the remaining $m-1$ cosets.

ii. x from a lone coset and other $m-2$ cosets, which is done successively up to the last lone coset with $m-(\psi+1)$ cosets.

iii. x from coset say A and other cosets not pairwise related to A.

iv. x and y within an ordinary coset, i.e x and y coming from a single coset which is neither H nor a lone coset.

Let $P_1 = \{(x,y)\ \in H\times N\ :\ xy \notin H\}$, with N a coset different from H. For each $(x,y)\ \in H\times N$, there exists $(y,x)\ \in H\times N$ since $H\times N\ \subseteq G\times G$. Since the order of H is β, then $|P_1| = 2\beta^2(m-1)$.

If m is even there could be a number of lone cosets, with each coset also of order β, let $P_2 = \{(x,y)\ \in L\times M\ :\ xy \notin H\}$ with x from a lone coset and y from the remaining cosets which are $m-2$ in number as H is now excluded. So for the next lone coset will be with $m-3$ cosets continuously up to the last lone coset with $m-(\psi+1)$ cosets.
Let P_3 be the set containing (x,y) with x from H and all lone cosets while y from the remaining $m-(\psi+1)$ cosets. So $|P_3| = 2\beta^2(m-i)$, with $i = 1,2,3.,.,.,(\psi+1)$.
Let $P_4 = \{(x,y)\ \in A\times Q\ :\ xy \notin H\}$ with Q a coset not related to coset A as shown in corollary 3, for every (x,y) we equally have (y,x) satisfying same condition. From Definition 2 the coset A has a related pair say B while coset Q has a related pair say R. This means x from A can also be paired with y from R in $2\beta^2$ ways, observe that coset B follows just like coset A. The pairing is done successively up to the last pair. So for the total pairing for two ordinary cosets not related out of all the ω coset gives $|P_4| = 8\beta^2(\omega-1)$.
Let $P_5 = \{(x,y)\ \in A\times A\ :\ xy \notin H\}$, in this case both x and y are from same coset, for a coset A the pairing is done in $2^{\beta}C_2$ ways since for each (x,y), (y,x) exists and for a pair of related cosets we get $4^{\beta}C_2$. Set P_5 also comprises of elements of the form (x,x) since $\forall\ x \notin H$, xx is also not in H, and for each pair of related cosets we have 2β of such pairs. So for all ω,

$|P_5| = 4\omega^{\beta}C_2 + 2\beta\omega = 2\omega(2^{\beta}C_2 + \beta)$.
The result is obtained by adding $|P_3|$, $|P_4|$ and $|P_5|$, then applying Definition 1 to get

$$P_{xy\,\notin H}(G) = \frac{2\beta^2 \sum_{i=1}^{\psi+1}(m-i) + 8\beta^2(\omega-1) + 2\omega(2\ {}^{\beta}C_2+\beta)}{|G|^2}.$$

□

Below are examples showing the non-closure degree of a finite group G using Theorem 3.

Example 8. Let $G = \mathbb{Z}_{12}$ with its normal subgroup $H = \{0, 4, 8\}$, the other cosets in G/H are $Q = \{1, 5, 9\}$, $L = \{2, 6, 10\}$ and $R = \{3, 7, 11\}$. Observe that L is a lone coset while Q and R are related cosets. Let $P_1 = \{(x, y) \in H \times N : xy \notin H\}$, where N is the set of elements from the other 3 cosets. So the elements of P_1 are $\{(0, 1), (0, 2), (0, 3), (0, 5), (0, 6), (0, 7), (0, 9), (0, 10), (0, 11), (1, 0), (2, 0), (3, 0), (5, 0), (6, 0), (7, 0), (9, 0), (10, 0), (11, 0), (4, 1), (4, 2), (4, 3), (4, 5), (4, 6), (4, 7), (4, 9), (4, 10), (4, 11), (1, 4), (2, 4), (3, 4), (5, 4), (6, 4), (7, 4), (9, 4), (10, 4), (11, 4), (8, 1), (8, 2), (8, 3), (8, 5), (8, 6), (8, 7), (8, 9), (8, 10), (8, 11), (1, 8), (2, 8), (3, 8), (5, 8), (6, 8), (7, 8), (9, 8), (10, 8), (11, 8)\}$, making $|P_1| = 54$.

Let $P_2 = \{(x, y) : L \times M : xy \notin H\}$, where M represents the set of elements of the two related cosets, the elements of P_2 are; $\{(2, 1), (2, 3), (2, 5), (2, 7), (2, 9), (2, 11), (1, 2), (3, 2), (5, 2), (7, 2), (9, 2), (11, 2), (6, 1), (6, 3), (6, 5), (6, 7), (6, 9), (6, 11), (1, 6), (3, 6), (5, 6), (7, 6), (9, 6), (11, 6), (10, 1), (10, 3), (10, 5), (10, 7), (10, 9), (10, 11), (1, 10), (3, 10), (5, 10), (7, 10), (9, 10), (11, 10)\}$. So, $|P_2| = 36$.

Let $P_3 = \{(x, y) \in Q \times Q : xy \notin H\}$, the elements of P_3 are; $\{(1, 5), (1, 9), (5, 9), (5, 1), (9, 1), (9, 5), (1, 1), (5, 5), (9, 9)\}$. Making $|P_3| = 9$. Similarly; $|P_4| = |\{(x, y) \in R \times R : xy \notin H\}| = 9$. Hence,

$$P_{xy\,\notin H}(G) = \frac{|P_1| + |P_2| + |P_3| + |P_4|}{|G|^2} = \frac{108}{144} = \frac{3}{4}.$$

Using the formula in Theorem 3 will be easier to obtain the result, in this case $\beta = 3,\ m = 4, \psi = 1$ and $\omega = 1$, leading to

$$\begin{aligned} P_{xy\,\notin H}(G) &= \frac{2\beta^2 \sum_{i=1}^{\psi+1}(m-i) + 8\beta^2(\omega-1) + 2\omega(2\ {}^{\beta}C_2+\beta)}{|G|^2} \\ &= \frac{2\beta^2[(m-1)+(m-2)] + 8\beta^2(0) + 2(2\ {}^{3}C_2+3)}{12^2} \\ &= \frac{18(5)+0+18}{144} = \frac{108}{144} = \frac{3}{4}. \end{aligned}$$

Example 9. Let $G = D_{12}$ and $H = \{1, a^3\}$, the remaining cosets in G/H will be $A = \{a, a^4\}$, $B = \{a^2, a^5\}$, $C = \{b, a^3b\}$, $D = \{ab, a^4b\}$ and $E = \{a^2b, a^5b\}$. Observe that the cosets A and B are related pair while C, D and E are lone cosets.

Here $\beta = 2,\ m = 6, \omega = 1$ and $\psi = 3$. Applying the formula in Theorem 3 gives;

$$\begin{aligned} P_{xy\ \notin H}(G) &= \frac{2\beta^2(m-i) + 8\beta^2(\omega - 1) + 2\omega(2\ {}^{\beta}C_2 + \beta)}{|G|^2} \\ &= \frac{2\beta^2[(m-1)+(m-2)+(m-3)+(m-4)] + 0 + 8}{144} \\ &= \frac{120}{144} = \frac{5}{6}. \end{aligned}$$

3 Bounds for non-closure degree for a group

Next are Theorems 4 and 5 showing the bounds for the non-closure degree of a finite group G relative to its normal subgroup H.

Theorem 4. *Let H be a normal subgroup of a finite group G, then the non-closure degree of G is equal to zero if and only if $H = G$.*

Proof. ($\Rightarrow$) Assume the probability is zero, i.e
$P_{xy\ \notin H}(G) = 0 = \frac{0}{|G|^2} = \frac{|\emptyset|}{|G|^2} = \frac{|\{(x,y)\ \in G\times G\ :\ xy\notin G\}|}{|G|^2} = P_{xy\ \notin G}(G)$ and this implies $H = G$.
($\Leftarrow$) Suppose $H = G$, by Definition 1 we have $P_{xy\ \notin H}(G) = P_{xy\ \notin G}(G) = \frac{|\{(x,y)\ \in\ G\times G\ :\ xy\ \notin G\}|}{|G|^2} = \frac{|\emptyset|}{|G|^2} = \frac{0}{|G|^2} = 0.$ ☐

For any finite set X, we denote the probability of selecting any pair of elements from the set such that a certain relation(condition) is satisfied by $P(X)$.

Theorem 5. *For a normal subgroup H of a finite group G, the non-closure degree for G is given by $P_{xy\ \notin H}(G) \leq \frac{|G|-1}{|G|}$.*

Proof. Let $X = \{(x,y)\ \in G \times G\ :\ xy \in G\}$, this implies that $P(X) = 1$. Let $Y = \{(x,y)\ \in G \times G\ :\ xy \in H = \{e\}\}$ and $Z = \{(x,y)\ \in G \times G\ :\ xy \notin H\}$, clearly $Y \subseteq X$, $Z \subseteq X$ and $(Y \cup Z) \subseteq X$.

The elements of Y are in the form $xy = e$ with $y = x^{-1}$, if π is the number of self inverse elements in G then there are $|G| - \pi$ elements which are not self inverse and this will lead to $2\frac{(|G|-\pi)}{2}$ number of (x,y) in $G \times G$. So $P(Y) = \frac{\pi + (|G|-\pi)}{|G|^2} = \frac{1}{|G|}$.

Since $(Y \cup Z) \subseteq X$, then we have $P(Y \cup Z) \leq P(X)$ which means $P(Y) + P(Z) \leq P(X)$ and $P(Z) \leq P(X) - P(Y)$. Substituting for $P(X)$ and $P(Y)$ gives $P(Z) \leq 1 - \frac{1}{|G|} = \frac{|G|-1}{|G|}$.
Observe that $P(Z) = P_{xy\ \notin H}(G)$. ☐

Theorems 4 and 5 above show that $0 \leq P_{xy\ \notin H}(G) \leq \frac{|G|-1}{|G|}$. When $H = \{e\}$ there are more pairs (x,y) in $G \times G$ such that $xy\ \notin H$ as such the non-closure degree of a group G is maximum when $H = \{e\}$. The following tables show the non-closure degree for $\mathbb{Z}_{12}$, Q_8 and D_{10} indicating the lower and upper bounds.

From Table 1 we can see that the non-closure degree for $\mathbb{Z}_{12}$ is greater when it is relative to the trivial subgroup. The trend is the same for Tables 2 and 3 with the non-closure degree as 0.5 when the subgroup is maximal.

Table 1: $P_{xy\notin H}(\mathbb{Z}_{12})$ relative to different subgroups of $\mathbb{Z}_{12}$.

S/N	Subgroup, H	$\|H\|$	$P_{xy\notin H}(G)$
1	$H=\{0\}$	1	0.9167
2	$H=\{0,6\}$	2	0.8333
3	$H=\{0,4,8\}$	3	0.7500
4	$H=\{0,3,6,9\}$	4	0.6667
5	$H=\{0,2,4,6,8,10\}$	6	0.5
6	$H=\mathbb{Z}_{12}$	12	0

Table 2: $P_{xy\notin H}(\mathbb{Q}_8)$ relative to different subgroups of Q_8.

S/N	Subgroup, H	$\|H\|$	$P_{xy\notin H}(G)$
1	$H=\{1\}$	1	0.8750
2	$H=\{1,-1\}$	2	0.7500
3	$H=\{1,-1,i,-i\}$	4	0.5000
4	$H=Q_8$	8	0

Table 3: $P_{xy\notin H}(D_{10})$ relative to different subgroups of D_{10}.

S/N	Subgroup, H	$\|H\|$	$P_{xy\notin H}(G)$
1	$H=\{1\}$	1	0.9000
2	$H=\{e,a,a^2,a^3,a^4\}$	5	0.5000
3	$H=D_5$	10	0

Conclusion

The work has to do with defining the non-closure degree for a group G relative to its normal subgroup H, which is the probability of selecting a pair of elements in G such that $xy \notin H$. The formula for computing the probability for finite groups are obtained using the elements of the quotient group G/H, illustrations are done using both abelian and non-abelian groups. It is shown that the probability lies in the interval $[0, \frac{|G|-1}{|G|})$ with the result as 0.5 when $|G|$ is even and H maximal.

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