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Remarks on the zero-set intersection graph

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Abstract. In this paper, we study the zero-set intersection graph $(\Gamma(C(X)))$ and its line graph $(L(\Gamma(C(X))))$. We showed that 0 is a cut vertex of $\Gamma(C(X))$ iff $|X| = 2$, and for a first countable space X, $\Gamma(C(X))$ is chordal iff $|X| = 2$ *or* $|X| = 3$. We stated some conditions for a maximal clique to be a maximal ideal. We obtained that two (first countable/real compact) topological spaces X and Y are homeomorphic iff $L(\Gamma(C(X)))$ is graph isomorphic to $L(\Gamma(C(Y)))$ iff $C(X)$ is isomorphic to $C(Y)$. We showed that $\{f, g\}$ is a dominating set of $\Gamma(C(X))$ iff $fg = 0$.

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1 Introduction and Preliminaries

Let $C(X)$ be the ring of real-valued continuous functions on a completely regular Hausdorff space X. In 2005, Azarpanah et. al [2] studied the zero divisor graph of $C(X)$ on when this graph will be triangulated, hypertriangulated, complemented, etc. In 2013, Alafifi et. al [1] studied the line graph of the zero divisor graph of $C(X)$. In 2020, B. Bose and A. Das [3] introduce a graph structure called zero-set intersection graph $\Gamma(C(X))$ on $C(X)$, where the set of vertices is the set $\mathcal{N}(X)$ of all non-units of the ring $C(X)$ and two distinct vertices f and g are adjacent if $Z(f) \bigcap Z(g) \neq \phi$, where $Z(f) = \{x \in X : f(x) = 0\}$. In this paper, we studied some other properties of $\Gamma(C(X))$ like hypertriangulated, chordal, complemented, etc. We showed that 0 is a cut vertex of $\Gamma(C(X))$ iff $|X| = 2$, and for a first countable space X, $\Gamma(C(X))$ is chordal iff $|X| = 2$ or $|X| = 3$. We give a bound for the clique number of $\Gamma(C(X))$. We also studied the line graph of $\Gamma(C(X))$, denoted by $L(\Gamma(C(X)))$. We showed that $L(\Gamma(C(X)))$ is connected with diameter 3. The eccentricity of any vertex of $L(\Gamma(C(X)))$ is either 2 or 3. We obtained that two (first countable/real compact) topological spaces X and Y are homeomorphic iff $L(\Gamma(C(X)))$ is graph isomorphic to $L(\Gamma(C(Y)))$ iff $C(X)$ is isomorphic to $C(Y)$. Finally, in section 4, we

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studied the dominating set of $\Gamma(C(X))$ and $L(\Gamma(C(X)))$, and the relationship between the zero divisor graph of $C(X)$ and the zero-set intersection graph.

For any two vertices x and y of a simple graph G, the distance $d(x, y)$ is the length of the shortest path between x and y. The diameter of G is $Diam(G) = sup\{d(x, y) : x, y \in V(G)\}$. $c(x, y)$ is the length of the shortest cycle containing both x and y. The eccentricity of a vertex x denoted by $e(x)$, is defined as $e(x) = sup\{d(x, y) : x \neq y\ , y \in V(G)\}$. The radius of G is defined as $\rho(G) = inf\{e(x) : x \in V(G)\}$. A vertex x is a central vertex of G if $e(x) = \rho(G)$. The center of a graph G denoted by $C(G)$, is the set of all central vertices of G. A vertex x is said to be a cut vertex of G if $G\backslash\{x\}$ is not connected. A separating set of a graph G is a set $S \subseteq V(G)$ such that $G\backslash S$ is not connected. A graph in which all vertices are pairwise adjacent is called a complete graph. A complete subgraph of a graph G is called a clique. A maximal clique is a clique that is maximal with respect to inclusion. The clique number of G is given by $\omega(G) = sup\{|V(H)| : H \text{ is a complete subgraph of } G\}$. A graph G is said to be triangulated (hypertriangulated) if each vertex (edge) is a vertex (an edge) of a triangle. A graph is chordal if each of its cycles of length at least 4 has a chord, which is an edge joining two vertices of a cycle but is not itself an edge of the cycle. The line graph of a graph G denoted by $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge in G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common vertex in G.

For undefined terms concerning $C(X)$ and graph theory, the reader is referred to [6] and [5]. If $|X| = 1$, then $\mathcal{N}(X)$ is a singleton. In this paper, we assume that $|X| > 1$ and X is a completely regular Hausdorff space.

2 Other properties of the zero-set intersection graph

In this section, we study other properties of the zero-set intersection graph $\Gamma(C(X))$.

Theorem 1. *Let f and g be two distinct vertices of $\Gamma(C(X))$. Then,*
(i) $d(f, g) = 1$ iff $Z(f) \bigcap Z(g) \neq \phi$.
(ii) $d(f, g) = 2$ iff $Z(f) \bigcap Z(g) = \phi$.

Proof. (i) Trivial.
(ii) Suppose $d(f, g) = 2$, then by (i) we must have $Z(f) \bigcap Z(g) = \phi$. Conversely, suppose $Z(f) \bigcap Z(g) = \phi$, then $f - fg - g$ is a path of length 2. So, $d(f, g) = 2$. □

Definition 1 ([2])**.** *A graph G is said to be complemented if for each vertex x there is a vertex y of G such that x and y are adjacent and there is no vertex w of G which is adjacent to both x and y.*

Remark 1. *It is easy to see that if f and g are two distinct vertices of $\Gamma(C(X))$. Then, there exists $h \in \mathcal{N}(X)$ such that $Z(h) \bigcap Z(f) \neq \phi$ and $Z(h) \bigcap Z(g) \neq \phi$. Hence, it follows that the zero-set intersection graph is never complemented. It was shown in [3] that $\Gamma(C(X))$ is triangulated. It is also easy to see that $\Gamma(C(X))$ is hypertriangulated.*

Theorem 2. *Let $f \in \mathcal{N}(X)$. Then,*

$$e(f) = \begin{cases} 1, & \text{if } f = 0 \\ 2, & \text{if } f \neq 0. \end{cases} \tag{1}$$

Proof. If $f = 0$, then f is adjacent to every non-unit. So, $e(f) = 1$. If $f \neq 0$, then there exists $p \in X$ such that $p \notin Z(f)$. As X is completely regular, there exists $g \in C(X)$ such that $g(p) = 0$ and $g(Z(f)) = 1$. This implies that $Z(f) \bigcap Z(g) = \phi$ and so by Theorem 1, we have $e(f) = 2$. □

Remark 2. *$\rho(\Gamma(C(X))) = 1$ and $f = 0$ is the central vertex of $\Gamma(C(X))$. Hence, the center of $\Gamma(C(X))$ is $C(\Gamma(C(X))) = \{0\}$.*

Theorem 3. *Let $f,\ g \in \mathcal{N}(X)$ be two distinct vertices. Then,*
(i) $c(f,g) = 3$ iff $Z(f) \bigcap Z(g) \neq \phi$.
(ii) If $|X| \geq 3$, then $c(f,g) = 4$ iff $Z(f) \bigcap Z(g) = \phi$.

Proof. (i) Trivial.
(ii) Suppose $c(f,g) = 4$, then the result follows from (i). Conversely, suppose $Z(f) \bigcap Z(g) = \phi$, then by (i) no cycle of length 3 contains both f and g. So, $c(f,g) > 3$. As $|X| \geq 3$, there exist $x, y, z \in X$ such that $x \in Z(f)$, $y \in Z(g)$, and $z \notin \{x, y\}$. By the complete regularity of X, there exists $h \in C(X)$ such that $h(z) = 1$ and $h(\{x, y\}) = 0$. For $0 \neq r \in \mathbb{R}$, $f - h - g - rh - f$ is a cycle of length 4. So, $c(f,g) = 4$. □

Remark 3. *If $|X| = 2$, say $X = \{x, y\}$. Then, it is easy to see that there is no cycle containing both f and g, where $Z(f) = \{x\}$ and $Z(g) = \{y\}$.*

Theorem 4. *0 is a cut vertex of $\Gamma(C(X))$ iff $|X| = 2$.*

Proof. Suppose $|X| \geq 3$. Let $f, g \in \Gamma(C(X))\backslash\{0\}$. As $f, g \in \mathcal{N}(X)$, there exist $a, b \in X$ such that $a \in Z(f)$ and $b \in Z(g)$. As $|X| \geq 3$, there exist $c \in X$ such that $c \notin \{a, b\}$. By the complete regularity of X, there exists $h \in C(X)$ such that $h(c) = 1$ and $h(\{a, b\}) = 0$. This implies that $f - h - g$ is a path that joins f and g. This shows that $\Gamma(C(X))\backslash\{0\}$ is connected, which contradicts that 0 is a cut vertex. Conversely, if $|X| = 2$, say $X = \{a, b\}$. Then, it is easy to see that $\Gamma(C(X))\backslash\{0\}$ is not connected and vertex set of $\Gamma(C(X))\backslash\{0\}$ is $N_a \bigcup N_b$, where $N_a = \{f \in C(X) :\ f(a) = 0 \textit{ and } f(b) \neq 0\}$, $N_b = \{f \in C(X) :\ f(b) = 0 \textit{ and } f(a) \neq 0\}$. □

Theorem 5. *Let X be a first countable space. Then, the zero-set intersection graph is chordal iff $|X| = 2$ or $|X| = 3$.*

Proof. If $|X| \geq 4$, then there exist $a, b, c, d \in X$. As X is first countable T_2 space, so every point is G_δ-point. From [6, 3.11] there exist functions $f_x \in C(X)$ such that $Z(f_x) = \{x\}$, where $x \in \{a, b, c, d\}$. Consider the function $g_{x,y} = f_x f_y$, where $x, y \in \{a, b, c, d\}$. Then, $g_{a,b} - g_{b,c} - g_{c,d} - g_{a,d} - g_{a,b}$ is a cycle of length 4 with no chord. Conversely, if $|X| = 2$ say $X = \{a, b\}$, then $\mathcal{N}(X) = N_a \bigcup N_b \bigcup \{0\}$ where $N_a = \{f \in C(X) :\ f(a) = 0 \textit{ and } f(b) \neq 0\}$, $N_b = \{f \in C(X) :\ f(b) = 0 \textit{ and } f(a) \neq 0\}$. Let C be a cycle of length greater than 3. If 0 is on the cycle, then C has a chord. Suppose 0 is not on the cycle, then we see that all vertices on the cycle C are either in N_a or all vertices on the cycle C are either in N_b. So, this cycle C has a chord. If $|X| = 3$, say $X = \{a, b, c\}$, if 0 is on the cycle, then C has a chord. Suppose 0 is not on the cycle, let $g_1 - g_2 - g_3 - g_4$, $g_i \in \mathcal{N}(X)$, $i \in \{1, 2, 3, 4\}$ be a path on a cycle C. Then, $Z(g_1) \bigcap Z(g_2) \neq \phi$, $Z(g_2) \bigcap Z(g_3) \neq \phi$, and $Z(g_3) \bigcap Z(g_4) \neq \phi$. If g_1 is adjacent to g_3 or g_2 is adjacent to g_4, then C has a chord. Suppose $Z(g_1) \bigcap Z(g_3) = \phi$ and $Z(g_2) \bigcap Z(g_4) = \phi$. As X has only three elements and $g_1 - g_2 - g_3 - g_4$ be a path, so we must have $|Z(g_2)| = |Z(g_3)| = 2$. Now, if g_1 and g_4 are adjacent, then the cycle $g_1 - g_2 - g_3 - g_4 - g_1$ must have a chord. Suppose g_1 and g_4 are not adjacent, then there is a vertex h which is adjacent to g_4. If $Z(h)$ is a singleton, then there is a chord joining h and g_3. Similarly, if $Z(h)$ is not a singleton, there is a chord in this cycle C. Hence, $\Gamma(C(X))$ is chordal. □

Let βX be the Stone-Čech Compactification of X [6, 6.6]. For each $p \in \beta X$, $M^p = \{f \in C(X) : p \in \ cl_{\beta X} Z(f)\}$ is a maximal ideal of $C(X)$ [6, 7.3]. The density $d(X)$ of a topological space X is the least cardinality of a dense subset of X. It was shown in [3] that every maximal ideal is a maximal clique, but the converse is not true. The following theorem gives some conditions under which a maximal clique is a maximal ideal.

Theorem 6. *Let M be a maximal clique in $\Gamma(C(X))$. If $\bigcap_{f \in M} Z(f) \neq \phi$, then M is a maximal ideal in $C(X)$.*

Proof. Let $S = \bigcap_{f \in M} Z(f) \neq \phi$. Then, for any $f \in M$, $S \subseteq Z(f)$. It is easy to show that M is an ideal of $C(X)$. As M is an ideal so $M \subseteq M^p$ for some maximal ideal M^p. But M^p being a maximal ideal is also a maximal clique. As M is a maximal clique. So, $M = M^p$. Hence, M is a maximal ideal of $C(X)$. □

Theorem 7. *Let X be an infinite space. Then, for each $p \in \beta X$, $|M^p| \leq \omega(\Gamma(C(X))) < 2^{d(X)}$, where $d(X)$ is the density of X.*

Proof. As maximal ideals are maximal cliques in $\Gamma(C(X))$ [3, Theorem 3.2]. So, $|M^p| \leq \omega(\Gamma(C(X)))$. Also, $\omega(\Gamma(C(X))) < |C(X)|$ and $|C(X)| \leq 2^{d(X)}$ [4, Theorem 2.4]. Hence, $|M^p| \leq \omega(\Gamma(C(X))) < 2^{d(X)}$. □

Example 1. If $X = \beta\mathbb{N}$, then it is easy to see that $|M^p| \geq \aleph_1$ and $2^{d(X)} = 2^{\aleph_0} = \aleph_1$, $\aleph_1$ denote the first infinite uncountable cardinal number. So, $\omega(\Gamma(C(X))) = \aleph_1$

3 Line graph of the zero-set intersection graph $L(\Gamma(C(X)))$

Let $f, g \in \mathcal{N}(X)$. Then, $[f, g]$ is a vertex of $L(\Gamma(C(X)))$ iff $Z(f) \bigcap Z(g) \neq \phi$. In this section, we study the connectedness, diameter, and radius of $L(\Gamma(C(X)))$. The main objective of this section is to study the relationships between topological properties of X, algebraic properties of $C(X)$, and graph properties of $L(\Gamma(C(X)))$.

Theorem 8. *$L(\Gamma(C(X)))$ is connected and Diam($L(\Gamma(C(X)))$) $= 3$.*

Proof. Let $[f_1, f_2]$ and $[g_1, g_2]$ be two distinct vertices of $L(\Gamma(C(X)))$, where $f_1, f_2, g_1, g_2 \in \mathcal{N}(X)$. If $f_i = g_j$ for some $i, j \in \{1, 2\}$, then $[f_1, f_2]$ and $[g_1, g_2]$ are adjacent. If $f_i \neq g_j$ for every $i, j \in \{1, 2\}$, then either $Z(f_i) \bigcap Z(g_j) \neq \phi$ for some $i, j \in \{1, 2\}$ or $Z(f_i) \bigcap Z(g_j) = \phi$ for every $i, j \in \{1, 2\}$. If $Z(f_i) \bigcap Z(g_j) \neq \phi$ for some $i, j \in \{1, 2\}$, then $[f_1, f_2] - [f_i, g_j] - [g_1, g_2]$ is a path of length 2. If $Z(f_i) \bigcap Z(g_j) = \phi$ for every $i, j \in \{1, 2\}$, then $[f_1, f_2] - [f_i, 0] - [0, g_j] - [g_1, g_2]$ for some $i, j \in \{1, 2\}$, is a path of length 3; hence, $L(\Gamma(C(X)))$ is connected and $\text{Diam}(L(\Gamma(C(X)))) = 3$. □

Corollary 1. *Let $[f_1, f_2]$ and $[g_1, g_2]$ be two distinct vertices of $L(\Gamma(C(X)))$. Then, the following conditions hold:*
(i) $d([f_1, f_2], [g_1, g_2]) = 1$ iff $f_i = g_j$ for some $i, j \in \{1, 2\}$.
(ii) $d([f_1, f_2], [g_1, g_2]) = 2$ iff $f_i \neq g_j$ for every $i, j \in \{1, 2\}$ and $Z(f_i) \bigcap Z(g_j) \neq \phi$ for some $i, j \in \{1, 2\}$.
(iii)$d([f_1, f_2], [g_1, g_2]) = 3$ iff $f_i \neq g_j$ for every $i, j \in \{1, 2\}$ and $Z(f_i) \bigcap Z(g_j) = \phi$ for every $i, j \in \{1, 2\}$.

Theorem 9. *Let I be a proper ideal of $C(X)$. Then, $d([f_1, f_2], [g_1, g_2]) \leq 2$, where $f_i, g_j \in I$ for all $i, j \in \{1, 2\}$.*

Proof. Let $[f_1, f_2]$ and $[g_1, g_2]$ be two vertices of $L(\Gamma(C(X)))$ where $f_i, g_j \in I$ for all $i, j \in \{1, 2\}$. If $f_i = g_j$ for some $i, j \in \{1, 2\}$, then by Corollary 1(i), we have $d([f_1, f_2], [g_1, g_2]) = 1 \leq 2$. If $f_i \neq g_j$ for every $i, j \in \{1, 2\}$, by [3, 3.1] I is a clique in $\Gamma(C(X))$ and as $f_i, g_j \in I$ for every $i, j \in \{1, 2\}$, this implies that $Z(f_i) \bigcap Z(g_j) \neq \phi$ for every $i, j \in \{1, 2\}$ and so by Corollary 1(ii), $d([f_1, f_2], [g_1, g_2]) = 2$. □

Theorem 10. *Let $[f_1, f_2]$ be a vertex in $L(\Gamma(C(X)))$. Then,*

$$e([f_1, f_2]) = \begin{cases} 2, & \textit{if } Z(f_1) \bigcup Z(f_2) = X \\ 3, & \textit{otherwise.} \end{cases} \tag{2}$$

Proof. Let $[g_1, g_2]$ be a vertex of $L(\Gamma(C(X)))$. If $[g_1, g_2]$ is adjacent to $[f_1, f_2]$, then $d([f_1, f_2], [g_1, g_2]) = 1$. Suppose $[g_1, g_2]$ is not adjacent to $[f_1, f_2]$.
Case 1: When $Z(f_1) \bigcup Z(f_2) = X$. Since g_1, g_2 are adjacent in $\Gamma(C(X))$, so $Z(g_1) \bigcap Z(g_2) \neq \phi$. This suggested that there exists some element x of X such that $x \in Z(g_1) \bigcap Z(g_2) \subseteq X = Z(f_1) \bigcup Z(f_2)$. This shows that $x \in Z(f_i)$ for some $i \in \{1, 2\}$ and $Z(f_i)$ intersect both $Z(g_1)$ and $Z(g_2)$. By Corollary 1(ii), $d([f_1, f_2], [g_1, g_2]) = 2$. So, $e([f_1, f_2]) = 2$.
Case 2: When $Z(f_1) \bigcup Z(f_2) \neq X$. Let $p \in X$ such that $p \notin Z(f_1) \bigcup Z(f_2)$. As X is completely regular, there exists $g_1 \in C(X)$ such that $g_1(p) = 0$ and $g_1(Z(f_1) \bigcup Z(f_2)) = 1$. Consider $g_2 = rg_1$ for $0 \neq r \in \mathbb{R}$, then $Z(f_i) \bigcap Z(g_j) = \phi$ for all $i, j \in \{1, 2\}$. By Corollary 1(iii), $d([f_1, f_2], [g_1, g_2]) = 3$. So, $e([f_1, f_2]) = 3$. Hence, we have,

$$e([f_1, f_2]) = \begin{cases} 2, & \text{if } Z(f_1) \bigcup Z(f_2) = X \\ 3, & \text{otherwise.} \end{cases} \tag{3}$$

□

Corollary 2. *(i) $\rho(L(\Gamma(C(X)))) = 2$.*
(ii) $[f_1, f_2]$ is a central vertex of $L(\Gamma(C(X)))$ iff $Z(f_1) \bigcup Z(f_2) = X$.
(iii) The center of $L(\Gamma(C(X)))$ is given by $C(L(\Gamma(C(X)))) = \{[f_1, f_2] \in V(L(\Gamma(C(X)))) : Z(f_1) \bigcup Z(f_2) = X\}$.

Theorem 11. *The graph $L(\Gamma(C(X)))$ is never complemented.*

Proof. Let $[f_1, f_2]$ and $[g_1, g_2]$ be an arbitrary vertex of $L(\Gamma(C(X)))$. If $[f_1, f_2]$ and $[g_1, g_2]$ are not adjacent, then we are through. If $[f_1, f_2]$ and $[g_1, g_2]$ are adjacent, then $f_i = g_j$ for some $i, j \in \{1, 2\}$. In particular, $f_1 = g_1$. If $f_1 = g_1 \neq 0$, then we can always get another vertex $h = rg_1$ for some $0 \neq r \in \mathbb{R}$ and we see that $[h, g_1]$ is adjacent to both $[f_1, f_2]$ and $[g_1, g_2]$ (since $f_1 = g_1$). Similarly, if $f_1 = g_1 = 0$, then it is easy to get another vertex h distinct from 0, f_2 and g_2 such that $[h, g_1]$ is adjacent to both $[f_1, f_2]$ and $[g_1, g_2]$. This shows that $L(\Gamma(C(X)))$ is not complemented. □

Theorem 12. *The graph $L(\Gamma(C(X)))$ is both triangulated and hypertriangulated, hence its girth is* 3*.*

Proof. Let $[f_1, f_2]$ be any vertex in $L(\Gamma(C(X)))$. If $f_1 \neq 0$ and $f_2 \neq 0$, then $[f_1, h] - [f_1, f_2] - [h, f_2] - [f_1, h]$ is a triangle, where $h = f_1 f_2$. If $f_1 = 0$, then $[f_1, f_2] - [f_1, 2f_2] - [f_1, 3f_2] - [f_1, f_2]$ is a triangle. Similarly, if $f_2 = 0$. Next, let $[f, h] - [h, g]$ be an edge of $L(\Gamma(C(X)))$. If $h \neq 0$, then for $r \neq 0$, $[f, h] - [h, rh] - [h, g] - [f, h]$ is a triangle. If $h = 0$, then $[f, h] - [h, rf] - [h, g] - [f, h]$ is a triangle. So, $L(\Gamma(C(X)))$ is both triangulated and hypertriangulated. □

Remark 4. *It is also easy to see that just like in [1, Theorem 10], $L(\Gamma(C(X)))$ is never chordal.*

The next theorem is for finding the length of the shortest cycle in $L(\Gamma(C(X)))$ that contains distinct vertices $[f_1, f_2]$ and $[g_1, g_2]$. The proof is parallel to the proof of Theorem 9 of [1], hence it is omitted.

Theorem 13. *Let $[f_1, f_2]$ and $[g_1, g_2]$ be two distinct vertices of $L(\Gamma(C(X)))$. Then, the following conditions hold:*

(i) $c([f_1, f_2], [g_1, g_2]) = 3$ iff $f_i = g_j$ for some $i, j \in \{1, 2\}$.

(ii) $c([f_1, f_2], [g_1, g_2]) = 4$ iff $f_i \neq g_j$ for every $i, j \in \{1, 2\}$ and for some $i \in \{1, 2\}$, $Z(f_i) \bigcap Z(g_j) \neq \phi$ for every $j \in \{1, 2\}$ or $Z(f_1) \bigcap Z(g_i) \neq \phi$ and $Z(f_2) \bigcap Z(g_j) \neq \phi$ where $\{i, j\} = \{1, 2\}$.

(iii) $c([f_1, f_2], [g_1, g_2]) = 5$ iff $f_i \neq g_j$ for every $i, j \in \{1, 2\}$ and for only one $i \in \{1, 2\}$,$Z(f_i) \bigcap Z(g_j) \neq \phi$ for only one $j \in \{1, 2\}$.

(iv) $c([f_1, f_2], [g_1, g_2]) = 6$ iff $f_i \neq g_j$ for every $i, j \in \{1, 2\}$ and $Z(f_i) \bigcap Z(g_j) = \phi$ for every $i, j \in \{1, 2\}$.

For $f \in \mathcal{N}(X)$, $N[f] = \{g \in C(X) : Z(f) \bigcap Z(g) \neq \phi\} \bigcup \{f\}$ is a closed neighbourhood of f and $N(f) = N[f] \backslash \{f\}$ is an open neighbourhood of f. Let $L'(N[f])$ be a subgraph of $L(\Gamma(C(X)))$, where $V(L'(N[f])) = \{[f, h] \in V(L(\Gamma(C(X)))) : h \in N(f)\}$. Then, every vertex of $L'(N[f])$ is adjacent since all vertex of $L'(N[f])$ share a common vertex f in $\Gamma(C(X))$. Hence, $L'(N[f])$ is a clique in $L(\Gamma(C(X)))$.

Theorem 14. *$V(L'(N[0]))$ is a separating set iff $|X| = 2$.*

Proof. If $|X| \geq 3$. Let $[f_1, f_2], [g_1, g_2]$ be two distinct vertices of $L(\Gamma(C(X)))\backslash V(L'(N[0]))$. Let $a \in Z(f_i)$ and $b \in Z(g_j)$ for some $i, j \in \{1, 2\}$. As $|X| \geq 3$, there exists $c \in X$. By the complete regularity of X, there exists $k \in C(X)$ such that $k(c) = 1$ and $k(\{a, b\}) = 1$. So, $[f_1, f_2] - [f_i, k] - [k, g_j] - [g_1, g_2]$ is a path joining $[f_1, f_2]$ and $[g_1, g_2]$. This shows that $L(\Gamma(C(X)))\backslash V(L'(N[0]))$ is connected, which is a contradiction. Conversely, suppose $|X| = 2$. It is easy to see that $L(\Gamma(C(X)))\backslash V(L'(N[0]))$ is disconnected, as it is the line graph of $\Gamma(C(X))\backslash\{0\}$ that is not connected by Theorem 4. □

Theorem 15. *Let $f, g \in \mathcal{N}(X)$ such that $f \neq g$. Then,*
(i) $V(L'(N[f])) \bigcap V(L'(N[g])) = \phi$ iff $Z(f) \bigcap Z(g) = \phi$.
(ii) $V(L'(N[f])) \bigcap V(L'(N[g])) = \{[f, g]\}$ iff $Z(f) \bigcap Z(g) \neq \phi$.

Proof. The proof is straightforward. □

Theorem 16. *For each $f \in \mathcal{N}(X)$, $L'(N[f])$ is a maximal clique in $L(\Gamma(C(X)))$.*

Proof. Suppose $L'(N[f])$ is not a maximal clique, then there exists $[h_1, h_2] \notin V(L'(N[f]))$ such that $[h_1, h_2]$ is adjacent to every vertex of $L'(N[f])$.
Claim: $h_i = f$ for some $i \in \{1, 2\}$. Suppose $h_i \neq f$ for all $i \in \{1, 2\}$. Let $[f, g_1]$ be vertex of $L'(N[f])$ for some $g_1 \in N[f]$, but $[h_1, h_2]$ is adjacent to every vertex of $L'(N[f])$ so $[h_1, h_2]$ is adjacent to $[f, g_1]$. Since $h_i \neq f$ for all $i \in \{1, 2\}$, so we must have $h_i = g_1$ for some $i \in \{1, 2\}$. In particular, $h_1 = g_1$. Again, $[h_1, h_2]$ is adjacent to $[f, g_2]$ for some $g_2 \in N[f]$. We must have $h_2 = g_2$ since $h_i \neq f$ for all $i \in \{1, 2\}$ and $g_1 \neq g_2$. Again, $[h_1, h_2]$ is adjacent to $[f, g_3]$ for some $g_3 \in N[f]$ where g_1, g_2, g_3 are all distinct. But this will force $h_i = f$ for some $i \in \{1, 2\}$ since $h_1 = g_1$ and $h_2 = g_2$, which contradicts our assumption. Hence, $h_i = f$ for some $i \in \{1, 2\}$, which shows that $[h_1, h_2] \in V(L'(N[f]))$. This shows that $L'(N[f])$ is a maximal clique in $L(\Gamma(C(X)))$. □

Theorem 17. *$N[[f, g]] = V(L'(N[f])) \bigcup V(L'(N[g]))$, i.e., the close neighbourhood of $[f, g]$ is a union of two maximal cliques.*

Proof. Clearly, $V(L'(N[f])) \bigcup V(L'(N[g])) \subseteq N[[f, g]]$. If $[h_1, h_2] \in N[[f, g]]$, then $[h_1, h_2]$ is adjacent to $[f, g]$. This shows that $h_i = f$ or $h_i = g$ for some $i \in \{1, 2\}$ and $[h_1, h_2] \in V(L'(N[f])) \bigcup V(L'(N[g]))$. As $[h_1, h_2]$ is an arbitrary vertex, we have, $N[[f, g]] \subseteq V(L'(N[f])) \bigcup V(L'(N[g]))$. Hence, $N[[f, g]] = V(L'(N[f])) \bigcup V(L'(N[g]))$. □

Using Theorem 3.10.1 and 3.10.3 of [8] and Theorem 5.7, 5.8, 5.9, 5.10, 5.11 of [3], we have the following theorem:

Theorem 18. *Let X and Y be (First countable/ Real Compact) topological spaces. The following conditions are equivalent:*
(i) X is homeomorphic to Y.
(ii) $L(\Gamma(C(X)))$ is graph isomorphic to $L(\Gamma(C(Y)))$.
(iii) $\Gamma(C(X))$ is graph isomorphic to $\Gamma(C(Y))$.
(iv) $C(X)$ is isomorphic to $C(Y)$ as a ring.

4 Dominating set

In a graph G, a dominating set is a set D of vertices such that every vertex not in D is adjacent to at least one member of D. The dominating number of a graph G is defined as $dt(G)) = inf\{|D| : D \textit{ is a dominating set of } G\}$. Clearly, 0 is adjacent to every $f \in \mathcal{N}(X)$. So, $\{0\}$ is a dominating set and hence $dt((\Gamma(C(X))) = 1$.

Theorem 19. *Let f and g be two non-zero distinct vertices of $\Gamma(C(X))$. Then, $\{f, g\}$ is a dominating set of $\Gamma(C(X))$ iff $fg = 0$.*

Proof. Suppose $\{f, g\}$ is a dominating set of $\Gamma(C(X))$. If $fg \neq 0$, then there exists some $p \notin Z(fg)$. By the complete regularity of X, there exists $h \in C(X)$ such that $h(p) = 0$ and $h(Z(fg)) = 1$. This implies that h is not adjacent to both f and g, which contradicts that $\{f, g\}$ is a dominating set. Conversely, for any vertex $h \in \mathcal{N}(X)$, h is adjacent to $0 = fg$. This implies that $Z(h) \bigcap Z(f) \neq \phi$ or $Z(h) \bigcap Z(f) \neq \phi$. This shows that $\{f, g\}$ dominates $\Gamma(C(X))$. □

Remark 5. *In the zero-divisor graph of $C(X)$ [2], two vertices f, g are adjacent if and only if $fg = 0$. Thus, we conclude that two vertices f, g in the zero-divisor graph are adjacent if and only if the set $\{f, g\}$ dominates the zero-set intersection graph.*

Theorem 20. *Suppose $\{f, g\}$, $f, g \in \mathcal{N}(X) \backslash \{0\}$ dominates $\Gamma(C(X))$. Then, there exists $0 \neq h \in C(X)$ such that both $\{f, h\}$ and $\{h, g\}$ dominates $\Gamma(C(X))$ iff $int(Z(f) \bigcap Z(g)) \neq \phi$, where $int(Z(f) \bigcap Z(g))$ is the set of all interior points of $(Z(f) \bigcap Z(g))$.*

Proof. Suppose there exists $0 \neq h \in C(X)$ such that both $\{f, h\}$ and $\{h, g\}$ dominate $\Gamma(C(X))$. Then, $fh = 0$ and $gh = 0$. This implies that $\phi \neq X \backslash Z(h) \subseteq int(Z(f) \bigcap Z(g))$. Conversely, suppose $int(Z(f) \bigcap Z(g)) \neq \phi$, say $p \in int(Z(f) \bigcap Z(g))$. So, $p \notin (X \backslash int(Z(f) \bigcap Z(g)))$. Therefore, by the complete regularity of X, there exists $h \in C(X)$ such that $h(p) = 1$ and $h(X \backslash int(Z(f) \bigcap Z(g))) = 0$. This implies that $fh = 0$ and $gh = 0$. □

A zero set Z in X is called a middle zero set if there exist two proper zero sets Z_1 and Z_2 such that $Z = Z_1 \bigcap Z_2$ and $X = Z_1 \bigcup Z_2$ [2]. If every non-empty middle zero set in X has a non-empty interior, then X is called a middle P-space. Using Remark 5 and Theorem 20 in Proposition 2.1 of [2], we have the following remarks:

Remark 6. *(i) For every zero divisor $f \neq 0$, there exist h, $g \in \mathcal{N}(X) \backslash \{0\}$ such that $\{f, g\}$, $\{g, h\}$ and $\{h, f\}$ dominates the zero-set intersection graph iff X has no isolated points.*
(ii) For every dominating set $\{f, g\}$, ($f, g \in \mathcal{N}(X) \backslash \{0\}$) of zero-set intersection graph $int(Z(f) \bigcap Z(g)) \neq \phi$ iff X is a connected middle P-space.

Remark 7. *From [6, 7.2 (b)] we have for any $f \in \mathcal{N}(X)$, $f \in O^p$ iff there exist $g \notin M^p$ such that $\{f, g\}$ dominate $\Gamma(C(X))$, where $O^p = \{f \in C(X) : \ cl_{\beta X} Z(f) \textit{ is a neighbourhood of } p\}$*

For any $f \in C(X)$, the hull of f is given by $H(f) = \{P \in Min(C(X)) : \ f \in P\}$, where $Min(C(X))$ is the space of minimal prime ideal of $C(X)$ [7]. Lemma 5.4 (a) and Corollary 5.5 of [7] can be rewritten as:

Lemma 1. *For any $f, g \in \mathcal{N}(X)$, $H(Ann(g)) \subseteq H(f)$ or $H(Ann(f)) \subseteq H(g)$ iff $\{f, g\}$ dominate $\Gamma(C(X))$.*

Corollary 3. *The space $Min(C(X))$ is compact iff for any $f \in \mathcal{N}(X)$ there exist $g \in \mathcal{N}(X)$ such that $\{f, g\}$ dominate $\Gamma(C(X))$ and $int(Z(f)) \bigcap int(Z(g)) = \phi$.*

Theorem 21. *Let $f \in \mathcal{N}(X)$. The set $D_f = \{[f, g] \in V(L(\Gamma(C(X)))) : g \in \mathcal{N}(X)\}$ is a dominating set of $L(\Gamma(C(X)))$ iff $f = 0$.*

Proof. Suppose D_f is a dominating set of $L(\Gamma(C(X)))$. To show that $f = 0$. If possible, let $f \neq 0$. Then, there exists $p \in X$ and $p \notin Z(f)$. By the complete regularity of X, there exists $h \in C(X)$ such that $h(p) = 0$ and $h(Z(f)) = 1$, that is, h and f are not adjacent. Taking $k = 2h$, then $[h, k]$ is a vertex in $L(\Gamma(C(X)))$, but since D_f is a dominating set. This implies that $[h, k]$ is adjacent to $[f, g]$ for some $g \in \mathcal{N}(X)$. This shows that $g = h$ or $g = k$ as $f \neq h$ and $f \neq k$. But if $g = h$, we get a contradiction, as h is not adjacent to f. Similarly, if $g = k$, then we get a contradiction. Hence, we must have $f = 0$. Conversely, suppose $f = 0$, then $D_0 = \{[0, g] \in L(\Gamma(C(X))) : g \in \mathcal{N}(X)\}$. Here we see that for any vertex $[h, k]$ in $L(\Gamma(C(X)))$, $[h, k]$ is adjacent to $[0, k]$ and $[h, 0]$. But $[0, k]$ and $[h, 0]$ are elements of D_0. This shows that D_0 is a dominating set of $L(\Gamma(C(X)))$. □

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