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Numerical pricing of American options under a nonlinear Black-Scholes framework with mixed fractional Brownian motion

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Abstract. Transaction costs significantly impact option pricing and trading strategies in financial markets. This study investigates the valuation of American options under transaction costs, modeled as a linear function of the underlying asset price. To capture long-range dependence in asset returns, the underlying dynamics are described by a mixed fractional Brownian motion (fBm). The model incorporates dividend-paying stocks, along with time-varying interest and dividend rates. A compact finite difference scheme is developed to solve the resulting nonlinear Black-Scholes equation, ensuring numerical stability and accuracy. The proposed framework offers an efficient approach for pricing American options in realistic market conditions.

Keywords: American options, compact difference scheme, mixed fractional Brownian motion, transaction costs.
AMS Subject Classification 2010: 91G20, 60G22, 34A34.

1 Introduction

Transaction costs in financial markets, particularly in the context of option pricing, play a significant role in determining market behavior and pricing mechanisms. Transaction costs refer to the expenses incurred when buying or selling financial instruments, such as options. These costs can include brokerage fees, bid-ask spreads, commissions, taxes, and costs associated with market impact:

- The brokerage fees are charges that imposed by brokers for executing trades. These fees can be fixed or variable and can significantly affect profitability, especially for frequent traders.
- The bid-ask spread, representing the difference between the buying (ask) and selling (bid) prices of an option, is a key indicator of transaction costs in financial markets. Wider spreads imply higher

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costs for entering and exiting positions, potentially hindering market efficiency and discouraging trading activity.

- Capital gains taxes on profits from trading options can also be considered a transaction cost, impacting the net return on investment.

- The market impact costs are costs when a large order moves the market price. For example, buying a significant number of options may drive up their price, leading to higher costs for subsequent trades. These high transaction costs can affect market efficiency.

In addition to affecting market efficiency, liquidity, and trading frequency, transaction costs also affect pricing models, hedging strategies, and arbitrage opportunities. For instance, traditional option pricing models are based on the assumption of frictionless markets, where transaction costs are considered negligible. In practice, however, the presence of transaction costs can cause significant deviations between theoretical option prices and actual market prices.

Also, transaction costs can limit arbitrage opportunities. In theory, arbitrage involves taking advantage of price discrepancies in different markets. However, if transaction costs are too high, the potential profits from arbitrage may not cover these costs, discouraging traders from executing such strategies. Moreover, investors and traders often use options to hedge against potential losses in their portfolios. However, the effectiveness of these hedging strategies is influenced by transaction costs. If transaction costs are high, the cost of implementing and maintaining a hedge might outweigh the benefits. Hence, in the reality of the financial market, investors were faced with considerable and non-ignorable transaction costs.

Options are one of the derivatives that are closed on stocks and securities assets and are not immune to these costs. Thus, transaction costs are one of the important factors that affect the option price. Leland [17] was among the first to examine the pricing and hedging of investment portfolios in the presence of transaction costs. Due to the infinite variation of geometric Brownian motion, transaction costs in a continuous-time fully hedging strategy would become unbounded. To address this issue, Leland proposed a delta hedging approach that accounts for transaction costs and operates within discrete time intervals, deviating from the classical Black-Scholes model's reliance on the no-arbitrage assumption.

In addition to Leland, many other scholars have investigated the option pricing model of contingent claims under transaction costs. Barles and Soner [5] developed a more intricate model that incorporates investor preferences, aligning with the exponential utility function. Amster et al. [3] introduced a variation of the Black-Scholes option pricing model that accounts for transaction costs. Additionally, Guasoni [14], Wang et al. [32], and Liu and Chang [18] investigated standard option pricing frameworks incorporating transaction costs within the framework of fBm.

In this paper, we suppose the transaction cost acts as a nonincreasing linear function $h(\nu_t) = \alpha - \beta \nu_t$ ($\alpha, \beta > 0$), depending on the trading stocks or underlying assets needed to hedge the replicating portfolio. Following Leland's thought [17], for portfolio $\Pi_t = V_t - \Delta S_t$, we get

$$\mathrm{d}\Pi_t = \mathrm{d}V_t - \Delta_t \mathrm{d}S_t - \Delta_t D_t S_t \mathrm{d}t - (\alpha - \beta|\nu_t|)|\nu_t|S_{t+\delta t},$$

where $V_t := V(S,t)$ is the option price, S_t is the underlying asset price, Δ_t is the units of underlying asset, D_t is the dividend yield, and ν_t is the number of shares of the underlying asset at the price S_t which are traded in order to maintain the equilibrium of the portfolio. The hedging portfolio Π_t is rebalanced every

δt where δt is a finite and fixed small time step. We assume that the parameter S_t in portfolio satisfies

$$\mathrm{d}S_t = (r_t - D_t)S_t\mathrm{d}t + \sigma S_t \mathrm{d}M_t^H = (r_t - D_t)S_t\mathrm{d}t + \lambda\sigma S_t\mathrm{d}B_t + \sigma S_t\mathrm{d}B_t^H, \tag{1}$$

where σ, r_t, B_t, and B_t^H are the volatility, interest rate, standard geometric Brownian motion, and fBm, respectively. The $M_t^H = \lambda B_t + B_t^H$ is a mixed fBm with parameters λ (a nonzero real constant) and H (Hurst exponent; $H \in (3/4, 1)$).

Empirical evidence suggests that fBm with $H \in (1/2, 1)$ effectively models the logarithmic returns of financial assets, capturing long-range dependence in financial time series [11, 13, 19, 36]. The key characteristics that make fBm particularly attractive for financial modeling are its self-similarity and long-range dependence. These properties have driven considerable interest in using fBm to represent the stochastic components in financial models. The fBm process was originally introduced by Mandelbrot and van Ness [20]. However, for values $H \neq 1/2$, fBm is neither a Markov process nor a semi-martingale. Consequently, it is not possible to construct an equivalent probability measure under which the process becomes a local martingale [29]. This structural property implies the existence of arbitrage opportunities in models based purely on fBm [7, 28]. To address this limitation, the mixed fBm model was proposed as an enhancement of the classical Black-Scholes framework. This model combines standard Brownian motion with fBm to incorporate long-range dependence while mitigating arbitrage opportunities [15, 22, 23, 35]. Specifically, when $H \in (3/4, 1)$, the mixed fBm model ensures an arbitrage-free market. Cheridito [6] demonstrated that, in this range of Hurst parameter values, the mixed fBm process is equivalent in law to standard Brownian motion, indicating that time-scaling effects and long-range dependence do not influence option pricing in complete markets without transaction costs.

In this study, we consider a stochastic model for the underlying asset price governed by equation (1), accounting for transaction costs. The corresponding deterministic differential equation derived from the stochastic differential equation (1), which governs option pricing under the mixed fBm framework with transaction costs, is as follows [17, 33]:

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\tilde{\sigma}^2(S,t)S^2\frac{\partial^2 V(S,t)}{\partial S^2} + (r(t) - D(t))S\frac{\partial V(S,t)}{\partial S} - r(t)V(S,t) = 0, \tag{2}$$

where

$$\tilde{\sigma}^2(S,t) = \sigma^2\Bigg[\lambda^2 + 2Ht^{2H-1} - \frac{2\alpha}{\sigma}\operatorname{sign}\left(\frac{\partial^2 V(S,t)}{\partial S^2}\right)\sqrt{\frac{2}{\pi}\left(\frac{\lambda^2}{\delta t} + (\delta t)^{2H-2}\right)} + 2\beta S\left(\lambda^2 + (\delta t)^{2H-1}\right)\frac{\partial^2 V(S,t)}{\partial S^2}\Bigg],$$

with the conditions

$$\text{Call option}: \begin{cases} V(S,T) = (S-E)^+, & 0 < S < S_f(t), \\ \frac{\partial V(S_f(t),t)}{\partial S} = 1, & 0 < t < T, \\ V(0,t) = 0, \; V(S_f,t) = S_f(t) - E, & 0 < t < T, \end{cases}$$

and

$$\text{Put option}: \begin{cases} V(S,T) = (E-S)^+, \ \ 0 < S_f(t) < S, \\ \frac{\partial V(S_f(t),t)}{\partial S} = -1, \ \ 0 < t < T, \\ V(S_f,t) = E - S_f(t), \ \ \lim_{S\to+\infty} V(S,t) = 0, \ \ 0 < t < T. \end{cases}$$

Here, E is the strike price, T is the maturity time, and $S_f(t)$ is the exercise boundary.

Using the time to maturity $\tau = T - t$, the final value problem (2) becomes the following initial value problem:

$$\frac{\partial W(S,\tau)}{\partial \tau} = \frac{1}{2}\hat{\sigma}^2(S,\tau)S^2\frac{\partial^2 W(S,\tau)}{\partial S^2} + (r(T-\tau) - D(T-\tau))S\frac{\partial W(S,\tau)}{\partial S} - r(T-\tau)W(S,\tau), \quad (3)$$

where $W(S,\tau) = V(S,t)$ and

$$\hat{\sigma}^2(S,\tau) = \tilde{\sigma}^2(S,t) = \sigma^2\left[\lambda^2 + 2H(T-\tau)^{2H-1} - \frac{2\alpha}{\sigma}\text{sign}\left(\frac{\partial^2 W(S,\tau)}{\partial S^2}\right)\sqrt{\frac{2}{\pi}\left(\frac{\lambda^2}{\delta t} + (\delta t)^{2H-2}\right)} + 2\beta S\left(\lambda^2 + (\delta t)^{2H-1}\right)\frac{\partial^2 W(S,\tau)}{\partial S^2}\right],$$

with the following initial and boundary conditions

$$\text{Call option}: \begin{cases} W(S,0) = (S-E)^+, \ \ 0 < S < S_f(\tau), \\ \frac{\partial W(S_f(\tau),\tau)}{\partial S} = 1, \ \ 0 < \tau < T, \\ W(0,\tau) = 0, \ \ V(S_f,\tau) = S_f(\tau) - E, \ \ 0 < \tau < T, \end{cases}$$

and

$$\text{Put option}: \begin{cases} W(S,0) = (E-S)^+, \ \ 0 < S_f(\tau) < S, \\ \frac{\partial W(S_f(\tau),\tau)}{\partial S} = -1, \ \ 0 < \tau < T, \\ W(S_f,\tau) = E - S_f(\tau), \ \ \lim_{S\to+\infty} W(S,\tau) = 0, \ \ 0 < \tau < T. \end{cases}$$

We are going to use (3) for pricing American option. This option can be based on various underlying assets, including stocks, indices, commodities, or other financial instruments. The most significant feature of American options is the flexibility they provide to the holder. An American option can be exercised at any time during its life, which allows the holder to take advantage of favorable market conditions at any moment. For example, if a stock pays dividends, a holder of a call option might choose to exercise the option early to capture the dividend, or an investor holding shares of a stock may purchase put options as a hedge against a potential decline in the stock's price. In the event of a price drop, the value of the put option rises, thereby offsetting a portion of the incurred losses. Thus, this flexibility of American options allows for dynamic hedging strategies where investors can adjust their positions as market conditions change. This adaptability is particularly useful in volatile markets. Hence, these options are widely used by investors and traders in various markets for hedging, speculation, and risk management.

There are different methods for American option pricing, such as the binomial models, moving boundary approach, and finite difference methods. Leisen [16] has priced American put option using binomial models. Chockalingam and Muthuraman [8] have applied the moving boundary method for pricing American option. Company et al. [9] have presented the finite difference schemes to price American option under the nonlinear Black-Scholes equation with rationality parameter. For more information on exchange-traded option valuation see [1, 2, 4, 21, 24, 25].

In this work, we employ a compact difference scheme to solve (3) for American option pricing. The term "compact difference scheme" typically refers to a numerical method used in the numerical solution of differential equations, particularly partial differential equations. These schemes are characterized by their use of compact stencils, which means they utilize a smaller number of grid points in the discretization process compared to traditional finite difference methods. This is particularly beneficial for resolving sharp gradients or discontinuities in the solution.

Many compact difference schemes are implicit, allowing for larger time step sizes in time-dependent problems. This feature can be particularly beneficial in applications where explicit methods necessitate prohibitively small time steps to ensure stability. Zhao et al. [37] developed three compact finite difference schemes for pricing American options and compared the resulting American put option prices with various methodologies, including the binomial and trinomial models, Crank-Nicolson projected successive over-relaxation, least-squares Monte Carlo, integral methods, and analytical approximations. Tangman et al. [31] introduced a high-order compact difference scheme designed for efficient pricing of American options, utilizing the transformed Black-Scholes equation within a singularity-separating framework.

The remainder of this paper is structured as follows: Section 2 introduces the compact finite difference scheme developed for pricing American options under the nonlinear Black-Scholes framework. Section 3 is devoted to the stability analysis of the proposed numerical method. Finally, Section 4 presents numerical examples for American call options, where the underlying asset dynamics are modeled using the mixed fBm process.

2 Compact difference scheme for American option pricing

For numerical pricing of American option, we introduce a uniform grid of mesh points (S_j, τ_k), with $S_j = j\Delta S$; $j = 0, 1, \cdots, N$, and $\tau_k = k\Delta\tau$; $k = 0, 1, \cdots, M$, where $N, M \in \mathbb{Z}^+$, $\Delta S = S_{\max}/N$; $S_{\max} = 3E$ is the mesh-width in price step and $\Delta\tau = \tau/M$ is the time step. At first, we approximate the time and space derivatives in (3) as follows:

$$\frac{\partial W(S_j,\tau_k)}{\partial \tau} = \frac{W(S_j,\tau_{k+1}) - W(S_j,\tau_k)}{\Delta\tau} + O(\Delta\tau), \tag{4a}$$

$$\frac{\partial W(S_j,\tau_k)}{\partial S} = \underbrace{\frac{W(S_{j+1},\tau_k) - W(S_{j-1},\tau_k)}{2\Delta S}}_{:=\delta_S W(S_j,\tau_k)} - \frac{\Delta S^2}{6}\frac{\partial^3 W(S_j,\tau_k)}{\partial S^3} + O\left(\Delta S^4\right), \tag{4b}$$

$$\frac{\partial^2 W(S_j,\tau_k)}{\partial S^2} = \underbrace{\frac{W(S_{j-1},\tau_k) - 2W(S_j,\tau_k) + W(S_{j+1},\tau_k)}{\Delta S^2}}_{:=\delta_S^2 W(S_j,\tau_k)} - \frac{\Delta S^2}{12}\frac{\partial^4 W(S_j,\tau_k)}{\partial S^4} + O\left(\Delta S^4\right). \tag{4c}$$

We consider a smoother approximation of second-order derivative as follows [12]:

$$\frac{\partial^2 W(S_j,\tau_k)}{\partial S^2} = \frac{W(S_{j-2},\tau_k) - 2W(S_j,\tau_k) + W(S_{j+2},\tau_k)}{(2\Delta S)^2} + O\left(\Delta S^2\right) := D^2_{2\Delta S} W(S_j,\tau_k).$$

We give the following difference scheme to solve (3) at grid point (S_j,τ_k):

$$\frac{\partial W(S_j,\tau_k)}{\partial \tau} = \frac{1}{2}\hat{\sigma}^2(S_j,\tau_k) S_j^2 \frac{\partial^2 W(S_j,\tau_k)}{\partial S^2} + (r(T-\tau_k) - D(T-\tau_k))S_j \frac{\partial W(S_j,\tau_k)}{\partial S} - r(T-\tau_k)W(S_j,\tau_k), \tag{5}$$

where

$$\hat{\sigma}^2(S_j,\tau_k) = \sigma^2 \Bigg[\lambda^2 + 2H(T-\tau_k)^{2H-1} - \frac{2\alpha}{\sigma}\operatorname{sign}\left(D^2_{2\Delta S} W_j^k\right)\sqrt{\frac{2}{\pi}\left(\frac{\lambda^2}{\delta t} + (\delta t)^{2H-2}\right)} + 2\beta S_j \left(\lambda^2 + (\delta t)^{2H-1}\right) D^2_{2\Delta S} W_j^k \Bigg].$$

Substituting (4a), (4b), and (4c) in (5), we obtain

$$\begin{aligned} \frac{W_j^k - W_j^{k-1}}{\Delta \tau} &= \frac{1}{2}\hat{\sigma}^2(S_j,\tau_k) S_j^2 \left[\delta_S^2 W(S_j,\tau_k) - \frac{\Delta S^2}{12}\frac{\partial^4 W(S_j,\tau_k)}{\partial S^4}\right] \\ &\quad + (r_k - D_k)S_j \left[\delta_S W(S_j,\tau_k) - \frac{\Delta S^2}{6}\frac{\partial^3 W(S_j,\tau_k)}{\partial S^3}\right] - r_k W_j^k + R_j^k, \end{aligned} \tag{6}$$

where the evaluation R_j^k holds in $|R_j^k| \le C(\Delta\tau + \Delta S^4)$. Also, from (3), (4b) and (4c), we have

$$\begin{aligned} \frac{\partial^3 W(S_j,\tau_k)}{\partial S^3} &= -\frac{4S_j^{-3}}{\hat{\sigma}^2(S_j,\tau_k)}\left[r_k W_j^k + \frac{\partial W(S_j,\tau_k)}{\partial \tau}\right] \\ &\quad + \left[\frac{4S_j^{-2}}{\hat{\sigma}^2(S_j,\tau_k)}(r_k - D_k) + \frac{2S_j^{-2}}{\hat{\sigma}^2(S_j,\tau_k)}D_k\right]\delta_S W(S_j,\tau_k) \\ &\quad + \frac{2S_j^{-2}}{\hat{\sigma}^2(S_j,t_k)}\left[\frac{\partial^2 W(S_j,\tau_k)}{\partial S \partial \tau} - (r_k - D_k)S_j \delta_S^2 W(S_j,\tau_k)\right] + O(\Delta S^2), \end{aligned} \tag{7}$$

and

$$\begin{aligned} \frac{\partial^4 W(S_j,\tau_k)}{\partial S^4} &= \frac{4S_j^{-4}}{\hat{\sigma}^2(S_j,\tau_k)}\left(3 + \frac{2(r_k - D_k)}{\hat{\sigma}^2(S_j,\tau_k)}\right)\left[r_k W_j^k + \frac{\partial W(S_j,\tau_k)}{\partial \tau}\right] \\ &\quad - \frac{4S_j^{-3}}{\hat{\sigma}^2(S_j,t_k)}\left[\left(3 + \frac{2(r_k - D_k)}{\hat{\sigma}^2(S_j,\tau_k)}\right)(r_k - D_k) + \left(2 + \frac{r_k - D_k}{\hat{\sigma}^2(S_j,t_k)}\right)D_k\right]\delta_S W(S_j,\tau_k) \\ &\quad + \frac{2S_j^{-2}}{\hat{\sigma}^2(S_j,\tau_k)}\left[2\left(2 + \frac{r_k - D_k}{\hat{\sigma}^2(S_j,t_k)}\right)(r_k - D_k) + 2D_k - r_k\right]\delta_S^2 W(S_j,\tau_k) \\ &\quad - \frac{4S_j^{-3}}{\hat{\sigma}^2(S_j,t_k)}\left(2 + \frac{r_k - D_k}{\hat{\sigma}^2(S_j,t_k)}\right)\frac{\partial^2 W(S_j,\tau_k)}{\partial S \partial \tau} + \frac{2S_j^{-2}}{\hat{\sigma}^2(S_j,\tau_k)}\frac{\partial^3 W(S_j,\tau_k)}{\partial S^2 \partial \tau} + O(\Delta S^2). \end{aligned} \tag{8}$$

Now, we substitute (7) and (8) in (6), that gives

$$
\begin{aligned}
&\left(\frac{1-a'_{j,K}-f'_{j,K}}{\Delta\tau}\right)W_j^{k+1}-\left(\frac{d'_{j,K}+g'_{j,K}}{\Delta\tau}\right)\left[W_{j+1}^{k+1}-W_{j-1}^{k+1}\right]+\frac{1}{12\Delta\tau}\left[W_{j-1}^{k+1}-2W_j^{k+1}+W_{j+1}^{k+1}\right]\\
&=\left(\frac{1}{\Delta\tau}+a'_{j,K}r_k-\frac{a'_{j,K}}{\Delta\tau}-\frac{f'_{j,K}}{\Delta\tau}+f'_{j,K}r_k-r_k\right)W_j^k\\
&\quad+\left(\frac{1}{2\Delta S^2}\hat{\sigma}^2(S_j,t_k)S_j^2+c'_{j,k}+\frac{1}{12\Delta\tau}+h'_{j,k}\right)\left[W_{j-1}^k-2W_j^k+W_{j+1}^k\right]\\
&\quad+\left(b'_{j,K}-\frac{d'_{j,K}}{\Delta\tau}+e'_{j,K}+(2r_k-D_k)g'_{j,K}-\frac{g'_{j,K}}{\Delta\tau}\right)\left[W_{j+1}^k-W_{j-1}^k\right]+R_j^k, \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
&a'_{j,K}:=-\frac{\Delta S^2}{6}S_j^{-2}\left(3+\frac{2(r_k-D_k)}{\hat{\sigma}^2(S_j,\tau_k)}\right),\\
&b'_{j,k}:=-\frac{\Delta S}{12}S_j^{-1}\left[\left(3+\frac{2(r_k-D_k)}{\hat{\sigma}^2(S_j,\tau_k)}\right)(r_k-D_k)+\left(2+\frac{r_k-D_k}{\hat{\sigma}^2(S_j,t_k)}\right)D_k\right],\\
&c'_{j,k}:=-\frac{1}{12}\left[2\left(2+\frac{r_k-D_k}{\hat{\sigma}^2(S_j,t_k)}\right)(r_k-D_k)+2D_k-r_k\right],\\
&d'_{j,k}:=\frac{\Delta S}{12}S_j^{-1}\left(2+\frac{r_k-D_k}{\hat{\sigma}^2(S_j,t_k)}\right),\quad e'_{j,k}:=\frac{(r_k-D_k)S_j}{2\Delta S},\quad f'_{j,k}:=\frac{2\Delta S^2}{3\hat{\sigma}^2(S_j,t_k)}(r_k-D_k)S_j^{-2},\\
&g'_{j,k}:=-(r_k-D_k)\frac{S_j^{-1}\Delta S}{6\hat{\sigma}^2(S_j,t_k)},\quad h'_{j,k}:=\frac{(r_k-D_k)^2}{3\hat{\sigma}^2(S_j,t_k)}.
\end{aligned}
$$

By rearranging equation (9), we have

$$
\begin{aligned}
&\left[\frac{1}{12\Delta\tau}-a_j^k\right]W_{j-1}^{k+1}+\left[b_j^k-\frac{1}{6\Delta\tau}\right]W_j^{k+1}+\left[\frac{1}{12\Delta\tau}+a_j^k\right]W_{j+1}^{k+1}\\
&\qquad=\left[d_j^k-e_j^k\right]W_{j-1}^k+\left[c_j^k-2d_j^k\right]W_j^k+\left[d_j^k+e_j^k\right]W_{j+1}^k+R_j^k,
\end{aligned}
$$

where

$$
\begin{aligned}
&a_j^k:=-\frac{d'_{j,K}+g'_{j,K}}{\Delta\tau},\quad b_j^k:=\frac{1-a'_{j,K}-f'_{j,K}}{\Delta\tau},\quad c_j^k:=\frac{1}{\Delta\tau}+a'_{j,K}r_k-\frac{a'_{j,K}}{\Delta\tau}-\frac{f'_{j,K}}{\Delta\tau}+f'_{j,K}r_k-r_k,\\
&d_j^k:=\frac{1}{2\Delta S^2}\hat{\sigma}^2(S_j,t_k)S_j^2+c'_{j,k}+\frac{1}{12\Delta\tau}+h'_{j,k},\\
&e_j^k:=b'_{j,K}-\frac{d'_{j,K}}{\Delta\tau}+e'_{j,K}+(2r_k-D_k)g'_{j,K}-\frac{g'_{j,K}}{\Delta\tau}.
\end{aligned}
$$

Finally, by eliminating R_j^k, we get the following compact difference scheme

$$
\begin{aligned}
&\left[\frac{1}{12\Delta\tau}-a_j^k\right]\widetilde{W}_{j-1}^{k+1}+\left[b_j^k-\frac{1}{6\Delta\tau}\right]\widetilde{W}_j^{k+1}+\left[\frac{1}{12\Delta\tau}+a_j^k\right]\widetilde{W}_{j+1}^{k+1}\\
&=\left[d_j^k-e_j^k\right]\widetilde{W}_{j-1}^k+\left[c_j^k-2d_j^k\right]\widetilde{W}_j^k+\left[d_j^k+e_j^k\right]\widetilde{W}_{j+1}^k, \qquad (10)
\end{aligned}
$$

where $\widetilde{W}$ is the exact solution of the equation (10) and W is the exact solution of the equation (3).

In the context of American option pricing, the option value remains strictly positive throughout its lifetime. Consequently, the holder has the right to exercise the option at any point between the initiation of the contract and its maturity. Therefore, for American call option pricing, an index $I(t_k)$ (for $k = 0,1,\cdots,M$) is found such that

$$\frac{\partial W(S_j,\tau_k)}{\partial \tau} = \frac{1}{2}\hat{\sigma}^2(S_j,\tau_k)S_j^2\frac{\partial^2 W(S_j,\tau_k)}{\partial S^2} + (r_k - D_k)S_j\frac{\partial V(S_j,\tau_k)}{\partial S} - r_k W(S_j,\tau_k),$$

where $W(S_j,\tau_k) > (S_j - E)^+$ for $j = 0,1,\cdots,I(\tau_k)$, and

$$-\frac{\partial W(S_j,\tau_k)}{\partial \tau} + \frac{1}{2}\hat{\sigma}^2(S_j,\tau_k)S_j^2\frac{\partial^2 W(S_j,\tau_k)}{\partial S^2} + (r_k - D_k)S_j\frac{\partial W(S_j,\tau_k)}{\partial S} - r_k W(S_j,\tau_k) < 0,$$

where $W(S_j,\tau_k) = (S_j - E)^+$ for $j = I(\tau_k)+1, I(\tau_k)+2,\cdots,N$. Thus, the option price is obtained via (10) for $j \le I(\tau_k)$ and is equal to the payoff function for $j > I(\tau_k)$. Also, American put option price is $W(S_j,\tau_k) = (E - S_j)^+$ for $j = 0,1,\cdots,I(\tau_k)-1$ such that

$$-\frac{\partial W(S_j,\tau_k)}{\partial \tau} + \frac{1}{2}\hat{\sigma}^2(S_j,\tau_k)S_j^2\frac{\partial^2 W(S_j,\tau_k)}{\partial S^2} + (r_k - D_k)S_j\frac{\partial W(S_j,\tau_k)}{\partial S} - r_k W(S_j,\tau_k) < 0,$$

and, an index $I(\tau_k)$ (for $k = 0,1,\cdots,M$) is found such that

$$\frac{\partial W(S_j,\tau_k)}{\partial \tau} = \frac{1}{2}\hat{\sigma}^2(S_j,\tau_k)S_j^2\frac{\partial^2 W(S_j,\tau_k)}{\partial S^2} + (r_k - D_k)S_j\frac{\partial W(S_j,\tau_k)}{\partial S} - r_k W(S_j,\tau_k),$$

where $W(S_j,\tau_k) > (E - S_j)^+$ for $j = I(\tau_k), I(\tau_k)+1,\cdots,N$. Therefore, American put option price is equal $E - S_j$ for $j < I(\tau_k)$ and is obtained by (10) for $j \ge I(\tau_k)$.

The existence and nature of the exercise boundary for an American call option depend entirely on dividend yield. If dividend yield is not paid on stock, then there is no exercise boundary. Since, the price of an American call option on a non-dividend-paying stock is equal to the price of its European option, holder of the American call option loses the time value of the option by exercising early and forfeites the profit he (she) could has earned on that money by pay the strike price early. It is always more profitable to sell American call option in the market than to exercise it. But in the case where dividend yield are paid on stock, there is an exercise boundary for American call option. It is defined as the lowest asset price at which immediate exercise is optimal. It can be optimal to exercise just before a dividend is paid. By exercising, you become the stockholder and are entitled to the dividend. This dividend income can be greater than the time value you sacrifice by exercising early.

The logic of exercise boundary in American call option is the reverse of an American put option. Exercise region is high stock prices in American call option and is low stock prices in American put option. Otherwise holding of option is optimal. To find the exercise boundary for a American call option, we look from the high stock price to low stock price and the lowest price where exercise is optimal will be exercise boundary. Similarly, for a American put option, we look from the low stock price to high stock price and the highest price where exercise is optimal will be exercise boundary.

The stencil associated with scheme (10) is illustrated in Figure 1.

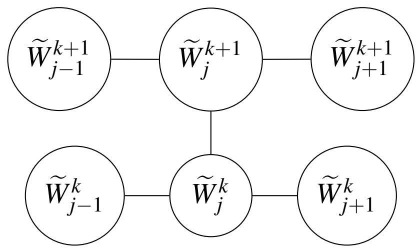

Figure 1: The graph of the scheme (10)

3 Stability analysis

In this section, by employing Fourier (Von Neumann) analysis as outlined in [30], the unconditional stability of the compact difference scheme given by equation (10) will be examined.

Let $\widehat{W}_j^k$ denote the numerical solution of the scheme (10), and define the error at each grid point and time level as:

$$\varepsilon_j^k = \widetilde{W}_j^k - \widehat{W}_j^k, \quad j = 0, 1, \ldots, N; \quad k = 0, 1, \ldots, M.$$

The error ε_j^k satisfies the following recurrence relation:

$$\begin{aligned} &\left[\frac{1}{12\Delta\tau} - a_j^k\right] \varepsilon_{j-1}^{k+1} + \left[b_j^k - \frac{1}{6\Delta\tau}\right] \varepsilon_j^{k+1} + \left[\frac{1}{12\Delta\tau} + a_j^k\right] \varepsilon_{j+1}^{k+1} \\ &= \left[d_j^k - e_j^k\right] \varepsilon_{j-1}^k + \left[c_j^k - 2d_j^k\right] \varepsilon_j^k + \left[d_j^k + e_j^k\right] \varepsilon_{j+1}^k, \end{aligned} \tag{11}$$

and $\varepsilon_0^k = \varepsilon_{I(\tau_k)+1}^k = \varepsilon_{I(\tau_k)+2}^k = \cdots = \varepsilon_N^k = 0$ (for the call option case; and $\varepsilon_0^k = \varepsilon_1^k = \cdots = \varepsilon_{I(\tau_k)-1}^k = \varepsilon_N^k = 0$ for the put option).

We construct a piecewise function on the spatial domain

$$\varepsilon^k(S) = \begin{cases} \varepsilon_j^k, & S \in \left(S_j - \frac{\Delta S}{2}, S_j + \frac{\Delta S}{2}\right], \\ 0, & S \in \left[0, \frac{\Delta S}{2}\right] \bigcup \left(S_{J(\tau_k)+1} - \frac{\Delta S}{2}, S_{J(\tau_k)+1} + \frac{\Delta S}{2}\right] \bigcup \cdots \bigcup \left(S_{\max} - \frac{\Delta S}{2}, S_{\max}\right], \end{cases}$$

and extend it periodically using a Fourier series with period $L = S_f(\tau)$ as follows:

$$\varepsilon^k(S) = \sum_{j=-\infty}^{+\infty} \eta_j^k e^{i\frac{2\pi jS}{L}} \quad (i^2 = -1), \quad k = 0, 1, \ldots, M,$$

$$\eta_j^k = \frac{1}{L} \int_0^L \varepsilon^k(S) e^{i\frac{2\pi jS}{L}} dS, \quad j \in \mathbb{Z}.$$

Suppose $\varepsilon^k = \left(\varepsilon_1^k, \varepsilon_2^k, \ldots, \varepsilon_{I(\tau_k)}^k\right)^t$, and define the following discrete L^2-norm of the error as

$$\left\|\varepsilon^k\right\|_2^2 = \sum_{j=1}^{I(\tau_k)} \Delta S \left|\varepsilon_j^k\right|^2 = \int_0^L \left|\varepsilon^k(S)\right|^2 dS = \left\|\varepsilon^k(S)\right\|_{L^2}^2, \quad k = 0, 1, \ldots, M.$$

Using the Parseval equality

$$\|\varepsilon^k(S)\|_{L^2}^2 = L\sum_{j=-\infty}^{+\infty}|\eta_j^k|^2, \quad k=0,1,\ldots,M,$$

which leads to

$$\|\varepsilon^k\|_2^2 = \sum_{j=1}^{I(\tau_k)}\Delta S\,|\varepsilon_j^k|^2 = L\sum_{j=-\infty}^{+\infty}|\eta_j^k|^2, \quad k=0,1,\ldots,M. \tag{12}$$

Based on the above theory and $S_j = j\Delta S$, we assume that equation (10) has a solution of the following form:

$$\varepsilon_j^k = \eta^k e^{iqj\Delta S}, \quad q=\frac{2\pi l}{L}, \quad l\in\mathbb{Z}. \tag{13}$$

Considering (11) subject to the solution form (13), results

$$\begin{aligned}&\left[\frac{1}{12\Delta\tau}-a_j^k\right]\eta^{k+1}e^{-iq\Delta S}+\left[b_j^k-\frac{1}{6\Delta\tau}\right]\eta^{k+1}+\left[\frac{1}{12\Delta\tau}+a_j^k\right]\eta^{k+1}e^{iq\Delta S}\\&=\left[d_j^k-e_j^k\right]\eta^k e^{-iq\Delta S}+\left[c_j^k-2d_j^k\right]\eta^k+\left[d_j^k+e_j^k\right]\eta^k e^{iq\Delta S}.\end{aligned}$$

Using $\sin^2(\frac{q\Delta S}{2}) = -\frac{1}{4}\left(e^{iq\Delta S}-2+e^{-iq\Delta S}\right)$, the above expression simplifies to

$$\left[-\frac{1}{3\Delta t}\sin^2(\frac{q\Delta S}{2})+b_j^k+2ia_j^k\sin(q\Delta S)\right]\eta^{k+1} = \left[-\frac{1}{4}d_j^k\sin^2(\frac{q\Delta S}{2})+c_j^k+2ie_j^k\sin(q\Delta S)\right]\eta^k.$$

This yields

$$\eta^{k+1} = \left[-\frac{1}{4}d_j^k\sin^2(\frac{q\Delta S}{2})+c_j^k+2ie_j^k\sin(q\Delta S)\right]\Big/\left[-\frac{1}{3\Delta t}\sin^2(\frac{q\Delta S}{2})+b_j^k+2ia_j^k\sin(q\Delta S)\right]\times\eta^k. \tag{14}$$

Lemma 1. *The following inequality holds:*

$$\left|\left[-\frac{1}{4}d_j^k\sin^2(\frac{q\Delta S}{2})+c_j^k+2ie_j^k\sin(q\Delta S)\right]\Big/\left[-\frac{1}{3\Delta t}\sin^2(\frac{q\Delta S}{2})+b_j^k+2ia_j^k\sin(q\Delta S)\right]\right|\le 1.$$

Proof. See [10]. □

Lemma 2. *Assume that η^k $(k=1,2,\cdots,M)$ is a solutions of relation (14). Then, $|\eta^k|\le|\eta^0|$.*

Proof. The proof is based on mathematical induction. For $k=1$, relation (14) becomes

$$\eta^1 = \left[-\frac{1}{4}d_j^0\sin^2(\frac{q\Delta S}{2})+c_j^0+2ie_j^0\sin(q\Delta S)\right]\Big/\left[-\frac{1}{3\Delta t}\sin^2(\frac{q\Delta S}{2})+b_j^0+2ia_j^0\sin(q\Delta S)\right]\times\eta^0.$$

Based on Lemma 1, $|\eta^1|\le|\eta^0|$. Assume that $|\eta^n|\le|\eta^0|$, $n=2,3,\ldots,k-1$, and prove $|\eta^k|\le|\eta^0|$. Applying Lemma 1 and the relation (14) for $k\ge 2$, we get

$$\begin{aligned}|\eta^k| =& \left|\left[-\frac{1}{4}d_j^{k-1}\sin^2(\frac{q\Delta S}{2})+c_j^{k-1}+2ie_j^{k-1}\sin(q\Delta S)\right]\Big/\left[-\frac{1}{3\Delta t}\sin^2(\frac{q\Delta S}{2})+b_j^{k-1}+2ia_j^{k-1}\sin(q\Delta S)\right]\right|\\&\times|\eta^{k-1}|\le|\eta^{k-1}|\le|\eta^0|,\end{aligned}$$

thus $|\eta^k|\le|\eta^0|$. □

Theorem 1. *The scheme* (10) *exhibits unconditional stability.*

Proof. Based on Lemma 2 and relation (12),

$$\|\varepsilon^k\|_2^2 = L\sum_{j=-\infty}^{+\infty}|\eta_j^k|^2 \le L\sum_{j=-\infty}^{+\infty}|\eta_j^0|^2 = \|\varepsilon^0\|_2^2.$$

Taking the square root of both sides yields the inequality

$$\|\varepsilon^k\|_2 \le \|\varepsilon^0\|_2,$$

for $k = 1,2,\dots,M$. This confirms that the proposed scheme (10) is unconditionally stable. □

4 Numerical experiments

In this section, we present numerical experiments to evaluate the effectiveness of the proposed compact finite difference scheme for pricing American call options under transaction costs and fractional market dynamics characterized by the Hurst parameter. All simulations are performed using MATLAB R2015b. We adopt the following time-dependent interest rate and dividend yield functions in equation (1) as [34]:

$$r_t = 0.1 + 0.05e^{-t}, \qquad D_t = 0.03 + 0.001e^{0.01t}.$$

Parameters in model (3) take the values in Table 1. Additionally, we use the values in this table to examine the effect of each of the model parameters on the option price.

Figure 2 displays the computed prices of American call options under the mixed fBm model with transaction costs using the proposed compact scheme. The values of this option at time $t \in \{0, 0.25, 0.5, 0.75, 1\}$ and for underlying asset prices $S \in \{0, 37.5, 75, 112.5, 150\}$ are listed in Table 2. Figure 3 illustrates the option prices as a function of the remaining time to maturity. It is observed that the option price increases with a longer remaining lifetime, indicating higher time value. Similarly, Figure 4 presents option prices for contracts with varying maturity dates. It confirms that longer maturity leads to higher option prices, as the increased time horizon allows for a greater number of potential favorable price movements.

The option price comprises both intrinsic value and time value. As the option's lifetime increases, its time value grows, resulting in a higher total price. Moreover, in the case of call options, a lower strike price leads to a more expensive option. Conversely, for put options, a higher strike price results in a higher option price. This explains why options on the same underlying asset are available at various strike prices-allowing market participants to tailor their strategies by selecting suitable call or put positions. Figure 5 demonstrates this behavior clearly: the price of the call option increases as the strike price decreases.

Finally, we note that volatility, a fundamental measure of financial risk, plays a significant role in the pricing and hedging of derivative instruments. It reflects the uncertainty or variability of asset returns, and hence is a critical parameter in option valuation models. The high volatility of the base asset means that in a certain period of time, the stock can fluctuate in a larger range and go up and down. So there is a greater possibility that the derivative contract will change from out-the-money to in-the-money in a shorter period of time or vice versa. Also, high volatility provides more opportunities to fluctuate from contracts and move between contracts or exit positions to increase the efficiency of strategies. On the other hand, long-term investors prefer less volatility. Figure 6 illustrates that American call option price increases with increasing σ because the probability of making large gains in the future increases with increasing volatility.

Figures 7-11 show American call option price based on different parameters λ, δt, α, β, and H. According to Figures 7-11, if the parameters λ, δt, β, and H increase in the transaction cost problem under the mixed fBm model, the option price also increases. But, if the parameter α increases in this model, American call option price decreases. Table 3 shows the quantitative effect of each parameter of model (3), which is shown in Figures 4-11. These results are briefly stated in Table 4. Figures 4-11 and Table 4 show the importance of the value of each problem parameter on American call option price, which will be effective in the investor's decision.

Table 1: Parameter values of model (3)

Parameter	Value	Value Set for Trend Analysis
T	1 (year)	$\{0.25, 0.50, 0.75, 1\}$ (year)
E	50 \$	$\{40\,\$, 50\,\$, 60\,\$, 70\,\$\}$
σ	0.4	$\{0.2, 0.3, 0.4, 0.5\}$
λ	0.1	$\{0.1, 0.4, 0.7, 1\}$
δt	0.02	$\{0.01, 0.02, 0.03, 0.04\}$
α	0.009	$\{0.003, 0.006, 0.009, 0.012\}$
β	0.002	$\{0.001, 0.003, 0.005, 0.007\}$
H	0.95	$\{0.80, 0.85, 0.90, 0.95\}$

Table 2: American call option price under the mixed fBm model (3) when $N = M = 100$

S (\$)	0	37.5	75	112.5	150
t (year)	Option Price (\$)				
0.00	0	3.1948	29.9725	66.3147	106.9646
0.25	0	2.7353	28.9602	65.3264	104.9480
0.50	0	1.9545	27.7000	64.2529	103.1543
0.75	0	0.8589	26.2249	63.2496	101.5216
1.00	0	0	25	62.5000	100

The above proposed model reduces to the classical Black-Scholes model in the special case where $H = 0.5$, $\lambda = 1$, $\alpha = 0$, and $\beta = 0$. Therefore, we can compare the model in this paper with the other models such as time-fractional Black-Scholes equation with transaction costs under Leland's model using an implicit difference scheme [26]; modified Black-Scholes equation with time-dependent parameters using the Crank-Nicholson scheme [27]; classical Black-Scholes equation using Crank-Nicholson scheme; classical Black-Scholes equation using binomial method; time-fractional Black-Scholes equation with time-dependent parameters under CEV model using a compact difference scheme [27]. The parameters of these models are as follows:

I. Black-Scholes equation on an underlying described by the mixed fBm model under the transaction costs: $H = 0.5$, $\lambda = 1$, $\delta t = 0.02$, $\alpha = 0$, $\beta = 0$, $\sigma = 0.4$, $T = 1$ (year), $E = 50\,\$$, $r_t = 0.1 + 0.05e^{-t}$, $D_t = 0.03 + 0.001e^{0.01t}$, and $N = M = 100$.

Table 3: American call option price under the mixed fBm model (3) when $N = M = 100$

S ($)	0	37.5	75	112.5	150		0	37.5	75	112.5	150
T (year)			Option Price ($)			E ($)			Option Price ($)		
0.25	0	0.0083	26.1909	63.4024	101.8403	40	0	6.2276	37.9589	74.9893	115.5717
0.50	0	0.5030	27.2535	64.2393	103.6128	50	0	3.1948	29.9725	66.3147	106.9646
0.75	0	1.7098	28.4779	65.1986	105.3201	60	0	1.5945	22.9461	57.8254	98.3575
1.00	0	3.1948	29.9725	66.3147	106.9646	70	0	0.7931	17.1259	49.6908	89.7504
σ			Option Price ($)			λ			Option Price ($)		
0.2	0	0.6731	28.8809	65.5020	106.9646	0.1	0	3.1948	29.9725	66.3147	106.9646
0.3	0	1.8384	29.1650	65.8830	106.9646	0.4	0	3.4470	30.1698	66.4419	106.9646
0.4	0	3.1948	29.9725	66.3147	106.9646	0.7	0	4.1210	30.7547	66.8074	106.9646
0.5	0	4.6261	31.2140	66.9189	106.9646	1.0	0	5.1388	31.7633	67.4238	106.9646
δt			Option Price ($)			α			Option Price ($)		
0.01	0	3.1742	29.9570	66.3046	106.9646	0.003	0	3.2891	30.0450	66.3619	106.9646
0.02	0	3.1948	29.9725	66.3147	106.9646	0.006	0	3.2421	30.0086	66.3383	106.9646
0.03	0	3.2033	29.9789	66.3189	106.9646	0.009	0	3.1948	29.9725	66.3147	106.9646
0.04	0	3.2083	29.9827	66.3213	106.9646	0.012	0	3.1471	29.9366	66.2911	106.9646
β			Option Price ($)			H			Option Price ($)		
0.001	0	3.1945	29.9723	66.3147	106.9646	0.80	0	3.1009	29.9108	66.3726	106.9646
0.003	0	3.1951	29.9726	66.3147	106.9646	0.85	0	3.1393	29.9363	66.3541	106.9646
0.005	0	3.1956	29.9729	66.3148	106.9646	0.90	0	3.1702	29.9566	66.3347	106.9646
0.007	0	3.1962	29.9732	66.3149	106.9646	0.95	0	3.1948	29.9725	66.3147	106.9646

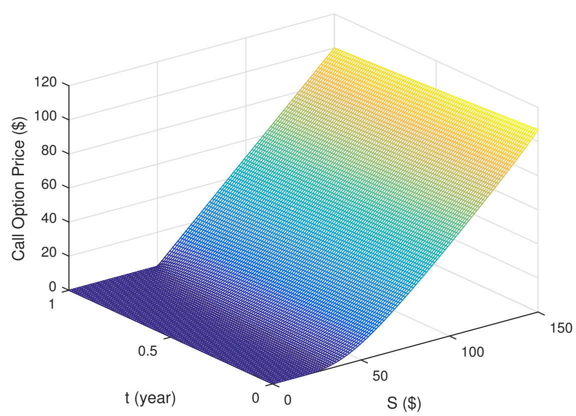

Figure 2: American call option price under mixed fBm model (3) when $N = M = 100$

II. Time-fractional Black-Scholes equation under Leland's transaction costs model [26]:

$$ {}_{0}^{C}D_{\tau}^{\alpha'} W(S,\tau) = \frac{1}{2}\sigma^2 \left(1 + \sqrt{\frac{2}{\pi}} \frac{\kappa}{\sigma\sqrt{\delta t}}\right) S^2 \frac{\partial^2 W}{\partial S^2} + [r-D]\, S \frac{\partial W}{\partial S} - rW, $$

where $\delta t = 0.02$, $\kappa = 0.05$, $\alpha' = 1$ (time-fracional order), $\sigma = 0.4$, $T = 1$ (year), $E = 50$ \$, $r = 0.1$, $D = 0.03$, and $N = M = 100$.

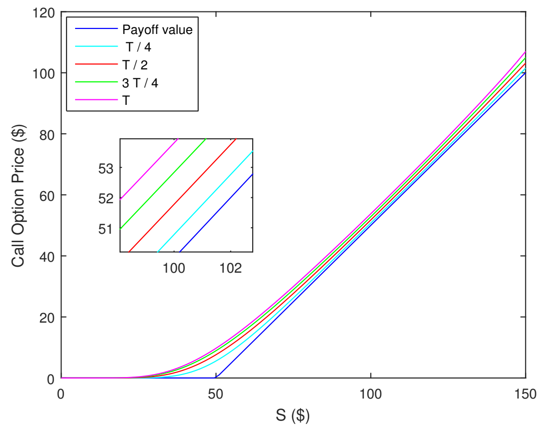

Figure 3: American option price based on remaining life to maturity when $N = M = 100$

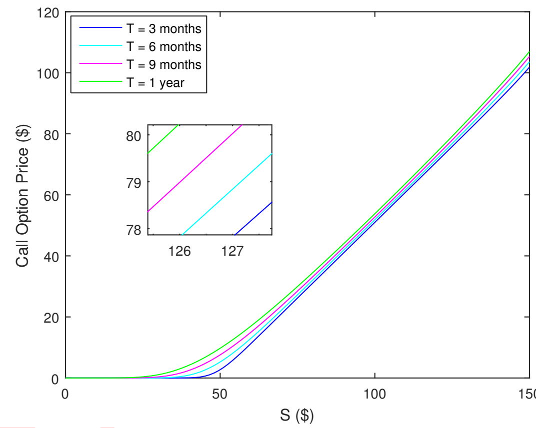

Figure 4: American option price based on different maturity times when $N = M = 100$

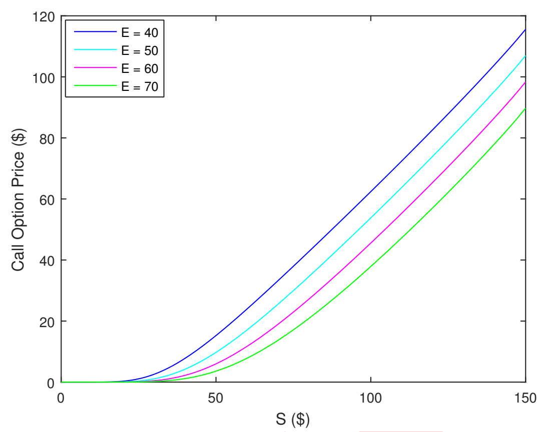

Figure 5: American option price based on different strike prices when $N = M = 100$

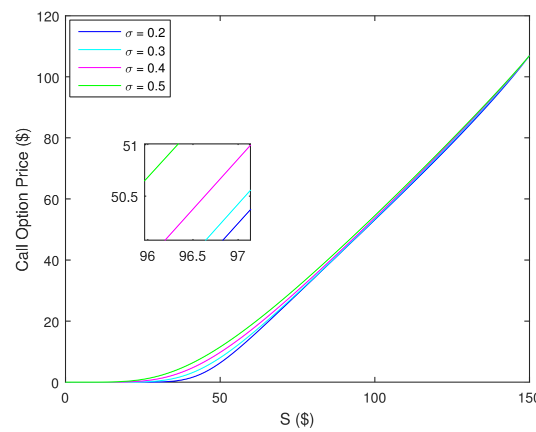

Figure 6: American option price based on different volatilities when $N = M = 100$

III. Modified Black-Scholes equation with time-dependent parameters [27]:

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + (r(t) - D(t))S\frac{\partial V(S,t)}{\partial S} - r(t)V(S,t) = 0,$$

where $\sigma = 0.4$, $T = 1$ (year), $E = 50$ \$, $r_t = 0.1 + 0.05e^{-t}$, $D_t = 0.03 + 0.001e^{0.01t}$, and $N = M = 100$.

IV. Classical Black-Scholes equation using Crank-Nicholson scheme:

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0,$$

where $\sigma = 0.4$, $T = 1$ (year), $E = 50$ \$, $r = 0.1$, $D = 0$, and $N = M = 100$.

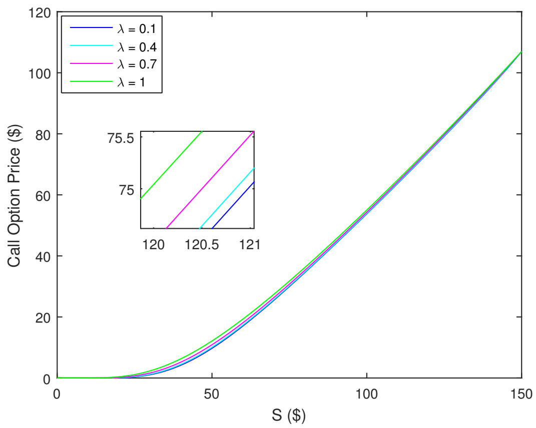

Figure 7: American option price based on different λ when $N = M = 100$

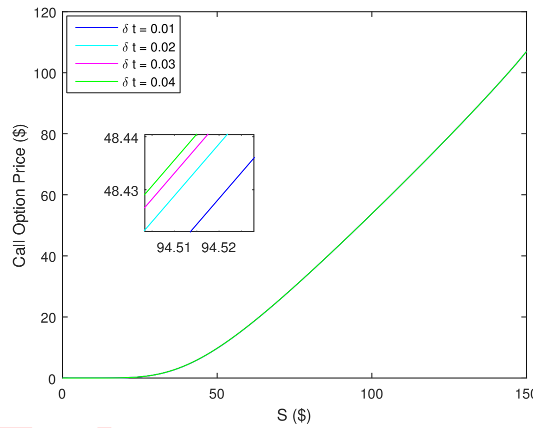

Figure 8: American option price based on different δt when $N = M = 100$

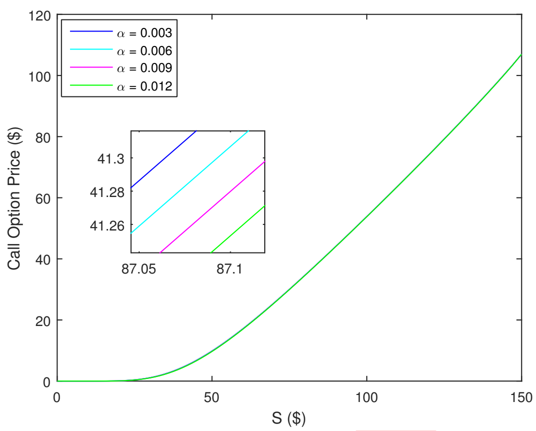

Figure 9: American option price based on different α when $N = M = 100$

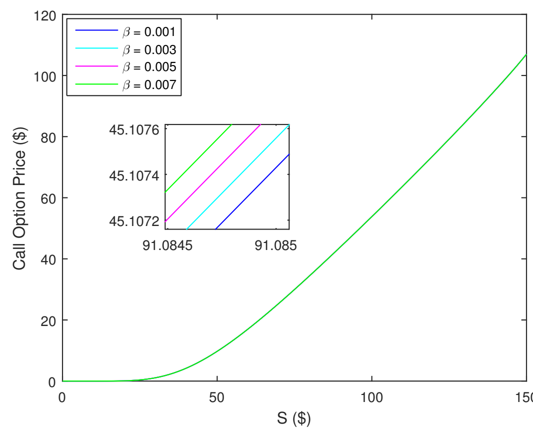

Figure 10: American option price based on different β when $N = M = 100$

V. Classical Black-Scholes equation using binomial method: $\sigma = 0.4$, $T = 1$ (year), $E = 50$ \$, $r = 0.1$, $D = 0$, and $N = M = 100$.

VI. Time-fractional Black-Scholes equation with time-dependent parameters under CEV model [27]:

$$ {}_0^C D_\tau^{\alpha'} W(S,\tau) = \frac{1}{2}\delta^2 S^{2\beta'+2}\frac{\partial^2 W}{\partial S^2} + \left(r(T-\tau) - D(T-\tau)\right) S\frac{\partial W}{\partial S} - r(T-\tau)W, $$

where $\alpha' = 1$ (time-fracional order), $\beta' = 0$ (elasticity of volatility), $\sigma_0 = 0.4$, $T = 1$ (year), $E = 50$ \$, $r_t = 0.1 + 0.05e^{-t}$, $D_t = 0.03 + 0.001e^{0.01t}$, and $N = M = 100$.

This comparison can be seen in Figures 12-16 and Table 5.

In the real world, every time the seller rebalances their hedge, they incur costs: brokerage commissions, bid-ask spreads, and taxes (in some markets). The classic Black-Scholes model assumes these

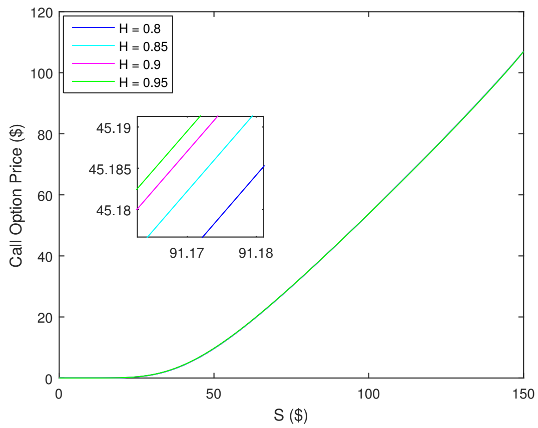

Figure 11: American option price based on different H when $N = M = 100$

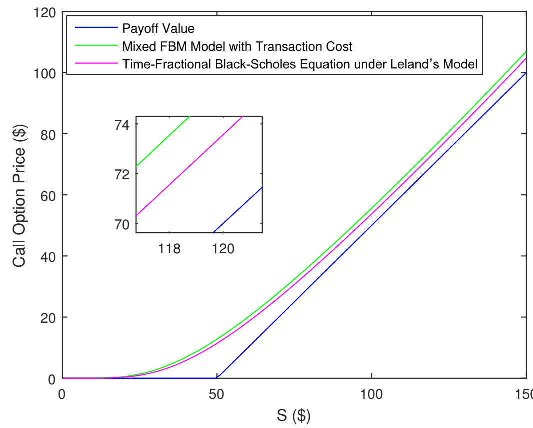

Figure 12: Comparison of the model I with the model II

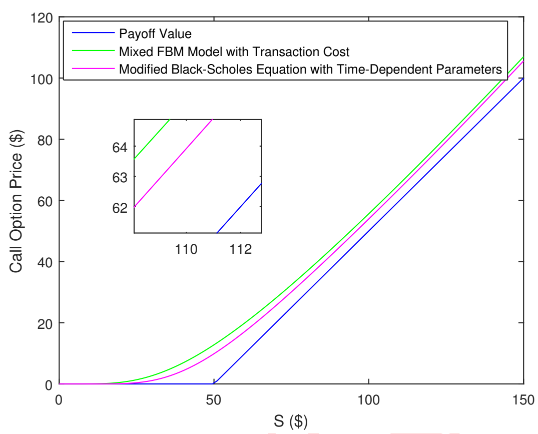

Figure 13: Comparison of the model I with the model III

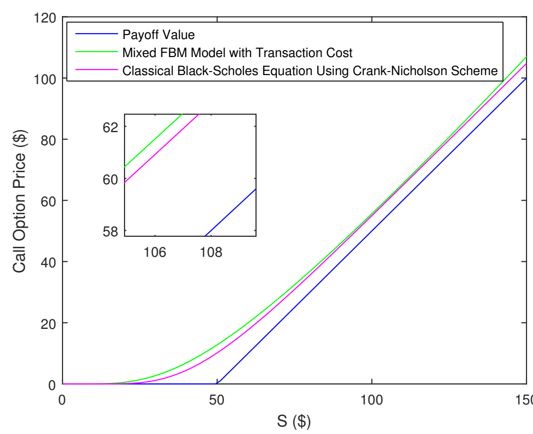

Figure 14: Comparison of the model I with the model IV

costs are zero. But, transaction costs exist in the markets and increase the option's price. When transaction costs exist, the process of dynamic hedging becomes expensive for the option seller. In the seller's primary risk management strategy (dynamic hedging), the more the stock price fluctuates (higher volatility), the more frequently the seller must rebalance, and the higher the total transaction costs. Therefore, as can be seen in Figures 12-16 and Table 5, the option pricing model with transaction costs is more expensive than the option pricing model without transaction costs. Also, as you can see in [26], the mixed fBm model with transaction cost is more expensive than the Leland's transaction cost model.

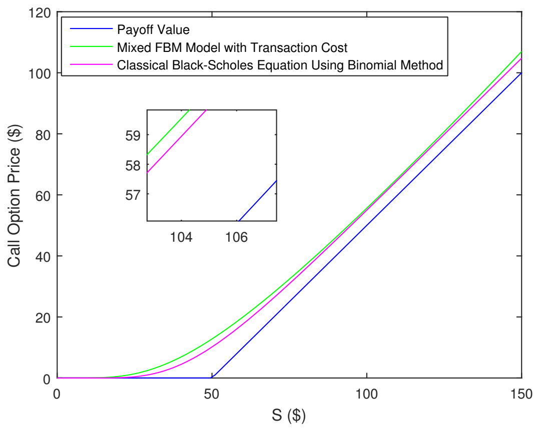

Figure 15: Comparison of the model I with the model V

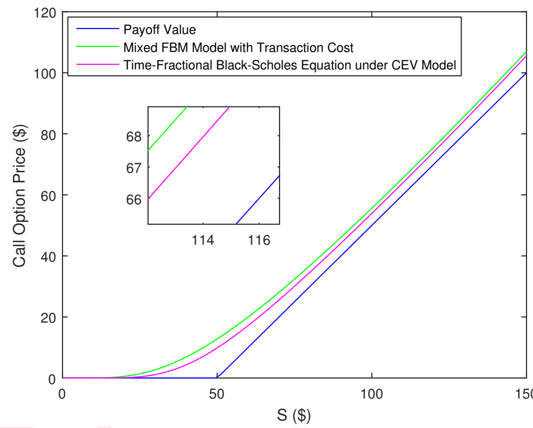

Figure 16: Comparison of the model I with the model VI

Table 4: Option price changes relative to parameters

Parameter	Parameter Changes	Option Price Changes
T	$\nearrow$	$\nearrow$
E	$\nearrow$	$\searrow$
σ	$\nearrow$	$\nearrow$
λ	$\nearrow$	$\nearrow$
δt	$\nearrow$	$\nearrow$
α	$\nearrow$	$\searrow$
β	$\nearrow$	$\nearrow$

5 Conclusion

In this study, motivated by the critical influence of transaction costs in financial markets, we investigate the pricing of American options under transaction costs in a mixed fBm framework–a model adept at capturing long-range dependence in asset prices. Employing a compact finite difference scheme for numerical valuation, we incorporate time-dependent interest rates and dividend yields to better reflect real-world market conditions. Our numerical results reveal that, within the mixed fBm setting with transaction costs, the price of an American call option exhibits a positive relationship with maturity (T), volatility (σ), the transaction cost coefficient (λ), time step size (δt), drift parameter (β), and the Hurst exponent (H). Conversely, the option price decreases with higher strike prices (E) and the parameter (α). These findings highlight the complex interplay between model parameters and transaction costs, providing valuable insights into the behavior of option prices under realistic market frictions.

Table 5: Comparison of the model I with the models II-IV

S ($)	15	30	45	60	75	90	105	120	135	150
t (year)	Black-Scholes equation on an underlying described by the mixed fBm model under the transaction costs ($)									
0.00	0.1391	2.7026	9.6458	19.9778	32.3900	46.0556	60.5288	75.5773	91.0782	106.9646
0.25	0.0477	1.7803	7.9200	17.9800	30.4150	44.1672	58.6913	73.7313	89.1691	104.9480
0.50	0.0074	0.8998	5.9505	15.7496	28.3972	42.4012	57.0633	72.1214	87.4923	103.1543
0.75	1.0444e-04	0.1956	3.5089	13.0799	26.3999	40.9275	55.7959	70.8357	86.0617	101.5216
1.00	0	0	0	10	25	40	55	70	85	100
t (year)	Time-fractional Black-Scholes equation under Leland's transaction costs model ($)									
0.00	0.0697	2.0639	8.4042	18.4059	30.6664	44.2443	58.6321	73.5698	88.9310	104.7581
0.25	0.0206	1.3324	6.9464	16.7708	29.1589	42.9118	57.4218	72.4185	87.7947	103.6128
0.50	0.0022	0.6380	5.2326	14.8819	27.5806	41.6513	56.3338	71.3698	86.7034	102.4385
0.75	8.7206e-06	0.1131	3.0675	12.5729	26.0222	40.6326	55.5149	70.5251	85.7076	101.2345
1.00	0	0	0	10	25	40	55	70	85	100
t (year)	Modified Black-Scholes equation with time-dependent parameters ($)									
0.00	0.0093	1.1193	6.8295	17.1730	30.0727	44.1711	58.9039	74.0811	89.6467	105.5827
0.25	0.0016	0.6336	5.5403	15.7576	28.8659	43.1367	57.9264	73.0762	88.5726	104.4273
0.50	6.5527e-05	0.2360	4.0322	14.1068	27.5803	42.0999	56.9488	72.0374	87.4202	103.1543
0.75	3.5623e-08	0.0212	2.1803	12.1126	26.2569	41.0825	55.9859	70.9779	86.1746	101.7069
1.00	0	0	0	10	25	40	55	70	85	100
t (year)	Classical Black-Scholes equation using Crank-Nicholson scheme ($)									
0.00	0.0095	1.1516	7.0313	17.6735	30.8888	45.1973	59.9314	74.8268	89.7823	104.7581
0.25	0.0016	0.6374	5.6126	16.0192	29.3767	43.8499	58.6869	73.6363	88.6198	103.6128
0.50	6.2521e-05	0.2319	4.0218	14.1775	27.8202	42.5159	57.4539	72.4416	87.4391	102.4385
0.75	3.2536e-08	0.0204	2.1472	12.0736	26.3032	41.2388	56.2347	71.2345	86.2345	101.2345
1.00	0	0	0	10	25	40	55	70	85	100
t (year)	Classical Black-Scholes equation using binomial method ($)									
0.00	0.0083	1.1409	7.0326	17.6780	30.8952	45.1929	59.9295	74.8278	89.7855	104.7701
0.25	0.0011	0.6357	5.6263	16.0288	29.3689	43.8494	58.6852	73.6342	88.6201	103.6151
0.50	2.0084e-05	0.2195	4.0231	14.1818	27.8224	42.5108	57.4522	72.4412	87.4389	102.4386
0.75	0	0.0167	2.1686	12.0830	26.2958	41.2377	56.2346	71.2345	86.2345	101.2345
1.00	0	0	0	10	25	40	55	70	85	100
t (year)	Time-fractional Black-Scholes equation with time-dependent parameters under CEV model ($)									
0.00	0.0097	1.1224	6.8186	17.1587	30.0639	44.1672	58.9024	74.0804	89.6465	105.5827
0.25	0.0017	0.6384	5.5295	15.7431	28.8589	43.1348	57.9263	73.0762	88.5727	104.4273
0.50	8.2201e-05	0.2419	4.0217	14.0932	27.5766	42.0997	56.9495	72.0382	87.4208	103.1543
0.75	8.1293e-08	0.0248	2.1711	12.1039	26.2579	41.0812	55.9859	70.9802	86.1769	101.7069
1.00	0	0	0	10	25	40	55	70	85	100

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