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Convex optimization approach for the path-following problem of two collaborative robots

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Abstract. This paper investigates the time-optimal path-tracking problem for a collaborative robotic system, considering some limitations and dynamic characteristics. The problem is formulated for a robotic system consisting of two-link planar manipulators with and without bar cases along a predetermined geometric path in minimum time. The main challenges are to satisfy both the co-position and co-time conditions of the end-effector movement, as well as the physical limitations of the torque applied to the joints. Through discretization and convexification, we convert the problem into a convex conic optimization problem. The numerical example confirms the effectiveness of the method.

Keywords: Convex problem, mathematical modeling, collaborative robots, optimization, control.
AMS Subject Classification 2010: 70E60, 49N90, 90C20, 93C85

1 Introduction

In recent decades, the development of advanced robotic systems to improve precision, speed, and efficiency in industrial and service tasks has become one of the key topics in engineering research. One of the challenges in this field is designing robot trajectories such that physical limitations are satisfied. Numerous studies have been conducted on the modeling and approximation of robotic motion trajectories. The use of polynomial functions for trajectory approximation, iterative learning algorithms, and the transformation of nonlinear problems are among the prominent approaches explored in recent research.

In [2], the increase in degrees of freedom is investigated by approximating the robot path using polynomial functions. This study demonstrated that polynomial-based modeling can effectively represent and approximate robotic trajectories, thereby enhancing the flexibility and precision of motion control.

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One important problem in robotics is the time-optimal path tracking problem, where motion along predefined paths is achieved with minimum time and energy consumption. Trajectory and time optimization not only enhance productivity but also significantly affect the durability of mechanical components and the overall quality of task execution. The time-optimal path tracking problem for a single robotic arm is reformulated as a convex optimization problem through a nonlinear change of variables in [8]. In addition to minimizing motion time, the authors also considered secondary objectives such as minimizing the absolute value of the torque rate and thermal energy generation. In [1], the time-optimal path tracking problem for a six-degrees-of-freedom industrial manipulator is reformulated as a convex optimization problem that includes velocity constraints.

Recently, the cooperation between multiple robots to jointly execute one or more tasks has given rise to a new research field known as collaborative robotics. The main goal of these systems is to enable efficient task execution using two or more robots, especially for operations that are beyond the capability of a single robot in terms of complexity, accuracy, or efficiency. Moreover, a team of cooperating robots generally outperforms individual robots in terms of productivity and precision [5], and their use is essential for optimizing energy, time, and cost consumption. Collaborative robots (Cobots) have been increasingly adopted in industrial environments due to their significant benefits. Cobots assist human workers in performing demanding or hazardous tasks, creating a symbiotic relationship between humans and robots [1]. They enable flexible automation in manufacturing with applications such as assembly [3], material handling [7], painting, palletizing, packaging, and welding [6]. This collaboration has led to improved product quality, reduced physical strain, and enhanced operational efficiency in production environments [5]. In collaborative robotic systems, time-optimal planning and energy-efficient control of manipulators are key challenges. Recently, in [5], the focus is on path-tracking for two manipulators carrying a bar, but it has only focused on time optimization. Despite these valuable contributions, there still exists a notable research gap in addressing time-optimal and energy-efficient path tracking for multiple collaborative manipulators handling a shared load. While [8] employs second-order cone programming (SOCP) for path tracking, it does not consider multi-robot cooperation or load sharing, and the optimization objective in [5] is limited to minimizing motion time.

In this study, inspired by the formulation in [8], the time-optimal path-tracking problem is extended to the case of two cooperative manipulators carrying a shared bar. The proposed optimization framework incorporates multiple objectives, including thermal energy and the absolute value of the rate of torque change, along with corresponding weighting factors to regulate heat generation and smooth control effort. The main contribution of this paper is addressing the optimization problem of two cooperating arms through mathematical modeling and reformulating it as an SOCP problem that can be solved efficiently. This paper is structured as follows: The first section outlines the time-optimal path-tracking problem for a single robot, which is then extended to two cooperative manipulators in the second section. Subsequently, the problem of two cooperating arms carrying a bar connected to their end-effectors is formulated and analyzed. The third section presents numerical experiments that demonstrate the effectiveness of the proposed method. Finally, conclusions and directions for future research are discussed in the last section.

2 Time-optimal trajectory tracking problem for a robot

This section presents the time-optimal path tracking problem for a single manipulator. Its movement is significantly influenced by two key factors: the kinematics and dynamics of the manipulator. The purpose of forward kinematics is to determine the coordinates and orientation of the end-effector based on the robot's joint variables. However, inverse kinematics aims to obtain the robot's joint variables according to the location and orientation of the end-effector. Obtaining the joint torque using the location, velocity and acceleration of the robot's joints is known as inverse dynamics. This can be performed using the Lagrange formulation to determine the required torques for a desired motion. By generating the appropriate solution in the joint space, it is ensured that the end-effector will pass the specified geometric path. The end-effector can move along a straight line or another predefined geometric curve from the starting point to the endpoint.

Assume a geometric path $q(s)$, specified in joint space coordinates. The motion of the manipulator along a defined trajectory can be expressed as a function of a scalar s. We denote the relationship between time and path with $s(t)$ and assume that the path begins from $t=0$ and ends at $t=T$:

$$0 = s(0) \leq s(t) \leq s(T) = 1. \tag{1}$$

By applying the chain derivative rule, a geometric path $q(s)$ in joint space allows for the determination of the joint velocities and accelerations as follows:

$$\dot{q}(s) = q'(s)\dot{s}, \tag{2}$$

$$\ddot{q}(s) = q'(s)\ddot{s} + q''(s)\ddot{s}, \tag{3}$$

where

$$q''(s) = \frac{\partial^2 q(s)}{\partial s^2},\;\; q'(s) = \frac{\partial q(s)}{\partial s},\;\; \ddot{s} = \frac{d^2 s}{dt^2},\;\; \dot{s} = \frac{ds}{dt}.$$

In this scenario, the dynamic equation of a manipulator, whose degree of freedom is n, can be expressed as:

$$\tau = \mathbf{M}(q)\ddot{q} + \mathbf{C}(q,\dot{q})\dot{q} + \mathbf{G}(q), \tag{4}$$

where $\tau \in \mathbb{R}^n$ is the joint torque, $M(q) \in \mathbb{R}^{n\times n}$ denotes the positive definite inertia matrix, $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ represents the matrix of Coriolis and centrifugal forces, and $G(q) \in \mathbb{R}^n$ illustrates the vector of gravitational torque. This equation shows how to create movement in the robot by the torque exerted by the actuators [8].

2.1 Modeling the problem

First, we examine the convex optimization problem for a two-link arm. The dynamics of a robotic manipulator can be formulated from its equations of motion. These equations incorporate joint velocities, torques, and accelerations while considering the imposed joint constraints, as follows:

$$\begin{aligned}
\min_{a(0),b(0),\tau(0)} & \int_0^1 \frac{1}{\sqrt{b(s)}} ds \\
\text{s.t.} \quad & \tau(t) = m(s(t))a(s) + c(s)b(s) + g(s(t)), \\
& b(0) = \dot{s}_0^2, \\
& b(1) = \dot{s}_T^2,
\end{aligned}$$

$$
\begin{aligned}
&b'(s) = 2a(s),\\
&b(s) \geqslant 0,\\
&\underline{\tau}(s) \leq \tau(s) \leq \bar{\tau}(s), \quad \text{for } s \in [0,1].
\end{aligned} \tag{5}
$$

The variables $a(s)$ and $b(s)$ are optimization parameters, and the velocity at both the beginning and the end of the trajectory is set to zero. For torque, both upper and lower limits are considered. In addition to optimizing time, other goals can be incorporated in the problem. The thermal energy produced by the actuators and the integral of the absolute magnitude of torque change can be added to the objective function. Reducing thermal energy and rapid torque variations helps avoid torque jumps and improves system control [8]. We define thermal energy as $\frac{(\tau_i^k)^2}{\bar{\tau}_i^2}$ and absolute value of the rate of torque change as $\frac{|\Delta\tau_i^k|}{|\bar{\tau}_i|}$. The γ_1 and γ_2 are weighting factors.

Along with torque constraints, other constraints may also be considered. Specifically velocity and acceleration limits are enforced by applying symmetrical upper and lower bounds on the joints. The generalized optimal control problem is convex due to the convex nature of the objective function and inequality constraints, as well as the linear characteristics of the dynamic system and equality constraints. By employing a direct transcription method, the problem is transformed into a finite-dimensional optimization problem. Initially, we discretize the path into nodal points and determine the values of the functions $a(s)$, $b(s)$, and $\tau(s)$ at the nodal points and in between them [8]. By introducing the shorthand notation $N_k = 0,1,2,...,k$ and $N_{k-1} = 0,1,2,...,k-1$, problem (5) can be reformulated as the following large-scale discretized optimization problem:

$$
\begin{aligned}
\min_{a^k,b^k,c^k} \quad & \sum_{k=0}^{K-1} \frac{2\Delta s^k \left(1+\gamma_1 \sum_{i=1}^{n} \frac{(\tau_i^k)^2}{\bar{\tau}_i^2}\right)}{\sqrt{b^{k+1}}+\sqrt{b^k}} + \gamma_2 \sum_{k=1}^{K-1} \left(\sum_{i=1}^{n} \frac{|\Delta\tau_i^k|}{|\bar{\tau}_i|} \right)\\
\text{s.t.} \quad & \tau^k = m(s^{\frac{k+1}{2}})a^k + c(s^{\frac{k+1}{2}})b^{\frac{k+1}{2}} + g(s^{\frac{k+1}{2}}),\\
& b^0 = \dot{s}_0^2,\\
& b^k = \dot{s}_T^2,\\
& b^{k+1} - b^k = 2a^k\Delta s^k,\\
& b^k \geqslant 0,\\
& b^k \leq \bar{b}(s^k),\\
& \underline{\tau}(s^{\frac{k+1}{2}}) \leq \tau^k \leq \bar{\tau}(s^{\frac{k+1}{2}}), \quad \text{for } k \in N_{k-1}.
\end{aligned} \tag{6}
$$

Considering two slack variables e^k and d^k, the following relations hold:

$$
\left(\frac{\mathbf{1}+\gamma_1 \sum\limits_{i=1}^{n} \frac{(\tau_i^k)^2}{\bar{\tau}_i^2}}{\sqrt{b^{k+1}}+\sqrt{b^k}} \right) \leq d^k, \; -e^k \leq \begin{pmatrix} \frac{|\Delta\tau_1^k|}{|\bar{\tau}_1|} \\ \vdots \\ \frac{|\Delta\tau_n^k|}{|\bar{\tau}_n|} \end{pmatrix} \leq e^k, \tag{7}
$$

which is equivalent to the following constraints:

$$\left(\frac{1 + \gamma_1 \sum_{i=1}^{n} \frac{(\tau_i^k)^2}{\bar{\tau}_i^2}}{c^{k+1} + c^k} \right) \leq d^k, \quad c^k \leq \sqrt{b^k}. \tag{8}$$

The objective function can be written as $\sum_{k=0}^{K-1} 2\Delta s^k d^k + \gamma_2 \sum_{k=1}^{K-1} \mathbf{1}^T e^k$, where $\mathbf{1} \in \mathbb{R}^n$ is a vector with all elements equal to one. To transform the constraint into a quadratic conic form, the problem is formulated as a large-scale discretized optimality problem, which finally leads to an SOCP as follows:

$$\begin{aligned}
\min_{a^k, b^k, \tau^k, c^k, d^k, e^k} \quad & \sum_{k=0}^{K-1} 2\Delta s^k d^k + \gamma_2 \sum_{k=1}^{K-1} \mathbf{1}^T e^k, \\
\text{s.t.} \quad & \tau^k = m(s^{\frac{k+1}{2}}) a^k + c(s^{\frac{k+1}{2}}) b^{\frac{k+1}{2}} + g(s^{\frac{k+1}{2}}), \\
& b^0 = \dot{s}_0^2, \\
& b^k = \dot{s}_T^2, \\
& b^{k+1} - b^k = 2a^k \Delta s^k, \\
& b^k \geqslant 0, \\
& b^k \leq \bar{b}(s^k), \\
& \underline{\tau}(s^{\frac{k+1}{2}}) \leq \tau^k \leq \bar{\tau}(s^{\frac{k+1}{2}}), \quad for \;\; k \in N_{k-1}, \\
& \left\| \begin{pmatrix} 2 \\ 2\sqrt{\gamma_1} \, \dfrac{\tau_1^k}{\bar{\tau}_1} \\ \vdots \\ 2\sqrt{\gamma_n} \, \dfrac{\tau_n^k}{\bar{\tau}_n} \\ c^{k+1} + c^k - d^k \end{pmatrix} \right\|_2 \leq c^{k+1} + c^k + d^k, \\
& -e^k \leq \begin{pmatrix} \frac{|\Delta \tau_1^k|}{|\bar{\tau}_1|} \\ \vdots \\ \frac{|\Delta \tau_n^k|}{|\bar{\tau}_n|} \end{pmatrix} \leq e^k, \quad for \;\; k \in N_{k-1}, \\
& \left\| \begin{pmatrix} 2c^k \\ b^k - 1 \end{pmatrix} \right\|_2 \leq b^k + 1, \\
& b^k \geqslant 0, \quad \text{for} \;\; k \in N_k.
\end{aligned} \tag{9}$$

3 Two manipulators

In this section, we examine the movement of the two arms of two-link robots and investigate their motion of the end-effector along a specified geometric path.

Using the objective function of the single-arm problem, and considering similar variables and constraints, the problem is transformed into a quadratic conic problem. Consider two arms, each with two degrees of freedom, whose end-effectors are supposed to traverse the path in the minimum co-time and

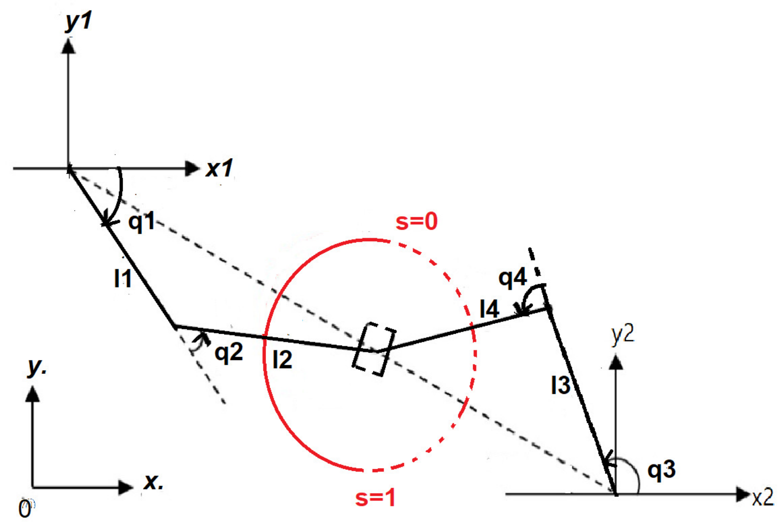

Figure 1: Two-link robots model

co-position (same time and same position). To ensure that both robots follow the path simultaneously, their motion is synchronized step by step, thereby satisfying the co-position and co-time conditions.

The path is discretized into multiple points, and each robot moves along it point by point. The key aspect of co-position and co-time is that it makes the problem dependent on the path. The equations of the intended path in this article are considered in Cartesian space, then the joint angles are calculated using inverse kinematics for two two-link robots (see Figure 1). The length of each link is 0.5m. The given path is a semicircle with a radius of 0.56m, and the path function is defined over the interval $[0,1]$.

For the robot on the left, we denote the angle of the first link with respect to the x-axis as q_1, and the angle of the second link relative to the extension of the first link as q_2. The angle between the two dotted lines and the x-axis is 45 °. For the robot on the right, the angle of the first link with the x-axis is denoted as q_3, and the angle of the second link with the extension of the first link is denoted as q_4. The angle of the line connecting the two links in the right robot with the x-axis is 135°. If the movement direction of q_1 is counterclockwise, q_2 is negative, and if it is clockwise, q_2 is positive. The same convention applies to the second robot. Since the dynamic equations are expressed for a single robot, the same equations can be applied to the second robot as well. Each robot traverses the path within a specific time frame. The key is that both robots must move along the path together. In this case, we ensure co-position and co-time.

The main point in co-time and co-position is to make the problem depend on the path $s(t)$. Therefore, the dynamic equations for the robot's motion take the following form:

$$\tau_{\mathbf{i}} = \mathbf{M_i}(q_i)\ddot{q}_i + \mathbf{C_i}(q_i,\dot{q}_i)\dot{q}_i + \mathbf{G_i}(q_i), \quad i = 1,2. \tag{10}$$

Given that the end-effectors are in the same location at every moment and traverse the same trajectory, we assume that the velocities and accelerations of both actuators are identical.

In the two-robot problem, similar to the single-robot problem, the objective function and constraints can be determined. Considering the time, thermal energy, and torque change rate in the objective function, along with the torque, velocity and acceleration limitations, and using discretization, the conic quadratic programming problem has been transformed into the following form:

$$
\begin{aligned}
\min\{ & \sum_{k=0}^{K-1} 2\Delta s^k d_1^k + \gamma_2 \sum_{k=0}^{K-1} 1^T e_1^k + \sum_{k=0}^{K-1} 2\Delta s^k d_2^k + \gamma_2 \sum_{k=0}^{K-1} 1^T e_2^k \} \\
\text{s.t.} \quad & \tau_i^k = m_i(s^{\frac{k+1}{2}})a_i^k + c_i(s^{\frac{k+1}{2}})b_i^{\frac{k+1}{2}} + g_i(s^{\frac{k+1}{2}}), \\
& b_i^0 = \dot{s}_0^2, \\
& b_i^K = \dot{s}_T^2, \\
& b_i^{k+1} - b_i^k = 2a_i^k \Delta s^k, \\
& \underline{\tau}_i(s^{\frac{k+1}{2}}) \leq \tau_i^k \leq \bar{\tau}_i(s^{\frac{k+1}{2}}), \\
& \left\| \begin{pmatrix} 2 \\ 2\sqrt{\gamma_1}\dfrac{\tau_{1i}^k}{\bar{\tau}_{1i}} \\ \vdots \\ 2\sqrt{\gamma_n}\dfrac{\tau_{ni}^k}{\bar{\tau}_{ni}} \\ c_i^{k+1} + c_i^k - d_i^k \end{pmatrix} \right\|_2 \leq c_i^{k+1} + c_i^k + d_i^k, \quad k \in N_{k-1}, \\
& b_i^k \geq 0, \\
& b_i^k \leq \bar{b}_i(s^k), \quad k \in N_k, \\
& -e_i^k \leq \begin{pmatrix} \dfrac{|\Delta\tau_{1i}^k|}{|\bar{\tau}_{1i}|} \\ \vdots \\ \dfrac{|\Delta\tau_{ni}^k|}{|\bar{\tau}_{ni}|} \end{pmatrix} \leq e_i^k, \quad k \in N_{k-1}, \\
& \left\| \begin{pmatrix} 2c_i^k \\ b_i^k - 1 \end{pmatrix} \right\|_2 \leq b_i^k + 1, \quad i = 1, 2.
\end{aligned}
\tag{11}
$$

4 Collaborative robot problem of a bar

In this part, the movement of two robots carrying a bar along a common path is examined (see Figure 2). To address the collaborative robot problem in the presence of the bar, a strip is connected to the end-effectors. The impact of the bar's movement on the manipulators is divided into two components and subsequently integrated with the dynamics of the respective manipulators [5].

4.1 Bar torque and end-effector

First, we describe the movement of the bar by examining the forces acting on it, which including transitional forces and rotational forces. If we denote the direction of movement of the bar by $P(s)$, then using

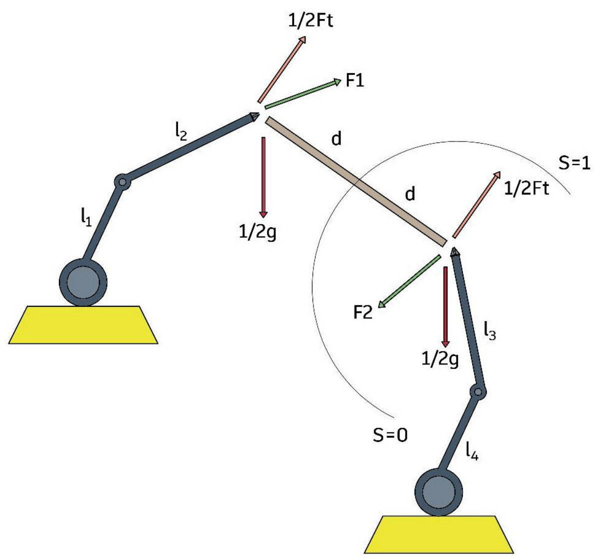

Figure 2: Schematic representation of the manipulators

the chain rule, the acceleration is described by the following equation:

$$\ddot{P}(s) = P'(s)\ddot{s} + P''(s)\dot{s}^2. \tag{12}$$

The forces exerted on the bar are separated into two components: transitional and the rotational forces. The transitional force is given by

$$F_t(s) = m_b \ddot{P}(s), \tag{13}$$

where m_b is bar'mass and $\ddot{p}(s)$ represents the acceleration along the path. By applying the chain rule, the acceleration can be reformulated as [5]:

$$F_t(s) = m_b P'(s)\ddot{s} + m_b P''(s)\dot{s}^2. \tag{14}$$

This force is the same for all parts of the bar. Therefore, it can be divided into two forces acting at the ends of the bar. Gravitational force is also applied uniformly to all parts of the bar. Since the force is applied at both ends of the rod and the path is horizontal, the net effect of gravity along the path is zero. To check the rotational force, we first consider the applied torque on the bar, which is obtained by the following equation [5]:

$$\tau_b(s) = \frac{d}{dt}\left(\mathbf{R}(s)\mathbf{I}\mathbf{R}(s)^T \boldsymbol{\omega}(s)\right), \tag{15}$$

where **I** refers to the inertial matrix, **R(s)** is the rotation matrix that converts the coordinates of each point in the device to the reference coordinates, and $\omega(s)$ is the angular velocity. This torque results from two forces, F_1 and F_2, which are applied at each rod end at a right angle to the bar, with the equal magnitude and in contrary direction. Therefore, bar movement equations are divided into two groups of equations.

Because, these forces are parallel to the tangent vector to the path, their magnitude has the following relationship with the magnitude of the bar moment [5]:

$$2d|F_{\tau_i}(s)| = |\tau_b(s)|, \quad i = 1,2, \tag{16}$$

where d represents the distance from the center of mass of the bar to the end-effector. Since we have two sets of dynamic equations, we denote them with indices 1 and 2. So, the net force applied to each of the end-effectors is obtained as follows, by adding the transitional and gravitational force, and is denoted by F_i^0 [5]:

$$F_i^0 = \frac{1}{2}F_t(s) + F_{\tau_i}(s) + \frac{1}{2}F_g, \tag{17}$$

$$F_t(s) = m_b p'(s)\ddot{s} + m_b p''(s)\dot{s}^2. \tag{18}$$

4.2 Joint torque

The vector of force and torque applied to i-th arm(joint) is as follows [5]:

$$F_i = [F_{ix}, F_{iy}, F_{iz}, n_{ix}, n_{iy}, n_{iz}]. \tag{19}$$

We denote the torque generated by the joint corresponding to the i-th (participating) arms by τ_i. Using the Jacobian of the cooperative arms, the following relationship between the moment produced at the joint and the applied force is established [5]:

$$\bar{\tau}_i(s) = J_i(q_i(s))^T F_i(s). \tag{20}$$

The J_i matrix is a six-by-n matrix that represents the number of joints. To obtain the joint angle, we can use inverse kinematics. In this case, there is no torque in the actuators, and the forces are applied only in along the direction of the x and y axes. For two cooperative manipulators, the Jacobian matrix is given by the following [5]:

$$J_i(q_i(s)) = \begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \tag{21}$$

where $h_1 = -l_{i1}\sin(q_{i1}) - l_{i2}\sin(q_{i1}+q_{i2}),\ h_2 = -l_{i2}\sin(q_{i1}+q_{i2}), h_3 = l_{i1}\cos(q_{i1}) + l_{i2}\cos(q_{i1}+q_{i2})$ and $h_4 = l_{i2}\cos(q_{i1}+q_{i2})$.

Since there is no transitional motion along the z-axis, the third row of the Jacobian matrix becomes zero [5]. Additionally, since there is no torque along the x and y axes, the fourth and fifth rows of the Jacobian matrix also become zero. This matrix consists of two components: translational velocity and angular velocity. Therefore, the total torque of the system, obtained by adding the torque of the bar, is expressed as follows [5]:

$$\begin{aligned}\bar{\tau}_i(s) = &\left(m_i(s) - \bar{J}_i(q_i(s))^T R_0^i \bar{m}_i(s)\right) a(s) + \left(c_i(s) - \bar{J}_i(q_i(s))^T R_0^i \bar{c}_i(s)\right) b(s) \\ &+ \left(g_i(s) - \bar{J}_i(q_i(s))^T R_0^i \bar{g}\right). \end{aligned} \tag{22}$$

4.3 Problem modeling

Based on the extracted torque of the entire system with the bar, the optimization problem addressed in the second section of the article aims to minimize time, actuator thermal energy, and torque change rate. This is done subject to constraints on speed, acceleration, and torque, and is formulated as the following convex quadratic conic problem:

$$
\begin{aligned}
\min & \left\{ \sum_{k=0}^{k-1} 2\Delta s^k d_1^k + \gamma_2 \sum_{k=0}^{k-1} 1^T e_1^k + \sum_{k=0}^{k-1} 2\Delta s^k d_2^k + \gamma_2 \sum_{k=0}^{k-1} 1^T e_2^k \right\} \\
\text{s.t.} \quad & \tau_i(s) = \big(m_i(s)\bar{J}_i(q_i(s))^T R_0^i m_i(s)\big)a(s) + (c_i(s)\bar{J}_i(q_i(s))^T R_0^i \bar{k}_i(s) b(s) \\
& \qquad + m_i(s) - (J_0(s) - \bar{J}_i(q_i(s))^T R_0^i \bar{J}), \\
& b_i^0 = \dot{s}_0^2, \\
& b_i^K = \dot{s}_T^2, \\
& b_i^{k+1} - b_i^k = 2a_i^k \Delta s^k, \\
& \underline{\tau_i}(s^{\frac{k+1}{2}}) \le \tau_i^k \le \bar{\tau}_i(s^{\frac{k+1}{2}}), \\
& \left\| \begin{pmatrix} 2 \\ 2\sqrt{\gamma_1}\, \dfrac{\tau_{1i}^k}{\bar{\tau}_{1i}} \\ \vdots \\ 2\sqrt{\gamma_n}\, \dfrac{\tau_{ni}^k}{\bar{\tau}_{ni}} \\ c_i^{k+1} + c_i^k - d_i^k \end{pmatrix} \right\|_2 \le c_i^{k+1} + c_i^k + d_i^k, \quad \text{for } k \in N_{k-1} \qquad (23) \\
& b_i^k \geqslant 0, \\
& b_i^k \le \bar{b}_i(s^k), \quad \text{for } k \in N_k \\
& -e_i^k \le \begin{pmatrix} \frac{|\Delta \tau_{1i}^k|}{|\bar{\tau}_{1i}|} \\ \vdots \\ \frac{|\Delta \tau_{ni}^k|}{|\bar{\tau}_{ni}|} \end{pmatrix} \le e_i^k, \\
& \left\| \begin{pmatrix} 2c_i^k \\ b_i^k - 1 \end{pmatrix} \right\|_2 \le b_i^k + 1, \ \text{for } k \in N_{k-1}, \ \text{for } i = 1, 2.
\end{aligned}
$$

5 Simulation

Here, the numerical results are presented for two cooperative manipulators. Table 1 provides the dynamic parameters used in the simulation for both manipulators. The parameter d is defined as 0.5m and the bar has a mass of 0.5kg. The optimization problem is structured as an SOCP and solved using CVX [4] in MATLAB. The results with $\gamma_1 = \gamma_2 = 0.001$ are shown in Figures 3 to 13. Figure 3 shows the motion of the end-effectors on a semicircular path with a radius of 0.65m.

The joint torques and their bounds for the first and second manipulators are plotted as a function of the time and path coordinate in Figures 4 and 5. Upon examining the plots, it becomes clear that at

Table 1: Dynamic parameters for the manipulators

Parameters	Values
Number of points	300
Length of links	$l_1 = l_2 = l_3 = l_4 = 0.5(m)$
Mass of links	$m_1 = m_2 = m_3 = m_4 = 0.5(kg)$
Joint stiffness tensor(link)	$I = 0.08(kg.m^2)$
Upper and lower torque	$\tau = [-1, 1](N.m)$

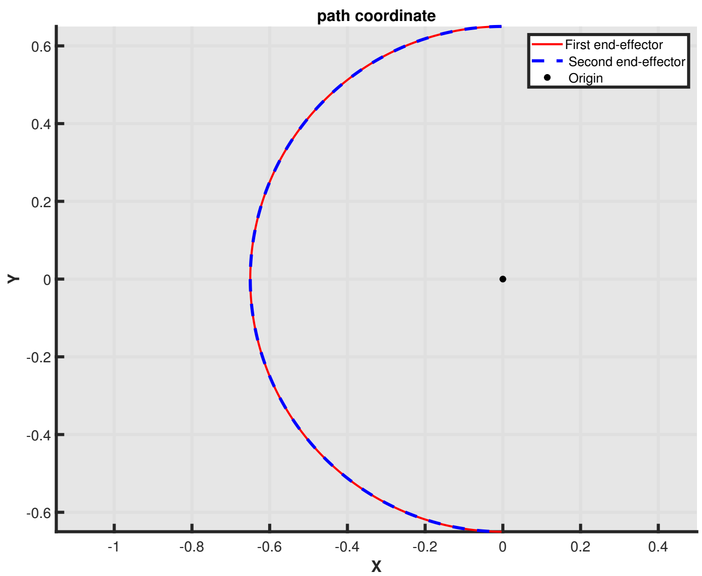

Figure 3: The motion of the end-effectors

each point along the path (or time), one of the joint torques reaches saturation, consistent with the results reported in the literature for minimum-time control of single manipulator [8].

Figures 6-9 illustrate the pseudo velocities (b_1 and b_2) and pseudo accelerations (a_1 and a_2) of the robot arms with respect to the path coordinate s. The initial and final velocities are chosen to be zero. The first arm has the highest joint velocities in the path coordinates from 0.5 to 0.6, while the second arm reaches its peak velocities in the range from 0.3 to 0.4. Maximum accelerations in both arms are observed during the time interval from 0 to 0.1 seconds.

Figures 10 and 11 present the angular displacement of the each robot arm as a function of time. The joint angles of the robotic arms may take positive or negative values over time due to the motion of the links. By convention, clockwise rotations are considered negative, whereas counterclockwise rotations are considered positive.

Figure 12 illustrates the relationship between path and time, while Figure 13 provides its three-dimensional representation with respect to time. To obtain a more concise figure, the time axis is scaled by a factor of 0.01. The time associated with each node point can be explicitly determined. Both arms

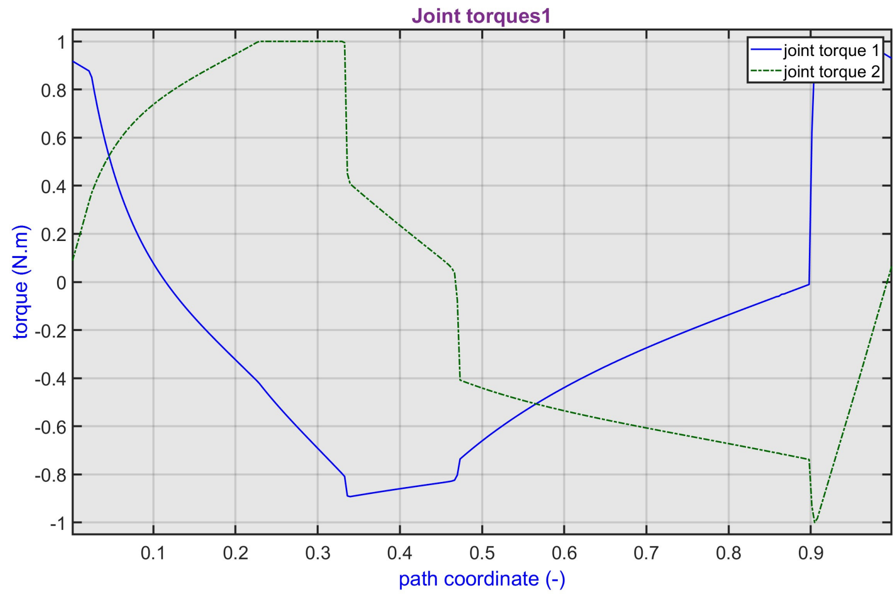

Figure 4: The joint torques of the first manipulators

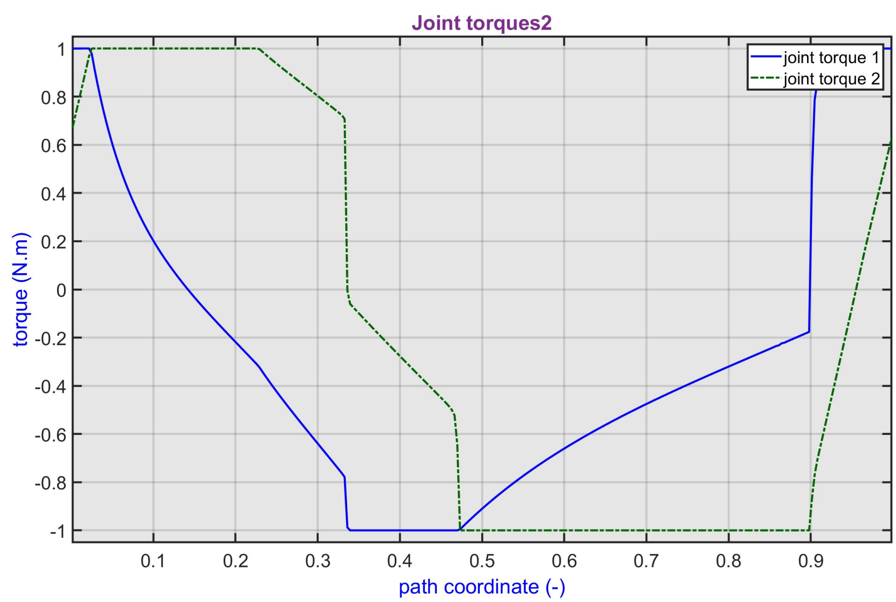

Figure 5: The joint torques of the second manipulators

move simultaneously from the initial to the terminal position, completing the trajectory in 36.0306 seconds. The objective function value at the optimal solution of problem (23) is 72.1471.

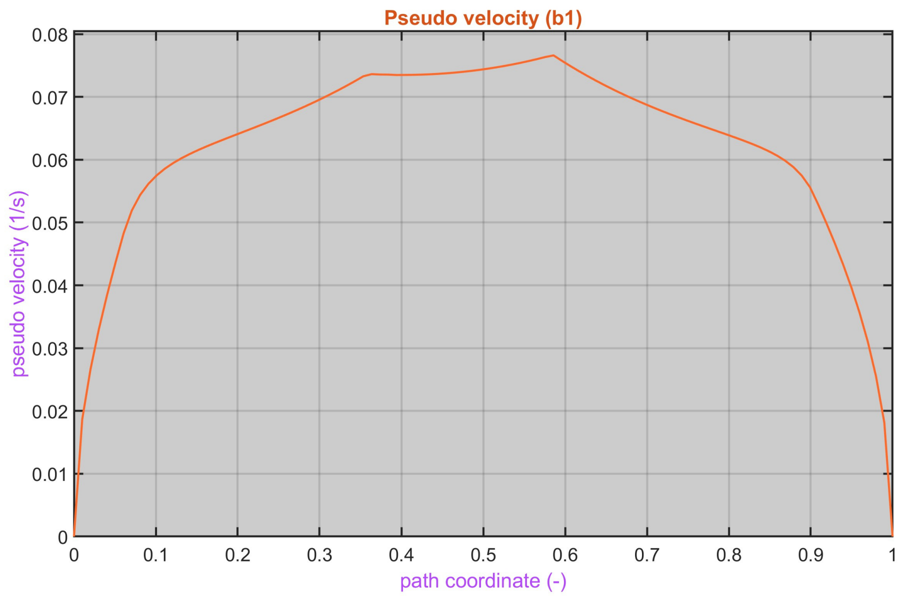

Figure 6: Joint velocities of first arm with respect to the path

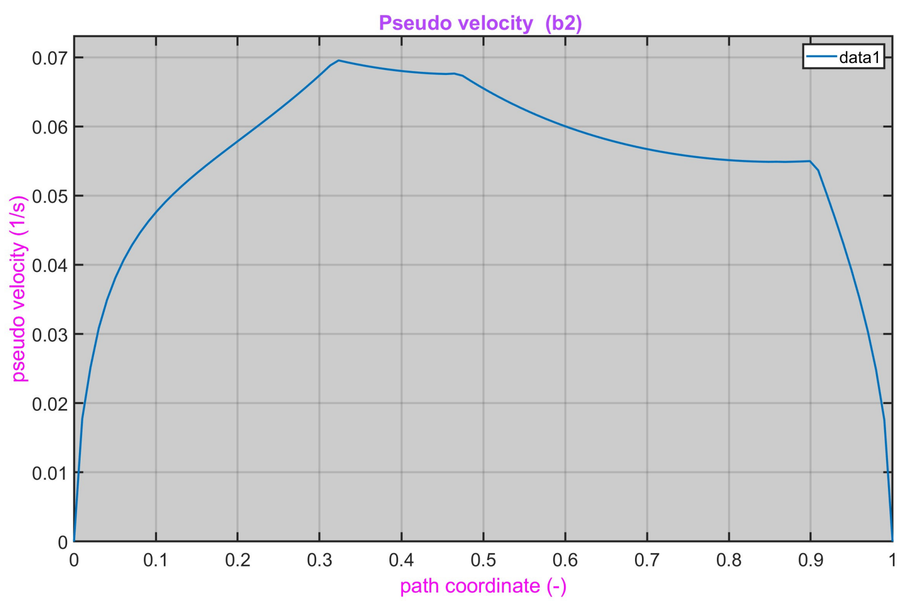

Figure 7: Joint velocities of second arm with respect to the path

6 Conclusion and suggestions

This article considers the problem of controlling collaborative robots, with a specific focus on the simultaneous minimization of time, thermal energy, and absolute value of the rate of torque change in

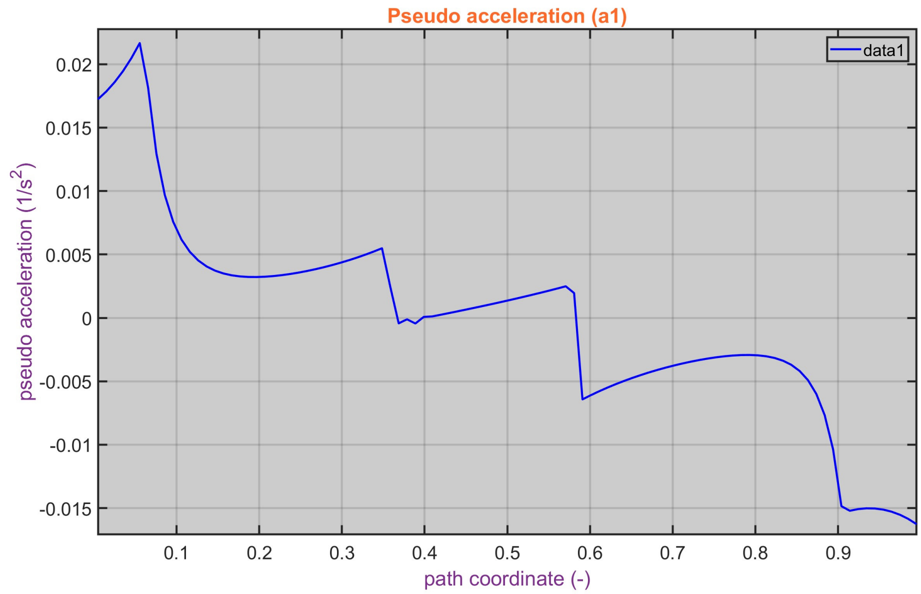

Figure 8: Joint accelerations of first arm with respect to the path

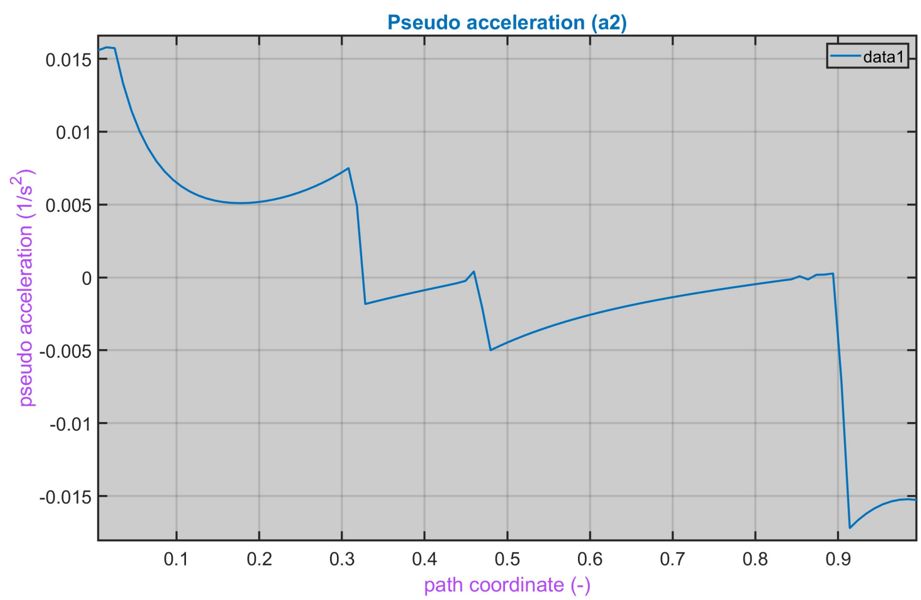

Figure 9: Joint accelerations of second arm with respect to the path

cooperative robots. The problem involves constraints on speed and acceleration and examines two scenarios, with and without a bar, for two two-jointed robots. The goal is to ensure their end-effectors move simultaneously along a common planar path. The challenges of co-location and co-time movement of the cooperative robots are modeled appropriately and implemented effectively. It is demonstrated that the

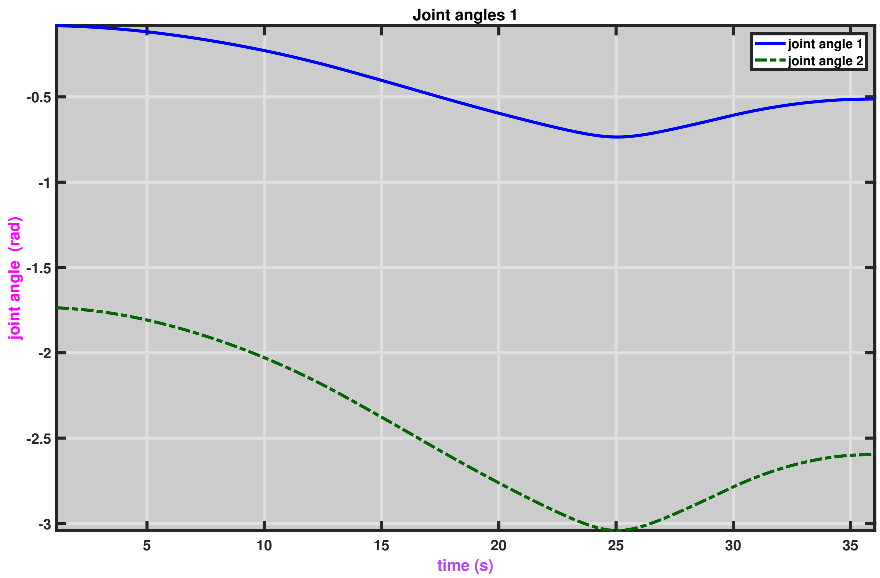

Figure 10: Angular displacements of first arm

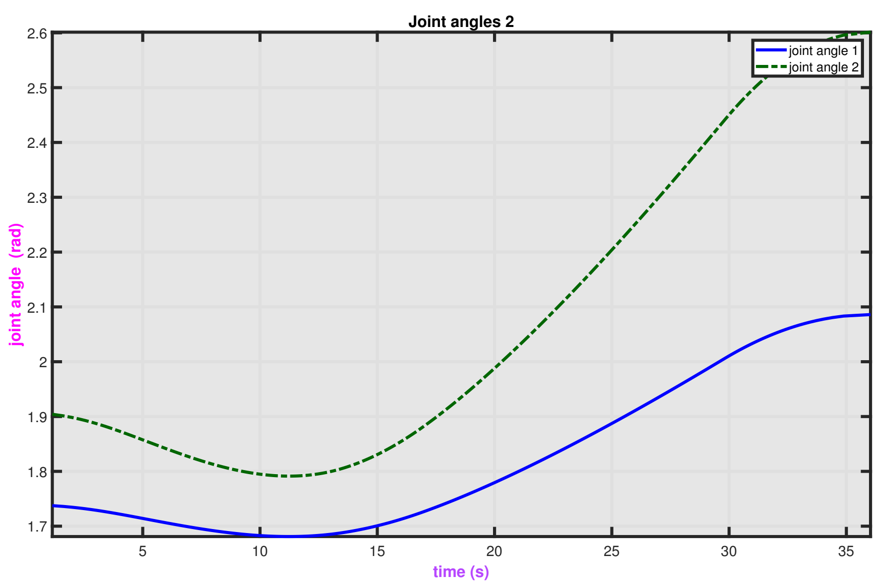

Figure 11: Angular displacements of second arm

problem can be formulated into a finite-dimensional conic optimization problem, which can be solved using conic optimization solvers. The numerical results indicate that the proposed method is reliable. In the simulations, it is observed that, for specific points of the path, joint torques in each arm reached saturation in both scenarios, confirming the optimality of the results.

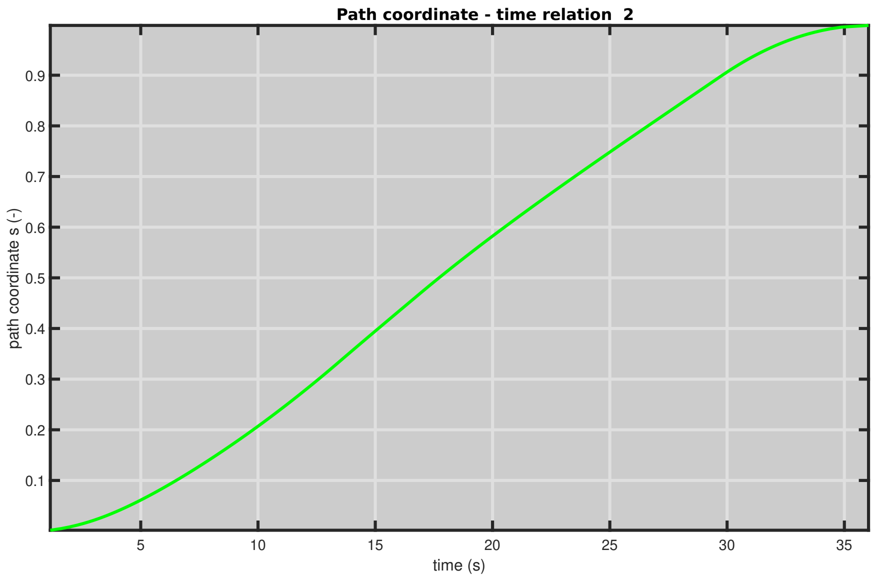

Figure 12: Arm path coordinates as a function of time

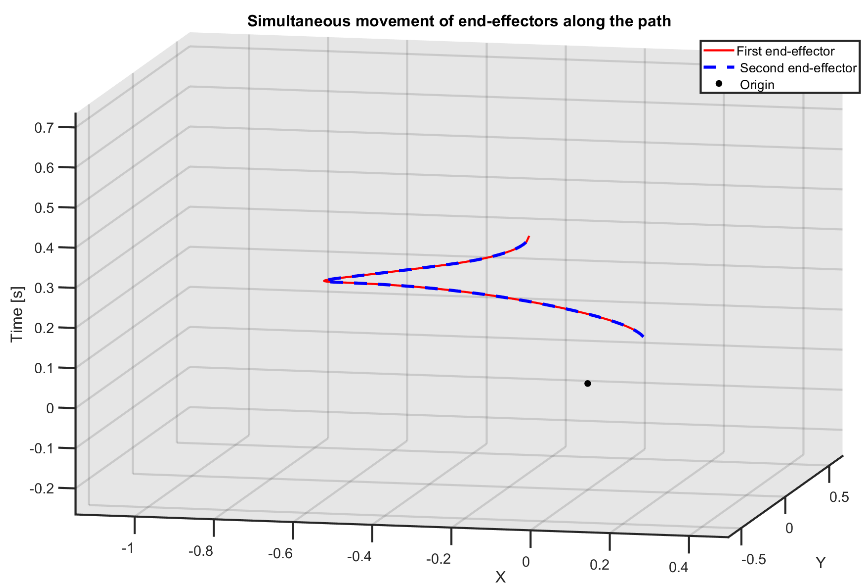

Figure 13: Arm path coordinates as a function of time

It is suggested to investigate the problem of minimum-time optimization on a non-flat path, taking into account friction and torque of the motors connected to the joints.

Table 2: Performance comparison of the proposed method

Weighting factors	Method in [2]	Proposed method
Thermal energy for first arm	231.8393	231.113
Thermal energy for second arm	363.7254	363.1151
Absolute value of the rate of torque change for first arm	11.6168	7.7488
Absolute value of the rate of torque change for second arm	10.5346	7.9576

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] T. Ardeshiri, M. Norrlof, J. Lofberg, A. Hansson, *Convex optimization approach for timeoptimal path tracking of robots with speed dependent constraints*, IFAC-Proc. **44** (2011) 14648-14653.

[2] C. Chung, S. Desa, C. Desilva, *Base Reaction Optimization of Redundant Manipulators for Space Application*, Journal contribution, Carnegie Mellon University, 2018.

[3] E. Gambao, M. Hernando, D. Surdilovic, *A new generation of collaborative robots for material handling*, Conference on Automation and Robotics **29** (2012) 1.

[4] M. Grant, S. Boyd, *CVX: Matlab software for disciplined convex programming*, CVX Research, 2014.

[5] H. Haghshenas, M. Norrlof, A. Hansson, *A convex optimization approach to the time-optimal path tracking problem for cooperative manipulators*, IEEE Trans. Autom. Control. **52** (2019) 400–405.

[6] A. Keshvarparast, D. Battini, O. Battaia, *Collaborative robots in manufacturing and assembly systems: literature review and future research agenda*, J. Intell. Manuf. **35** (2024) 2065–2118.

[7] C. Smith, Y. Karayiannidis, L. Nalpantidis, X. Gratal, P. Qi, D.V. Dimarogonas, D. Kragic, *Dual arm manipulation-A survey*, Robot. and Auton. Syst. **60** (2012) 1340–1353.

[8] D. Verscheure, B. Demeulenaere, J. Swevers, J. De Schutter, M. Diehl, *Time-optimal path tracking for robots: A convex optimization approach*, IEEE Trans. Autom. Control. **54** (2009) 2318–2327.