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Higher order numerical method for a class of singularly perturbed time dependent nonlinear reaction diffusion problems

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Abstract. Nonlinear science plays an important role in modern technology. Due to the limitations over the linear theories and the chaotic nature of the problems in this technological era, investigation of nonlinear problems has become indispensable to analyse the dynamics of complicated and multi-scale characteristics problems. This article aims at the analysis and implementation of a numerical method for a class of singularly perturbed time dependent nonlinear reaction diffusion problems. Together with the classical finite difference operators, a piecewise uniform Shishkin mesh in the spatial direction and a uniform mesh in the temporal direction are used to formulate a new numerical method to solve the class of problems. The method is proved to be second order convergent in space and first order convergent in time uniformly with respect to the perturbation parameter. Numerical experiments are included to support the theoretical results.

Keywords: Singularly perturbed time dependent nonlinear reaction diffusion problems, boundary layers, finite difference scheme, Shishkin mesh, parameter uniform convergence
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1 Introduction

Occurrence of small quantities in numerous physical problems induce a chaotic behaviour in the solution of the problems and/or in the derivatives of their solution. For an instance, the solution of physical problems involving Prandtl number, Reynolds number, Peclet number and Rayleigh number undergoes a rapid change in some portion of the domain of the problems; fluid which experiences a boundary layer

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flow is a precise example for this. Such problems could be modeled using differential equations with prescribed conditions. Precisely, a differential equation together with a small positive parameter in its higher order derivative and boundary/initial conditions find applications in fluid dynamics [8], control systems [17], chemical transport models [11] and bio-fluid mechanics [4]. Such differential equations are termed as Singular Perturbation Problems (SPPs).

Since SPPs exclude exact solutions and further classical numerical techniques fail to resolve SPPs, new techniques are developed. Most of the SPPs of practical importance are nonlinear in nature. In the literature, very few asymptotic and numerical techniques are available for nonlinear SPPs; asymptotic techniques provide only qualitative theory for such problems and there are many limitations over the numerical methods. It is highly complicated to establish a parameter uniform numerical method for nonlinear SPPs; since nonlinear nature of the problems, occurrence of small quantities and the presence of the boundary layers lead to much complexities in the computational aspect. Designing parameter uniform, robust and layer resolving numerical methods for these problems is still in nascent level.

Nonlinear science plays a vital role in modern technology [2, 14]. Due to the limitations over the linear theories, the analysis of nonlinear problems has become very essential to investigate the dynamics of complicated and multi scale characteristics problems.

Several non-classical numerical methods are available in the literature for linear SPPs whereas only few robust and parameter uniform numerical methods are available in the literature for nonlinear SPPs. Precisely, only a handful of numerical methods are available in the literature for nonlinear parabolic SPPs. To list a few, in [3] an adaptive space-time Newton-Galerkin approach involving Newton's method is developed for a nonlinear parabolic SPP. Article [5] deals with the construction of monotone iterative algorithms for aforesaid problem whereas article [6] deals with the construction of inexact monotone method for the same problem.

In [7], a discontinuous Galerkin finite element method is designed for a nonlinear parabolic SPP. Article [9] is devoted to the construction of a kind of non-linear analytical technique called as variational iteration method for a nonlinear parabolic SPP. Applicability of Lagrange's multiplier for a nonlinear parabolic SPP is demonstrated in [10, 16]. Ali developed a collocation method for aforesaid problem in [1]; further, in [12, 13], Kopteva and Linss used an elliptic reconstruction approach to solve the same problem.

In the present article, a class of nonlinear parabolic SPPs with Dirichlet boundary conditions is considered. An algorithm which utilizes the continuation method is used to compute the numerical approximations for the class of problems under consideration. Many of the numerical methods reported in the literature utilizes Newton's method as a nonlinear solver. It is worth observing that Newton's method renders it useless in the limiting case $\varepsilon \to 0$ [8]. It is to be noted that in the present study no artificial condition is imposed either on the perturbation parameter or on the boundary conditions. Moreover, the boundary conditions are not necessarily zero in the present study which induces the complexity of ensuring the compatibility conditions.

For any continuous function ϕ on a domain Λ, $||\phi||_\Lambda = \sup_{(x,t)\in\Lambda} |\phi(x,t)|$ and for any mesh function Φ, $||\Phi||_{\Lambda^{M,N}} = \max_{j,k} |\Phi(x_j,t_k)|$. Throughout this article C denotes a positive constant which is independent of the variables x,t and the parameters ε, M, N.

2 The main problem

The class of singularly perturbed time dependent nonlinear reaction diffusion problems under consideration is

$$\mathfrak{B}_\varepsilon u(x,t) = \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t,u(x,t)) = 0,\ (x,t) \in D, \tag{1}$$

$$\text{with } u(x,0) = B_b(x) \text{ on } \Gamma_b,\ u(0,t) = B_l(t) \text{ on } \Gamma_l,\ u(1,t) = B_r(t) \text{ on } \Gamma_r, \tag{2}$$

where $B_b(x), B_l(t), B_r(t)$ are known smooth functions, $0 < \varepsilon \le 1$, $D = \Omega_x \times \Omega_t$, $\Omega_x = (0,1)$, $\Omega_t = (0,T]$, $\overline{D} = D \cup \Gamma$, $\Gamma = \Gamma_b \cup \Gamma_l \cup \Gamma_r$, $\Gamma_b = \{(x,0) : 0 \le x \le 1\}$, $\Gamma_l = \{(0,t) : 0 \le t \le T\}$ and $\Gamma_r = \{(1,t) : 0 \le t \le T\}$. It is assumed that for all $(x,t,\theta(x,t)) \in \overline{D} \times \mathbb{R}$, $f(x,t,\theta(x,t))$ is sufficiently smooth such that $\dfrac{\partial f(x,t,\theta(x,t))}{\partial \theta} \ge \alpha > 0$.

Under suitable compatibility and continuity conditions, a unique solution $u(x,t)$ of (1)-(2) exists [15]. The reduced problem corresponding to (1)-(2) is

$$\frac{\partial u_0(x,t)}{\partial t} + f(x,t,u_0(x,t)) = 0, (x,t) \in [0,1] \times (0,T], \tag{3}$$

$$\text{with } u_0 = u \text{ on } \Gamma_b. \tag{4}$$

From (3)-(4), for $(x,t) \in \overline{D}$,

$$\begin{aligned} &|u_0(x,t)| \le C, \quad |\tfrac{\partial^k u_0(x,t)}{\partial t^k}| \le C, \quad |\tfrac{\partial^k u_0(x,t)}{\partial x^k}| \le C,\ 1 \le k \le 4, \\ &|\tfrac{\partial^k u_0(x,t)}{\partial x^{k-1} \partial t}| \le C,\ k = 2,3,4. \end{aligned} \tag{5}$$

In general $u \ne u_0$ on $\Gamma_l \cup \Gamma_r$. Hence the solution $u(x,t)$ of (1)-(2) may exhibit boundary layers near both Γ_l and Γ_r.

3 Theoretical results

Let $\mathfrak{B}_\varepsilon^*$ be a linear operator such that

$$\mathfrak{B}_\varepsilon^* \psi(x,t) = \frac{\partial \psi(x,t)}{\partial t} - \varepsilon \frac{\partial^2 \psi(x,t)}{\partial x^2} + a(x,t)\psi(x,t),$$

where $a(x,t)$ is a smooth function such that $a(x,t) \ge \alpha$ on $\overline{D}$.

Theorem 1. *Let χ be any function in the domain of $\mathfrak{B}_\varepsilon^*$ such that $\chi \ge 0$ on Γ, $\mathfrak{B}_\varepsilon^* \chi \ge 0$ on D then $\chi \ge 0$ on $\overline{D}$.*

Proof. Let $\tilde{x}, \tilde{t}$ be such that $\chi(\tilde{x},\tilde{t}) = \min\limits_{x,t} \chi(x,t)$ and suppose $\chi(\tilde{x},\tilde{t}) < 0$. Then $(\tilde{x},\tilde{t}) \notin \Gamma$, $\frac{\partial \chi(\tilde{x},\tilde{t})}{\partial t} = 0$ and $\frac{\partial^2 \chi(\tilde{x},\tilde{t})}{\partial x^2} \ge 0$. Let $(\tilde{x},\tilde{t}) \in D$ and consider

$$\mathfrak{B}_\varepsilon^* \chi(\tilde{x},\tilde{t}) = \frac{\partial \chi(\tilde{x},\tilde{t})}{\partial t} - \varepsilon \frac{\partial^2 \chi(\tilde{x},\tilde{t})}{\partial x^2} + a(\tilde{x},\tilde{t})\chi(\tilde{x},\tilde{t}) < 0,$$

a contradiction. Hence, $\chi(\tilde{x},\tilde{t}) \ge 0$ which proves the theorem. □

Theorem 2. *Let χ be any function in the domain of $\mathfrak{B}_\varepsilon^*$. Then for $(x,t) \in \overline{D}$,*

$$|\chi(x,t)| \leq ||\chi||_\Gamma + \tfrac{1}{\alpha}||\mathfrak{B}_\varepsilon^*\chi||_D.$$

Proof. Let $\Theta = ||\chi||_\Gamma + \frac{1}{\alpha}||\mathfrak{B}_\varepsilon^*\chi||_D$ and $\Psi^\pm(x,t) = \Theta \pm \chi(x,t)$. Then $\Psi^\pm \geq 0$ on Γ and

$$\begin{array}{rcl} \mathfrak{B}_\varepsilon^*\Psi^\pm(x,t) & = & \Theta a(x,t) \pm \mathfrak{B}_\varepsilon^*\chi(x,t) \\ & \geq & \frac{1}{\alpha}||\mathfrak{B}_\varepsilon^*\chi||_D\, a(x,t) \pm \mathfrak{B}_\varepsilon^*\chi(x,t) \\ & \geq & 0 \text{ on } D. \end{array}$$

Hence from Theorem 1, $\Psi^\pm \geq 0$ on $\overline{D}$, which proves the theorem. □

Decompose $u(x,t)$ of (1)-(2) into $q(x,t)$ and $s(x,t)$ such that $u(x,t) = q(x,t) + s(x,t)$ where

$$\frac{\partial q(x,t)}{\partial t} - \varepsilon\frac{\partial^2 q(x,t)}{\partial x^2} + f(x,t,q(x,t)) = 0 \text{ on } D, \tag{6}$$

$$\text{with } q = u_0 \text{ on } \Gamma \tag{7}$$

and

$$\frac{\partial s(x,t)}{\partial t} - \varepsilon\frac{\partial^2 s(x,t)}{\partial x^2} + f(x,t,q(x,t)+s(x,t)) - f(x,t,q(x,t)) = 0 \text{ on } D, \tag{8}$$

$$\text{with } s = u - q \text{ on } \Gamma. \tag{9}$$

From (9), on Γ_b, $s = u - q = u - u_0 = 0$.

3.1 Bounds on $q(x,t)$ and its derivatives

Theorem 3. *For all $(x,t) \in \overline{D}$,*

$$\begin{array}{l} |q(x,t)| \leq C, \quad |\frac{\partial^k q(x,t)}{\partial t^k}| \leq C, \quad |\frac{\partial^k q(x,t)}{\partial x^k}| \leq C, \;\; k=1,2, \\ \left|\frac{\partial^k q(x,t)}{\partial x^k}\right| \leq C\left(1+\varepsilon^{1-\frac{k}{2}}\right), \;\; k=3,4, \quad |\frac{\partial^k q(x,t)}{\partial x^{k-1}\partial t}| \leq C, \;\; k=2,3. \end{array} \tag{10}$$

Proof. From (6) and (3),

$$\begin{array}{rcl} \mathfrak{B}_\varepsilon^* q(x,t) & = & \dfrac{\partial q(x,t)}{\partial t} - \varepsilon\dfrac{\partial^2 q(x,t)}{\partial x^2} + a_1(x,t)\,q(x,t) \\ & = & \dfrac{\partial u_0(x,t)}{\partial t} + a_1(x,t)\,u_0(x,t) \\ & = & f_1(x,t), \end{array} \tag{11}$$

where $a_1(x,t) = \dfrac{\partial f(x,t,\theta_1(x,t))}{\partial u}$ is an intermediate value. To establish the required results, decompose $q(x,t)$ further as follows

$$q(x,t) = u_0(x,t) + \varepsilon\, q_1(x,t), \tag{12}$$

where $u_0(x,t)$ is the solution of the reduced problem (3)-(4) and $q_1(x,t)$ is the solution of the following problem

$$\mathfrak{B}_\varepsilon^* q_1(x,t) = \frac{\partial q_1(x,t)}{\partial t} - \varepsilon\frac{\partial^2 q_1(x,t)}{\partial x^2} + a_1(x,t)\,q_1(x,t) = \frac{\partial^2 u_0(x,t)}{\partial x^2} \text{ on } D, \tag{13}$$

$$\text{with } q_1 = 0 \text{ on } \Gamma. \tag{14}$$

Now the bounds for $q_1(x,t)$ and its derivatives are established. Let $A^{\pm}(x,t) = C \pm q_1(x,t)$. Then $A^{\pm} = C \geq 0$ on Γ and for a suitable choice of C, for $(x,t) \in D$,

$$\mathfrak{B}^*_{\varepsilon} A^{\pm}(x,t) = C a_1(x,t) \pm \mathfrak{B}^*_{\varepsilon} q_1(x,t) \geq 0.$$

Hence, using Theorem 1 with $A^{\pm}(x,t)$, for any $(x,t) \in \overline{D}$, $|q_1(x,t)| \leq C$. Differentiating (13) partially with respect to t and rearranging,

$$\mathfrak{B}^*_{\varepsilon} \frac{\partial q_1(x,t)}{\partial t} = \frac{\partial^3 u_0(x,t)}{\partial x^2 \partial t} - \frac{\partial a_1(x,t)}{\partial t} q_1(x,t)$$

and then using Theorem 2, for any $(x,t) \in \overline{D}$, $\left|\frac{\partial q_1(x,t)}{\partial t}\right| \leq C$. In the same way it can be established that for any $(x,t) \in \overline{D}$, $\left|\frac{\partial^2 q_1(x,t)}{\partial t^2}\right| \leq C$. Using the mean value theorem to $\frac{\partial q_1}{\partial x}$, for some $y \in I = [\eta, \eta + \sqrt{\varepsilon}]$,

$$\frac{\partial q_1(y,t)}{\partial x} = \frac{q_1(\eta + \sqrt{\varepsilon}, t) - q_1(\eta, t)}{\sqrt{\varepsilon}},$$

which gives

$$\left|\frac{\partial q_1(y,t)}{\partial x}\right| \leq 2\varepsilon^{-1/2} \|q_1\|_I \leq C\varepsilon^{-1/2}.$$

Now consider

$$\begin{aligned} \frac{\partial q_1(x,t)}{\partial x} &= \frac{\partial q_1(y,t)}{\partial x} + \int_y^x \frac{\partial^2 q_1(s,t)}{\partial x^2} ds \quad (15)\\ &= \frac{\partial q_1(y,t)}{\partial x} + \varepsilon^{-1} \int_y^x \left(\frac{\partial q_1(s,t)}{\partial t} + a_1(s,t) q_1(s,t) - \varepsilon \frac{\partial^2 u_0(s,t)}{\partial x^2} \right) ds. \quad (16) \end{aligned}$$

From (15), for any $(x,t) \in \overline{D}$, $\left|\frac{\partial q_1(x,t)}{\partial x}\right| \leq C\varepsilon^{-1/2}$. In the same way it can be established that for any $(x,t) \in \overline{D}$, $|\frac{\partial^k q_1(x,t)}{\partial x^{k-1} \partial t}| \leq C$, $k = 2,3$. From (13), for any $(x,t) \in \overline{D}$, $\left|\frac{\partial^2 q_1(x,t)}{\partial x^2}\right| \leq C\varepsilon^{-1}$. Differentiating (13) partially with respect to x once and twice and then using the required results, for any $(x,t) \in \overline{D}$, $\left|\frac{\partial^3 q_1(x,t)}{\partial x^3}\right| \leq C\varepsilon^{-3/2}$ and $\left|\frac{\partial^4 q_1(x,t)}{\partial x^4}\right| \leq C\varepsilon^{-2}$. Finally using the appropriate bounds of $u_0(x,t), q_1(x,t)$ and their derivatives with (12), the bounds of $q(x,t)$ and its derivatives follow. □

3.2 Bounds on $s(x,t)$ and its derivatives

From (8),

$$\mathfrak{B}^*_{\varepsilon} s(x,t) = \frac{\partial s(x,t)}{\partial t} - \varepsilon \frac{\partial^2 s(x,t)}{\partial x^2} + a_2(x,t) s(x,t) = 0, \tag{17}$$

where $a_2(x,t) = \dfrac{\partial f(x,t,\theta_2(x,t))}{\partial u}$ is an intermediate value. Decompose $s(x,t)$ as follows:

$$s(x,t) = s^l(x,t) + s^r(x,t), \tag{18}$$

where $\mathfrak{B}_{\varepsilon}^{*}s^{l}(x,t)=0$ on D with $s^{l}=s$ on Γ_{l}, $s^{l}=0$ on $\Gamma_{b}\cup\Gamma_{r}$ and $\mathfrak{B}_{\varepsilon}^{*}s^{r}(x,t)=0$ on D with $s^{r}=s$ on Γ_{r}, $s^{r}=0$ on $\Gamma_{b}\cup\Gamma_{l}$. Let, for any $x\in[0,1]$,

$$\mathbb{B}^{l}(x)=e^{-x\sqrt{\alpha}/\sqrt{\varepsilon}}\ \text{ and }\ \mathbb{B}^{r}(x)=e^{-(1-x)\sqrt{\alpha}/\sqrt{\varepsilon}}.$$

Theorem 4. *For all* $(x,t)\in\overline{D}$,

$$\begin{aligned}|\tfrac{\partial^{k}s^{l}(x,t)}{\partial t^{k}}| &\leq C\mathbb{B}^{l}(x),\ 0\leq k\leq 2,\\ |\tfrac{\partial^{k}s^{l}(x,t)}{\partial x^{k}}| &\leq C\varepsilon^{-k/2}\mathbb{B}^{l}(x),\ 1\leq k\leq 4.\end{aligned}$$

Analogous results hold for s^{r} and its derivatives with x replaced by $1-x$.

Proof. Let $E_{1}^{\pm}(x,t)=C\mathbb{B}^{l}(x)\pm s^{l}(x,t)$, $(x,t)\in\overline{D}$. Then for a proper choice of C, $E_{1}^{\pm}\geq 0$ on Γ and for $(x,t)\in D$,

$$\mathfrak{B}_{\varepsilon}^{*}E_{1}^{\pm}(x,t)=C\,(a_{2}(x,t)-\alpha)\,\mathbb{B}^{l}(x)\geq 0.$$

Thus using Theorem 1 with $E_{1}^{\pm}$, $E_{1}^{\pm}\geq 0$ on $\overline{D}$ which gives for any $(x,t)\in\overline{D}$, $|s^{l}(x,t)|\leq C\mathbb{B}^{l}(x)$. Now the bound on $\frac{\partial s^{l}}{\partial t}$ is established. Let $E_{2}^{\pm}(x,t)=C\mathbb{B}^{l}(x)\pm\frac{\partial s^{l}(x,t)}{\partial t}$, $(x,t)\in\overline{D}$. Note that $|\frac{\partial s^{l}(x,t)}{\partial t}|_{\Gamma}\leq C$. Thus for a proper choice of C, $E_{2}^{\pm}\geq 0$ on Γ. Differentiate $\mathfrak{B}_{\varepsilon}^{*}s^{l}=0$ partially with respect to t once and rearrange to get

$$\mathfrak{B}_{\varepsilon}^{*}\frac{\partial s^{l}}{\partial t}=\frac{\partial^{2}s^{l}}{\partial t^{2}}-\varepsilon\frac{\partial^{3}s^{l}}{\partial x^{2}\partial t}+a_{2}\frac{\partial s^{l}}{\partial t}=-\frac{\partial a_{2}}{\partial t}s^{l}. \tag{19}$$

From (19), for any $(x,t)\in D$,

$$\left|\mathfrak{B}_{\varepsilon}^{*}\frac{\partial s^{l}(x,t)}{\partial t}\right|\leq C\mathbb{B}^{l}(x).$$

Now for $(x,t)\in D$,

$$\mathfrak{B}_{\varepsilon}^{*}E_{2}^{\pm}(x,t)=C\,(a_{2}(x,t)-\alpha)\,\mathbb{B}^{l}(x)\pm\mathfrak{B}_{\varepsilon}^{*}\frac{\partial s^{l}(x,t)}{\partial t}\geq 0.$$

Thus using Theorem 1 with $E_{2}^{\pm}$, $E_{2}^{\pm}\geq 0$ on $\overline{D}$ which gives for any $(x,t)\in\overline{D}$, $|\frac{\partial s^{l}(x,t)}{\partial t}|\leq C\mathbb{B}^{l}(x)$. In the same way the bound on $\frac{\partial^{2}s^{l}(x,t)}{\partial t^{2}}$ can be established. Following the technique used to bound $\frac{\partial q_{1}(x,t)}{\partial x}$, for $(x,t)\in\overline{D}$, $\left|\frac{\partial s^{l}(x,t)}{\partial x}\right|\leq C\varepsilon^{-1/2}$. Further, following the technique used to bound $\frac{\partial s^{l}(x,t)}{\partial t}$ with the function $E_{3}^{\pm}(x,t)=C\varepsilon^{-1/2}\mathbb{B}^{l}(x)\pm\frac{\partial s^{l}(x,t)}{\partial x}$, for $(x,t)\in\overline{D}$, $\left|\frac{\partial s^{l}(x,t)}{\partial x}\right|\leq C\varepsilon^{-1/2}\mathbb{B}^{l}(x)$. Now from the equation $\mathfrak{B}_{\varepsilon}^{*}s^{l}(x,t)=0$, for $(x,t)\in\overline{D}$, $\left|\frac{\partial^{2}s^{l}(x,t)}{\partial x^{2}}\right|\leq C\varepsilon^{-1}\mathbb{B}^{l}(x)$. Following the technique used to bound $\frac{\partial s^{l}(x,t)}{\partial x}$, for $(x,t)\in\overline{D}$, $\left|\frac{\partial^{2}s^{l}(x,t)}{\partial x\partial t}\right|\leq C\varepsilon^{-1/2}\mathbb{B}^{l}(x)$ and $\left|\frac{\partial^{3}s^{l}(x,t)}{\partial x^{2}\partial t}\right|\leq C\varepsilon^{-1}\mathbb{B}^{l}(x)$. Differentiating $\mathfrak{B}_{\varepsilon}^{*}s^{l}(x,t)=0$ once and twice partially with respect to x and using the required bounds, the bounds on $\frac{\partial^{3}s^{l}(x,t)}{\partial x^{3}}$ and $\frac{\partial^{4}s^{l}(x,t)}{\partial x^{4}}$ follow. □

4 Mesh and discrete problem

4.1 The rectangular mesh

A rectangular mesh $\overline{D}^{M,N}$ is now constructed on $\overline{D}$. Let $D_x^N = \{x_j\}_{j=1}^{N-1}$, $\overline{D}_x^N = \{x_j\}_{j=0}^{N}$, $D_t^M = \{t_k\}_{k=1}^{M}$, $\overline{D}_t^M = \{t_k\}_{k=0}^{M}$, $D^{M,N} = D_t^M \times D_x^N$, $\overline{D}^{M,N} = \overline{D}_t^M \times \overline{D}_x^N$ and $\Gamma^{M,N} = \overline{D}^{M,N} \cap \Gamma$. On spatial domain $[0,1]$ a piecewise uniform Shishkin mesh $\overline{D}_x^N$ is constructed as follows, $[0,1] = [0,\lambda] \cup (\lambda, 1-\lambda] \cup (1-\lambda, 1]$ such that

$$\lambda = \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon}}{\sqrt{\alpha}} \ln N\right\}.$$

On both $[0,\lambda]$ and $(1-\lambda,1]$, a uniform mesh with $\frac{N}{4}$ mesh intervals is placed whereas on $(\lambda, 1-\lambda]$, a uniform mesh with $\frac{N}{2}$ mesh intervals is placed. Let h_1 be the step length in both $[0,\lambda]$ and $[1-\lambda,1]$ and h_2 be that of $[\lambda, 1-\lambda]$. Then $h_1 = \dfrac{4\lambda}{N}$, $h_2 = \dfrac{2(1-2\lambda)}{N}$, and

$$x_j = \begin{cases} jh_1, & \text{for } 0 \le j \le \dfrac{N}{4}, \\ \lambda + \left(j - \dfrac{N}{4}\right) h_2, & \text{for } \dfrac{N}{4} + 1 \le j \le \dfrac{3N}{4}, \\ (1-\lambda) + \left(j - \dfrac{3N}{4}\right) h_1, & \text{for } \dfrac{3N}{4} + 1 \le j \le N. \end{cases}$$

On the other hand, on the temporal domain $[0,T]$ a uniform mesh $\overline{D}_t^M$ is placed. Thus, for $0 \le k \le M$, $t_k = kh_t$ where $h_t = \frac{T}{M}$.

4.2 The discrete problem

The discrete problem corresponding to (1)-(2) is, for $(x_j, t_k) \in D^{M \times N}$,

$$\begin{aligned} &\mathfrak{B}_\varepsilon^{M,N} U(x_j,t_k) = D_t^- U(x_j,t_k) - \varepsilon\, \delta_x^2 U(x_j,t_k) + f(x_j,t_k,U(x_j,t_k)) = 0, \\ &\text{with } \ U = u \ \text{ on } \Gamma^{M \times N}. \end{aligned} \tag{20}$$

Here
$D_t^- U(x_j,t_k) = \dfrac{U(x_j,t_k) - U(x_j,t_{k-1})}{h_t}$, $\delta_x^2 U(x_j,t_k) = \dfrac{(D_x^+ - D_x^-)U(x_j,t_k)}{\overline{h}_j}$,
$D_x^- U(x_j,t_k) = \dfrac{U(x_j,t_k) - U(x_{j-1},t_k)}{h_j}$, $D_x^+ U(x_j,t_k) = \dfrac{U(x_{j+1},t_k) - U(x_j,t_k)}{h_{j+1}}$, $h_j = x_j - x_{j-1}$ and $\overline{h}_j = \dfrac{h_{j+1} + h_j}{2}$ with $\overline{h}_0 = \dfrac{h_1}{2}$ and $\overline{h}_N = \dfrac{h_N}{2}$.

5 Error analysis

Let $P_1(x_j,t_k)$ and $P_2(x_j,t_k)$ be two mesh functions such that $P_1 = P_2$ on $\Gamma^{M\times N}$. Consider, for $(x_j,t_k) \in D^{M,N}$,

$$\begin{aligned}(\mathfrak{B}_\varepsilon^{M,N}P_1 - \mathfrak{B}_\varepsilon^{M,N}P_2)(x_j,t_k) &= D_t^-(P_1-P_2)(x_j,t_k) - \varepsilon\,\delta_x^2(P_1-P_2)(x_j,t_k)\\ &\quad + f(x_j,t_k,P_1(x_j,t_k)) - f(x_j,t_k,P_2(x_j,t_k))\\ &= D_t^-(P_1-P_2)(x_j,t_k) - \varepsilon\,\delta_x^2(P_1-P_2)(x_j,t_k)\\ &\quad + a_3(x_j,t_k)(P_1-P_2)(x_j,t_k)\\ &= \mathfrak{B}_\varepsilon^{M^*,N^*}(P_1-P_2)(x_j,t_k),\end{aligned} \tag{21}$$

where $a_3(x_j,t_x)$ is an intermediate value and $\mathfrak{B}_\varepsilon^{M^*,N^*}$ is the Frechet derivative of $\mathfrak{B}_\varepsilon^{M,N}$.

Theorem 5. *Let Ψ be any mesh function such that $\Psi \geq 0$ on $\Gamma^{M,N}$, $\mathfrak{B}_\varepsilon^{M^*,N^*}\Psi \geq 0$ on $D^{M,N}$, then $\Psi \geq 0$ on $\overline{D}^{M,N}$.*

Proof. Similar to Theorem 1. □

Theorem 6. *Let Ψ be any mesh function. Then for $(x_j,t_k) \in \overline{D}^{M,N}$,*

$$|\Psi(x_j,t_k)| \leq ||\Psi||_{\Gamma^{M,N}} + \frac{1}{\alpha}||\mathfrak{B}_\varepsilon^{M^*,N^*}\Psi||_{D^{M,N}}.$$

Proof. Similar to Theorem 2. □

Using Theorem 6 with (21),

$$\begin{aligned}|(P_1-P_2)(x_j,t_k)| &\leq ||P_1-P_2||_{\Gamma^{M,N}} + \frac{1}{\alpha}||\mathfrak{B}_\varepsilon^{M^*,N^*}(P_1-P_2)||_{D^{M,N}}\\ &= \frac{1}{\alpha}||\mathfrak{B}_\varepsilon^{M^*,N^*}(P_1-P_2)||_{D^{M,N}}\\ &= \frac{1}{\alpha}|(\mathfrak{B}_\varepsilon^{M,N}P_1 - \mathfrak{B}_\varepsilon^{M,N}P_2)(x_j,t_k)|_{D^{M,N}}.\end{aligned} \tag{22}$$

As in the continuous case, decompose U into Q and S such that $U = Q+S$ where Q is the solution of

$$\begin{aligned}\mathfrak{B}_\varepsilon^{M^*,N^*}Q(x_j,t_k) &= D_t^-Q(x_j,t_k) - \varepsilon\,\delta_x^2 Q(x_j,t_k) + a_1(x_j,t_k)Q(x_j,t_k)\\ &= f_1(x_j,t_k),\ (x_j,t_k) \in D^{M,N},\end{aligned} \tag{23}$$

$$\text{with } Q = q \text{ on } \Gamma^{M,N} \tag{24}$$

and S is the solution of

$$\begin{aligned}\mathfrak{B}_\varepsilon^{M^*,N^*}S(x_j,t_k) &= D_t^-S(x_j,t_k) - \varepsilon\,\delta_x^2 S(x_j,t_k) + a_2(x_j,t_k)S(x_j,t_k)\\ &= 0,\ (x_j,t_k) \in D^{M,N},\end{aligned} \tag{25}$$

$$\text{with } S = s \text{ on } \Gamma^{M,N}. \tag{26}$$

Note that for any smooth function $\eta(x_j,t_k)$,

$$\left|\left(\frac{\partial}{\partial t}-D_t^-\right)\eta(x_j,t_k)\right| \le C(t_k-t_{k-1})\max_{\zeta\in T_k}\left|\frac{\partial^2\eta(x_j,\zeta)}{\partial t^2}\right|, \tag{27}$$

$$\left|\left(\frac{\partial^2}{\partial x^2}-\delta_x^2\right)\eta(x_j,t_k)\right| \le C(x_{j+1}-x_{j-1})^2\max_{\zeta\in X_j}\left|\frac{\partial^4\eta(\zeta,t_k)}{\partial x^4}\right| \tag{28}$$

and

$$\left|\delta_x^2\eta(x_j,t_k)\right| \le \max_{\zeta\in X_j}\left|\frac{\partial^2\eta(\zeta,t_k)}{\partial x^2}\right|, \tag{29}$$

where $T_k=[t_{k-1},t_k]$ and $X_j=[x_{j-1},x_{j+1}]$.

5.1 The local truncation error in $Q-q$

Theorem 7. *Let q be the solution of (11),(7) and Q be that of (23),(24) then for any $(x_j,t_k)\in D^{M,N}$,*

$$\left|\mathfrak{B}_\varepsilon^{M^*,N^*}(Q-q)(x_j,t_k)\right| \quad\le\quad C(M^{-1}+N^{-2}). \tag{30}$$

Proof. From (11) and (23), for $(x_j,t_k)\in D^{M,N}$,

$$\begin{aligned}\mathfrak{B}_\varepsilon^{M^*,N^*}(Q-q)(x_j,t_k) &= f_1(x_j,t_k)-\mathfrak{B}_\varepsilon^{M^*,N^*}q(x_j,t_k)\\ &= \left(\mathfrak{B}_\varepsilon^{*}-\mathfrak{B}_\varepsilon^{M^*,N^*}\right)q(x_j,t_k)\\ &= \left(\tfrac{\partial}{\partial t}-D_t^-\right)q(x_j,t_k)-\varepsilon\left(\tfrac{\partial^2}{\partial x^2}-\delta_x^2\right)q(x_j,t_k).\end{aligned} \tag{31}$$

Since $t_k-t_{k-1}=TM^{-1}$ for all k and $x_{j+1}-x_{j-1}\le 2N^{-1}$ for any choice of λ, from (27), (28) and (31), for any $(x_j,t_k)\in D^{M,N}$,

$$\left|\mathfrak{B}_\varepsilon^{M^*,N^*}(Q-q)(x_j,t_k)\right| \le C\left(M^{-1}\left|\frac{\partial^2 q}{\partial t^2}\right|_{T_k}+\varepsilon N^{-2}\left|\frac{\partial^4 q}{\partial x^4}\right|_{X_j}\right). \tag{32}$$

Using Theorem 3 with (32), for any $(x_j,t_k)\in D^{M,N}$,

$$\left|\mathfrak{B}_\varepsilon^{M^*,N^*}(Q-q)(x_j,t_k)\right| \quad\le\quad C(M^{-1}+N^{-2}).$$

□

5.2 The local truncation error in $S-s$

Theorem 8. *Let s be the solution of (17),(9) and S be that of (25),(26), then for any $(x_j,t_k)\in D^{M,N}$*

$$\left|\mathfrak{B}_\varepsilon^{M^*,N^*}(S-s)(x_j,t_k)\right| \quad\le\quad C(M^{-1}+(N^{-1}\ln N)^2). \tag{33}$$

Proof. From (17) and (25), for $(x_j,t_k) \in D$,

$$\begin{aligned} \mathfrak{B}_\varepsilon^{M^*,N^*}(S-s)(x_j,t_k) &= 0-\mathfrak{B}_\varepsilon^{M^*,N^*}s(x_j,t_k) \\ &= \left(\mathfrak{B}_\varepsilon^{*}-\mathfrak{B}_\varepsilon^{M^*,N^*}\right)s(x_j,t_k) \\ &= \left(\tfrac{\partial}{\partial t}-D_t^-\right)s(x_j,t_k)-\varepsilon\left(\tfrac{\partial^2}{\partial x^2}-\delta_x^2\right)s(x_j,t_k). \end{aligned} \tag{34}$$

The result in (33) is established through $S-s^l$ and $S-s^r$. Following (34),

$$\mathfrak{B}_\varepsilon^{M^*,N^*}(S-s^l)(x_j,t_k) = \left(\frac{\partial}{\partial t}-D_t^-\right)s^l(x_j,t_k)-\varepsilon\left(\frac{\partial^2}{\partial x^2}-\delta_x^2\right)s^l(x_j,t_k). \tag{35}$$

The result in (33) similar for $S-s^l$ is proved for the choices $\lambda=\frac{1}{4}$ and $\lambda=\frac{2\sqrt{\varepsilon}}{\sqrt{\alpha}}\ln N$ separately. First let $\lambda=\frac{1}{4}$. In this case $\varepsilon^{-1}\le C(\ln N)^2$ and $x_{j+1}-x_{j-1}=N^{-1}$. Using (27), (28) and (35), for any $(x_j,t_k)\in D^{M,N}$

$$\left|\mathfrak{B}_\varepsilon^{M^*,N^*}(S-s^l)(x_j,t_k)\right| \le C\left(M^{-1}\left|\frac{\partial^2 s^l}{\partial t^2}\right|_{T_k}+\varepsilon N^{-2}\left|\frac{\partial^4 s^l}{\partial x^4}\right|_{X_j}\right). \tag{36}$$

Using Theorem 4 with (36), for any $(x_j,t_k)\in D^{M,N}$,

$$\begin{aligned} \left|\mathfrak{B}_\varepsilon^{M^*,N^*}(S-s^l)(x_j,t_k)\right| &\le C\left(M^{-1}+N^{-2}\varepsilon^{-1}\right) \\ &\le C\left(M^{-1}+(N^{-1}\ln N)^2\right). \end{aligned} \tag{37}$$

Now consider the second case $\lambda=\frac{2\sqrt{\varepsilon}}{\sqrt{\alpha}}\ln N$. For this choice of λ, the result is proved on the sub-domains $[0,\lambda]\times[0,T]$, $[1-\lambda,1]\times[0,T]$ and $[\lambda,1-\lambda]\times[0,T]$ separately. The spatial mesh length on both $[0,\lambda]$ and $[1-\lambda,1]$ is $h_1=4N^{-1}\lambda=\frac{8\sqrt{\varepsilon}N^{-1}\ln N}{\sqrt{\alpha}}$. Using (27) and (28) with (35) and then Theorem 4 for both $x_j\in[0,\lambda]$ and $x_j\in[1-\lambda,1]$ and $t_k\in D_t^M$, we have

$$\left|\mathfrak{B}_\varepsilon^{M^*,N^*}(S-s^l)(x_j,t_k)\right| \le C(M^{-1}+(N^{-1}\ln N)^2). \tag{38}$$

Finally consider $[\lambda,1-\lambda]$. In this case the spatial mesh length $h_2=\frac{2(1-2\lambda)}{N}\le CN^{-1}$. Using (27) and (28) with (35) and then Theorem 4, we get

$$\left|\mathfrak{B}_\varepsilon^{M^*,N^*}(S-s^l)(x_j,t_k)\right| \le C\left(M^{-1}+\varepsilon(x_{j+1}-x_{j-1})^2\left|\frac{\partial s^l}{\partial x^4}\right|_{X_j}\right). \tag{39}$$

Also using (27) and (29) with (35) and then Theorem 4,

$$\left|\mathfrak{B}_\varepsilon^{M^*,N^*}(S-s^l)(x_j,t_k)\right| \le C\left(M^{-1}+\varepsilon\left|\frac{\partial^2 s^l}{\partial x^2}\right|_{X_j}\right). \tag{40}$$

Note that

$$\max_{x \in X_j} \left\{ \mathbb{B}^l(x), \mathbb{B}^r(x) \right\} \leq CN^{-2}, \ \lambda \leq x_j \leq 1 - \lambda. \tag{41}$$

To establish the required result in the final case, consider the two inequalities $\varepsilon \geq N^{-2}$ and $\varepsilon \leq N^{-2}$. Suppose $\varepsilon \geq N^{-2}$. For aforementioned inequality, using Theorem 4 and (41) with (39), for any $x_j \in [\lambda, 1-\lambda]$ and $t_k \in D_t^M$,

$$\left| \mathfrak{B}_\varepsilon^{M^*,N^*}(S - s^l)(x_j, t_k) \right| \leq C(M^{-1} + N^{-2}). \tag{42}$$

Now consider the other inequality $\varepsilon \leq N^{-2}$. For this choice, using Theorem 4 and (41) with (40), for any $x_j \in [\lambda, 1-\lambda]$ and $t_k \in D_t^M$,

$$\left| \mathfrak{B}_\varepsilon^{M^*,N^*}(S - s^l)(x_j, t_k) \right| \leq C(M^{-1} + N^{-2}). \tag{43}$$

From (37), (38), (42) and (43), either $\lambda = \frac{1}{4}$ or $\lambda = \frac{2\sqrt{\varepsilon}}{\sqrt{\alpha}} \ln N$, for any $(x_j, t_k) \in D^{M,N}$,

$$\left| \mathfrak{B}_\varepsilon^{M^*,N^*}(S - s^l)(x_j, t_k) \right| \leq C(M^{-1} + (N^{-1}\ln N)^2). \tag{44}$$

Following similar technique, the same bound for $\mathfrak{B}_\varepsilon^{M^*,N^*}(S - s^r)(x_j, t_k)$ can be established. Using the triangle inequality and the results for $\mathfrak{B}_\varepsilon^{M^*,N^*}(S - s^l)(x_j, t_k)$ and $\mathfrak{B}_\varepsilon^{M^*,N^*}(S - s^r)(x_j, t_k)$, (33) holds for any $(x_j, t_k) \in D^{M,N}$. □

5.3 The error in $U - u$

Theorem 9. *Let u be the solution of* (1)-(2) *and U be that of* (20). *Then for any $(x_j, t_k) \in \overline{D}^{M,N}$,*

$$|U(x_j, t_k) - u(x_j, t_k)| \leq C(M^{-1} + (N^{-1}\ln N)^2). \tag{45}$$

Proof. For any $(x_j, t_k) \in D^{M,N}$,

$$\begin{aligned} \left| \mathfrak{B}_\varepsilon^{M^*,N^*}(U - u)(x_j, t_k) \right| \leq & \left| \mathfrak{B}_\varepsilon^{M^*,N^*}(Q - q)(x_j, t_k) \right| \\ & + \left| \mathfrak{B}_\varepsilon^{M^*,N^*}(S - s)(x_j, t_k) \right|. \end{aligned} \tag{46}$$

Using Theorems 7 and 8 with (46), for any $(x_j, t_k) \in D^{M,N}$,

$$\left| \mathfrak{B}_\varepsilon^{M^*,N^*}(U - u)(x_j, t_k) \right| \leq C(M^{-1} + (N^{-1}\ln N)^2). \tag{47}$$

Since $U = u$ on $\Gamma^{M,N}$, using Theorem 6 with (47), for any $(x_j, t_k) \in \overline{D}^{M,N}$,

$$|(U(x_j, t_k) - u(x_j, t_k)| \leq C(M^{-1} + (N^{-1}\ln N)^2).$$

□

6 Numerical experiments

In this section, the nonlinear time dependent problem (1)-(2) is solved numerically by using a variant of the continuation technique designed in [8] together with the computational technique provided in this article.

Notations $E^{M,N}$, $C_p^{M,N}$ and $p^{M,N}$ denote the parameter-uniform maximum pointwise error, parameter-uniform error constant and parameter-uniform rate of convergence, respectively and are given by

$$E^{M,N} = \max_{\varepsilon} E_{\varepsilon}^{M,N} \text{ where } E_{\varepsilon}^{M,N} = \begin{cases} \left|U^{M,N} - U^{M,2N}\right|, & \text{for } x \\ \left|U^{M,N} - U^{2M,N}\right|, & \text{for } t \end{cases}$$

$$p^{M,N} = \begin{cases} \log_2 \dfrac{E^{M,N}}{E^{M,2N}}, & \text{for } x \\ \log_2 \dfrac{E^{M,N}}{E^{2M,N}}, & \text{for } t, \end{cases} \quad C_p^{M,N} = \begin{cases} \frac{E^{M,N} N^{p^\star}}{1-2^{-p^\star}}, & \text{for } x \\ \frac{E^{M,N} M^{p^\star}}{1-2^{-p^\star}}, & \text{for } t \end{cases}$$

$$\text{where} \quad p^\star = \begin{cases} \min\limits_N p^{M,N}, & \text{for } x \\ \min\limits_M p^{M,N}, & \text{for } t. \end{cases}$$

Example 1. Consider the nonlinear time dependent problem

$$\frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + \left(u^3(x,t) + (1+x+t)u(x,t)\right) = 0, \ (x,t) \in D,$$

with $\Omega_t = (0,T] = (0,1]$, $u(x,0) = 0$ on Γ_b, $u(0,t) = \dfrac{t}{2}$ on Γ_l and $u(1,t) = t$ on Γ_r.

For Example 1, for different values of ε, the values of $E^{M,N}$, $C_p^{M,N}$ and $p^{M,N}$ in the variable x with a uniform mesh consists of 10 mesh intervals for the variable t is given in Table 1 and for different values of ε, the values of $E^{M,N}$, $C_p^{M,N}$ and $p^{M,N}$ in the variable t with a piecewise uniform Shishkin mesh consists of 128 mesh intervals for the variable x is given in Table 2. Moreover, the CPU time (in seconds) for each row of both the tables is given in the last column of the tables. For $M = 10$, $N = 64$ and $\varepsilon = 2^{-15}$, Figure 1 portraits the numerical approximations of $u(x,t)$; for the same values, Figure 2 is the rotated version of Figure 1. For $M = 10$, $N = 64$, $\varepsilon = 2^{-9}, 2^{-15}, 2^{-21}$, the mesh plot of $u(x,t)$ is presented in Figure 3 and for aforesaid values of M,N and $\varepsilon = 2^{-3}, 2^{-9}, 2^{-15}$, the cross section of solution $u(x,t)$ at $(x,1)$ is presented in Figure 4. Further, $log - log\, plots$ for the error are given in Figure 5 and Figure 6.

Example 2. Consider the nonlinear time dependent problem

$$\frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + \left(u^5(x,t) + (\sqrt{2} + x^2 + t/2)u(x,t)\right) = 0, \ (x,t) \in D,$$

with $\Omega_t = (0,T] = (0,1]$, $u(x,0) = 0$ on Γ_b, $u(0,t) = \dfrac{t}{\sqrt{\pi}}$ on Γ_l and $u(1,t) = \dfrac{t}{2}$ on Γ_r.

For Example 2, for different values of ε, the values of $E^{M,N}$, $C_p^{M,N}$ and $p^{M,N}$ in the variable x with a uniform mesh consists of 10 mesh intervals for the variable t is given in Table 3 and for different values of ε, the values of $E^{M,N}$, $C_p^{M,N}$ and $p^{M,N}$ in the variable t with a piecewise uniform Shishkin mesh consists of 128 mesh intervals for the variable x is given in Table 4. Moreover, the CPU time (in seconds) for each row of both the tables is given in the last column of the tables.

Table 1: *Values of* $E^{M,N}$, $p^{M,N}$ & $C_p^{M,N}$ *for* $M = 10$ & $\alpha = 0.9$

ε	N					CPU time
	64	128	256	512	1024	(in seconds)
2^{-3}	1.1964e-04	2.9965e-05	7.4946e-06	1.8739e-06	4.6848e-07	374.4625
2^{-6}	8.8734e-04	2.2622e-04	5.6796e-05	1.4217e-05	3.5551e-06	745.7579
2^{-9}	6.1498e-03	1.7095e-03	4.4487e-04	1.1214e-04	2.8109e-05	1115.9714
2^{-12}	3.3941e-03	2.4820e-03	1.0642e-03	5.4053e-04	2.2426e-04	1446.1152
2^{-15}	3.3765e-03	2.4745e-03	1.0663e-03	5.3970e-04	2.3479e-04	1544.8254
2^{-18}	3.3703e-03	2.4719e-03	1.0670e-03	5.3941e-04	2.3485e-04	1638.8346
2^{-21}	3.3682e-03	2.4709e-03	1.0673e-03	5.3931e-04	2.3487e-04	1732.6916
2^{-24}	3.3674e-03	2.4706e-03	1.0673e-03	5.3927e-04	2.3488e-04	1826.3198
2^{-27}	3.3671e-03	2.4705e-03	1.0674e-03	5.3926e-04	2.3488e-04	1919.7852
2^{-30}	3.3670e-03	2.4704e-03	1.0674e-03	5.3925e-04	2.3488e-04	2013.2693
2^{-33}	3.3670e-03	2.4704e-03	1.0674e-03	5.3925e-04	2.3488e-04	2106.7252
2^{-36}	3.3670e-03	2.4704e-03	1.0674e-03	5.3925e-04	2.3488e-04	2200.398
$E^{M,N}$	6.1498e-03	2.4820e-03	1.0674e-03	5.4053e-04	2.3488e-04	
$p^{M,N}$	9.8164e-01	1.2174e+00	9.8164e-01	1.2025e+00		
$C_p^{M,N}$	7.3876e-01	5.8877e-01	5.0000e-01	5.0000e-01	4.2903e-01	

Table 2: *Values of* $E^{M,N}$, $p^{M,N}$ & $C_p^{M,N}$ *for* $N = 128$ & $\alpha = 0.9$

ε	M					CPU time
	160	320	640	1280	2560	(in seconds)
2^{0}	7.5072e-05	1.8625e-04	1.8580e-04	1.2266e-04	6.8143e-05	468.1151
2^{-2}	4.0261e-04	3.0780e-04	1.8616e-04	1.1145e-04	6.4288e-05	931.1274
2^{-4}	4.0261e-04	3.0780e-04	1.8616e-04	1.1145e-04	6.4288e-05	1393.2488
2^{-6}	4.0261e-04	3.0780e-04	1.8616e-04	1.1145e-04	6.4288e-05	1855.5419
$E^{M,N}$	4.0261e-04	3.0780e-04	1.8616e-04	1.2266e-04	6.8143e-05	
$p^{M,N}$	3.8742e-01	7.2540e-01	6.0192e-01	8.4803e-01		
$C_p^{M,N}$	1.2213e-02	1.2213e-02	9.6621e-03	8.3272e-03	6.0513e-03	

7 Conclusion

In this article a numerical method involving classical finite difference operators, a piecewise uniform Shishkin mesh in the spatial direction and a uniform mesh in the temporal direction has been developed for a class of singularly perturbed time dependent nonlinear reaction diffusion problems with Dirichlet boundary conditions. It has been proved both theoretically and numerically that the developed method is robust, layer resolving and parameter uniform convergent.

Figures presented in this article reveal the fact that the boundary layers changes rapidly near both the boundaries Γ_l and Γ_r of the domain of the problem. From the tables in this article, it is evident that the maximum pointwise errors decreases monotonically through the diagonal. Further, the computed rate of convergence increases whereas the computed error constant decreases when the number of mesh points is increased; this shows the consistency of the proposed numerical technique.

The analysis presented in this article can be appropriately modified to problems with discontinuous

Table 3: *Values of* $E^{M,N}$, $p^{M,N}$ & $C_p^{M,N}$ *for* $M = 10$ & $\alpha = 0.9$

ε	N					CPU time (in seconds)
	64	128	256	512	1024	
2^{-3}	4.5437e-05	1.1371e-05	2.8433e-06	7.1086e-07	1.7772e-07	376.0943
2^{-6}	2.8607e-04	7.2303e-05	1.8105e-05	4.5280e-06	1.1321e-06	747.8975
2^{-9}	2.0621e-03	5.4802e-04	1.3871e-04	3.4840e-05	8.7171e-06	1115.2013
2^{-12}	2.4677e-03	1.4327e-03	6.6710e-04	2.8504e-04	1.0938e-04	1444.846
2^{-15}	2.4675e-03	1.4427e-03	6.6957e-04	2.8600e-04	1.1376e-04	1541.6231
2^{-18}	2.4675e-03	1.4462e-03	6.7044e-04	2.8634e-04	1.1383e-04	1635.5685
2^{-21}	2.4675e-03	1.4475e-03	6.7075e-04	2.8646e-04	1.1386e-04	1730.0049
2^{-24}	2.4674e-03	1.4479e-03	6.7086e-04	2.8651e-04	1.1387e-04	1824.0905
2^{-27}	2.4674e-03	1.4481e-03	6.7090e-04	2.8652e-04	1.1387e-04	1917.8662
2^{-30}	2.4674e-03	1.4481e-03	6.7091e-04	2.8653e-04	1.1387e-04	2011.909
2^{-33}	2.4674e-03	1.4481e-03	6.7092e-04	2.8653e-04	1.1387e-04	2106.3572
2^{-36}	2.4674e-03	1.4482e-03	6.7092e-04	2.8653e-04	1.1387e-04	2200.1872
$E^{M,N}$	2.4677e-03	1.4482e-03	6.7092e-04	2.8653e-04	1.1387e-04	
$p^{M,N}$	7.6898e-01	1.1100e+00	1.2274e+00	1.3313e+00		
$C_p^{M,N}$	1.4625e-01	1.4625e-01	1.1546e-01	8.4026e-02	5.6905e-02	

Table 4: *Values of* $E^{M,N}$, $p^{M,N}$ & $C_p^{M,N}$ *for* $N = 128$ & $\alpha = 0.9$

ε	M					CPU time (in seconds)
	160	320	640	1280	2560	
2^{0}	4.9600e-05	1.2721e-04	1.2870e-04	8.4370e-05	4.4241e-05	467.9497
2^{-2}	2.6542e-04	2.0529e-04	1.1765e-04	6.8207e-05	3.8520e-05	934.9692
2^{-4}	2.6542e-04	2.0529e-04	1.1765e-04	6.8207e-05	3.8520e-05	1401.7115
2^{-6}	2.6542e-04	2.0529e-04	1.1765e-04	6.8207e-05	3.8520e-05	1868.2465
$E^{M,N}$	2.6542e-04	2.0529e-04	1.2870e-04	8.4370e-05	4.4241e-05	
$p^{M,N}$	3.7064e-01	6.7363e-01	6.0923e-01	9.3133e-01		
$C_p^{M,N}$	7.6858e-03	7.6858e-03	6.2298e-03	5.2802e-03	3.5799e-03	

source terms and can be extended to nonlinear systems and nonlinear systems with discontinuous source terms.

Conflict of interest

The authors declare that they have no conflict of interest.

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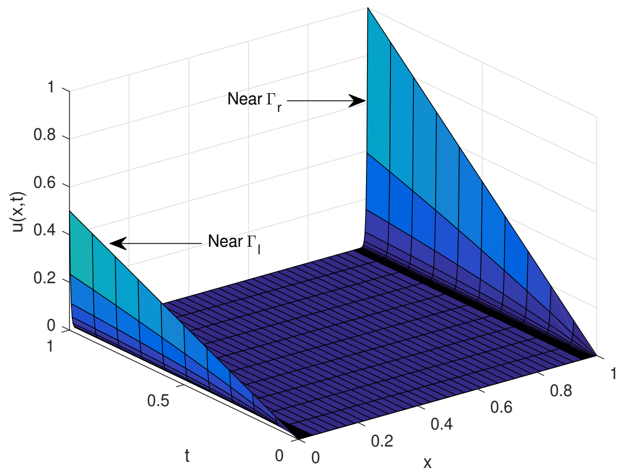

Figure 1: Solution profile $u(x,t)$ of Example 1

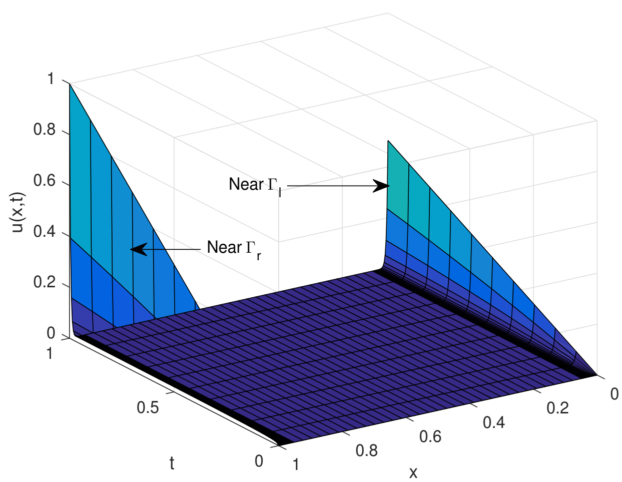

Figure 2: Rotated version of Figure 1

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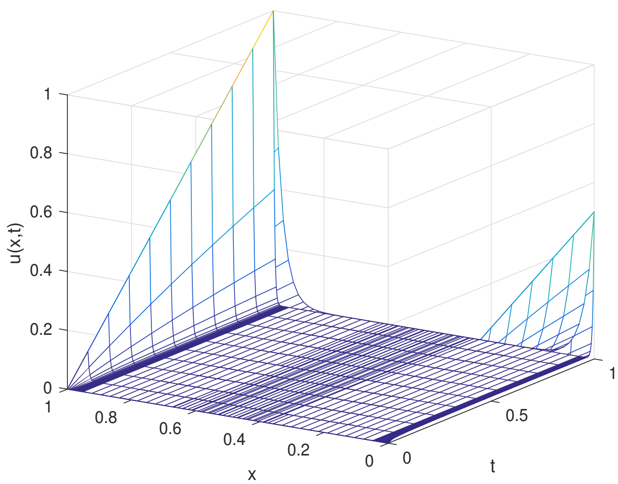

Figure 3: Mesh plot of solution $u(x,t)$ of Example 1 as $\varepsilon \to 0$

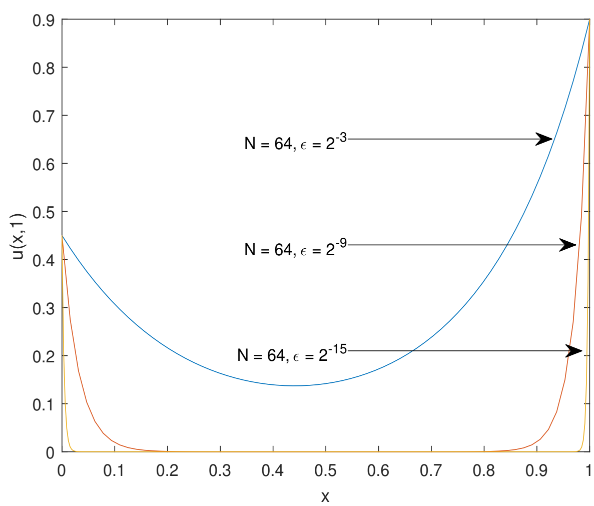

Figure 4: Cross section of solution $u(x,t)$ of Example 1 at $(x,1)$ as $\varepsilon \to 0$

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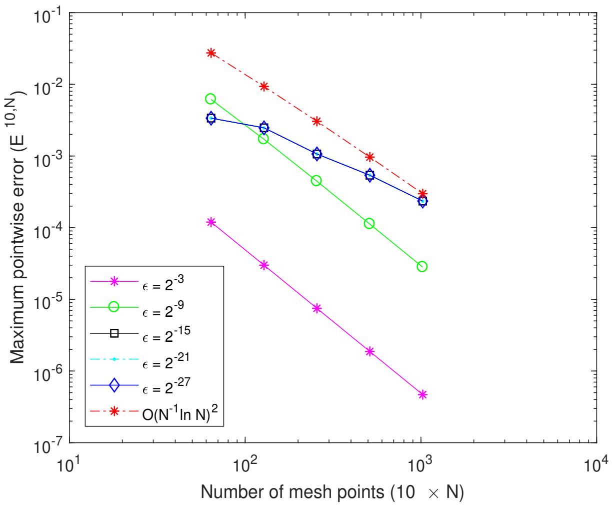

Figure 5: *Log − log plot* of the maximum pointwise errors corresponding to Table 1

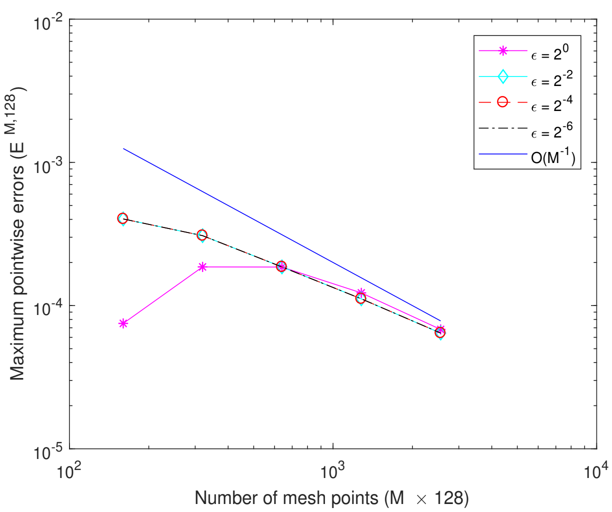

Figure 6: *Log − log plot* of the maximum pointwise errors corresponding to Table 2

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