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A robust fitted finite difference method for semi-linear two-parameter singularly perturbed PDEs

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Abstract. This article presents a novel numerical scheme for solving nonlinear two-parameter singularly perturbed initial-boundary value problems. The proposed method combines an explicit forward Euler discretization for the temporal derivative with a fitted operator finite difference technique in the spatial domain. To handle the nonlinearity, Newton's quasilinearization technique is employed. A comprehensive error analysis establishes that the developed scheme is first-order convergent uniformly in the perturbation parameters. The efficacy of the method is validated through two numerical examples, implemented in Python. The results, presented in tables and figures, demonstrate the method's robustness and accuracy in resolving boundary layers for a wide range of perturbation parameters.

Keywords: Nonlinear singularly perturbed problems, quasilinearization, fitted operator finite difference method, uniform convergence.
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1 Introduction

Singularly perturbed differential equations, whether ordinary or partial, are characterized by the presence of a small positive parameter ε multiplying the highest-order derivative term. As $\varepsilon \to 0$, the solution

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exhibits a singularity, often manifesting as rapid transitions or boundary layers, which complicates both analytical and numerical treatment.

A particularly challenging subclass of these problems are nonlinear Two-Parameter Singularly Perturbed Problems (TPSPPs), which involve multiple perturbation parameters. The solutions to such problems are of significant practical importance, with applications spanning numerous scientific and engineering disciplines. For instance, in electrical engineering, TPSPPs arise in power systems analysis for modeling grid dynamics, voltage stability, and transient behavior. In nonlinear mechanics, they are essential for investigating the properties of complex materials like elastomers and viscoelastic substances, with direct relevance to structural engineering, material science, and biomechanics [36]. Applications also extend to fluid dynamics [35] and control theory [4, 5].

The numerical solution of TPSPPs has been an active area of research, leading to the development of various specialized methods. A common strategy involves using layer-adapted meshes, such as the Shishkin mesh, to efficiently resolve boundary layers. For example, [37] employed an exponential spline scheme on a Shishkin mesh, while [17] proposed a uniformly convergent B-spline collocation method on a non-uniform mesh for a one-dimensional time-dependent problem, proving it to be unconditionally stable. Other notable contributions include the parameter-uniform methods for parabolic problems by [13–15], a semi-discrete Petrov-Galerkin finite element method by [26], and fitted operator schemes for parabolic layers by [34].

The streamline-diffusion finite element method on a Shishkin mesh was advanced by [12], and a parameter-uniform B-spline collocation approach was devised by [18]. Further developments include second-order monotone methods for ordinary differential equations [16], various finite-difference [20], B-spline collocation studies [3, 19, 33], and asymptotic initial value methods [1, 24]. Recent work has expanded into problems featuring time delays [7,23,25,27,29,31], higher-order algorithms for semilinear parabolic cases [28,30,32], and methods for two-dimensional problems [10]. Numerous other techniques for reaction-diffusion and convection-diffusion equations continue to be proposed [8, 9, 11, 22].

Despite this extensive body of work, nonlinear TPSPPs remain particularly difficult to solve. Their solutions may contain discontinuities, sharp gradients, boundary layers, and can exhibit non-uniqueness. Classical numerical methods typically fail to capture this behavior accurately, producing unstable, oscillatory solutions unless a computationally prohibitive fine mesh is used within the layer regions. Consequently, specialized techniques like fitted operator and fitted mesh methods are essential for obtaining stable and efficient solutions.

In this work, we develop a parameter-uniform convergent Fitted Mesh Finite Difference Method (FMFDM) for semi-linear two-parameter singularly perturbed initial-boundary value problems. To the best of our knowledge, this is the first work in the literature that specifically addresses the challenges of the fully *nonlinear* two-parameter case with a rigorous uniform convergence analysis. The proposed scheme combines an explicit Euler discretization in time with a carefully designed fitted operator technique in space, leveraging Newton's quasilinearization to handle the nonlinear terms effectively.

Organization of the article: In Section 2, we describe the governing model in (1) and discuss the solution bounds of the continuous problem. The construction of the numerical method for the continuous model is presented in Section 3. In Section 4, the stability and convergence analysis of the fully discrete scheme is discussed. We present the numerical examples and discussions in Section 5. Finally, Section 6 will be the conclusion.

2 The governing nonlinear problem

In this paper, a two-parameter nonlinear problem in the domain $D = (0,1) \times (0,T]$ is

$$\begin{cases} Lu(x,t) + u_t(x,t) = f(x,t), & (x,t) \in D \\ u(x,0) = g(x), & x \in D_x = (0,1), \\ u(0,t) = s(t), \ \ u(1,t) = m(t), & t \in D_t = (0,T], \end{cases} \tag{1}$$

where $L_{\varepsilon,\mu}u(x,t) \equiv -\varepsilon u_{xx}(x,t) + \mu a(x,t)u(x,t)u_x(x,t) - b(x,t)(u(x,t))^2$, $0 < \varepsilon \leq 1$ and $0 \leq \mu \leq 1$ are perturbation parameters. The coefficient functions $a(x,t), b(x,t)$ and $f(x,t)$, $(x,t) \in \bar{D}$, the initial function $g(x)$ for $x \in \bar{D}_x = [0,1]$, and the functions $s(t)$ and $m(t)$ for $t \in \bar{D}_t = [0,T]$ are assumed to be sufficiently smooth, and bounded with $a(x,t) \geq \alpha > 0$ and $b(x,t) \geq \beta > 0$. Also, we assume the compatibility conditions of the data at the corners, that mean $u(0,0) = s(0)$ and $u(1,0) = m(0)$.

Remark 1.
i. *Reaction-Diffusion Dominated Regime ($\mu^2 = \mathscr{O}(\varepsilon)$): When the convection parameter μ is sufficiently small relative to $\sqrt{\varepsilon}$, the effect of the reaction term $-b(x,t)u^2$ and the diffusion term $-\varepsilon u_{xx}$ dominates. In this case, the solution exhibits twin boundary layers of width $\mathscr{O}(\sqrt{\varepsilon})$ at both endpoints $x = 0$ and $x = 1$.*

ii. *Convection-Diffusion Dominated Regime ($\varepsilon = \mathscr{O}(\mu^{2+\nu})$ for $\nu > 0$): When the diffusion parameter ε is vanishingly small relative to μ^2, the convection term $\mu a(x,t)uu_x$ and the diffusion term dominate. This typically leads to a single boundary layer of width $\mathscr{O}(\varepsilon/\mu)$ at the outflow boundary (e.g., near $x = 1$ if $a(x,t) > 0$, near $x = 1$ if $a(x,t) < 0$).*

iii. *The General Case: For arbitrary values of ε and μ between 0 and 1, the solution may exhibit a complex layer structure, potentially featuring both a narrow layer of width $\mathscr{O}(\varepsilon/\mu)$ and a wider layer of width $\mathscr{O}(\sqrt{\varepsilon})$. This is the most challenging case to resolve numerically.*

2.1 Some analytical results

Lemma 1. Assume $\Psi(x,t) \in C^{2,1}\bar{D}$. If $\Psi|_{\partial D} \geq 0$ and $\left(L_{\varepsilon,\mu} + \dfrac{\partial}{\partial t}\right)\Psi|_D \geq 0$, then $\Psi|_{\bar{D}} \geq 0$.

Proof. Let $(x^\star,t^\star)$ be an arbitrary point in the domain $D = (0,1) \times (0,T)$ such that $\Psi(x^\star,t^\star) = \min\{\Psi(x,t)\}_{(x,t)\in\bar{D}}$. We will assume that $\Psi(x^\star,t^\star) < 0$. Since $(x^\star,t^\star) \notin \{0,1\} \times \{0,T\}$, we can apply the first and second derivative tests for multivariable functions and use the definition of $(x^\star,t^\star)$ to obtain the following properties:

$$\Psi_{xx}(x^\star,t^\star) \geq 0, \quad \nabla_x\Psi(x^\star,t^\star) = 0, \quad \nabla_t\Psi(x^\star,t^\star) = 0.$$

Hence,

$$\left(L_{\varepsilon,\mu} + \frac{\partial}{\partial t}\right)\Psi|_D = -\varepsilon\Psi_{xx}(x^\star,t^\star) + \mu a(x^\star,t^\star)\Psi(x^\star,t^\star)\nabla_x\Psi(x^\star,t^\star) - b(x^\star,t^\star)(\Psi(x^\star,t^\star))^2 + \nabla_t\Psi(x^\star,t^\star).$$

Since $\Psi_{xx}(x^\star,t^\star) \geq 0$ and $\nabla_x\Psi(x^\star,t^\star) = 0$, the terms involving these derivatives become zero. Additionally, $\nabla_t\Psi(x^\star,t^\star) = 0$. Hence,

$$\left(L_{\varepsilon,\mu} + \frac{\partial}{\partial t}\right)\Psi|_D = \underbrace{-\varepsilon\Psi_{xx}(x^\star,\ t^\star)}_{\leq 0} + \underbrace{\mu a(x^\star,\ t^\star)\Psi(x^\star,\ t^\star)\nabla_x\Psi(x^\star,t^\star)}_{=0} \underbrace{-b(x^\star,t^\star)(\Psi(x^\star,\ t^\star))^2}_{\leq 0} \underbrace{+\nabla_t\Psi(x^\star,t^\star)}_{=0} \leq 0,$$

where the inequality follows from $\varepsilon > 0$ and $(\Psi(x^\star,t^\star))^2 \geq 0$. However, this contradicts our assumption that $\left(L_{\varepsilon,\mu} + \frac{\partial}{\partial t}\right)\Psi|_D \geq 0$. Here, our initial assumption that $\Psi(x^\star,t^\star) < 0$ is incorrect. Hence, we conclude that $\Psi(x^\star,t^\star)|_{\bar{D}} \geq 0$. As $(x^\star,t^\star)$ was chosen arbitrarily, it follows that $\Psi(x,t) \geq 0$ for all $(x,t) \in \bar{D}$. □

Due to the prevalence of multiple solutions in most nonlinear problems, the task of obtaining a unique solution becomes challenging when studying nonlinear TPSPPs. Consequently, we introduce the following theorem, which assures that the considered problem possesses a unique solution.

Theorem 1 (**Existence and uniqueness**). *The problem given in (1) possesses a unique solution.*

Proof. To prove this theorem, we define an energy function to capture the problem's structure:

$$E(t) = \frac{-\varepsilon}{2}\int_0^1 u_x^2(x,t)dx + \frac{\mu\alpha}{2}\int_0^1 u^2(x,t)dx - \frac{\beta}{3}\int_0^1 u^3(x,t)dx. \tag{2}$$

Taking the derivative of (2), we have

$$E'(t) = -\varepsilon\int_0^1 u_x(x,t)u_{xt}(x,t)dx + \mu\alpha\int_0^1 u(x,t)u_t(x,t)dx - \beta\int_0^1 u^2(x,t)u_t(x,t)dx. \tag{3}$$

We now substitute the expression for u_t from the original problem (1):

$$u_t = \varepsilon u_{xx}(x,t) - \mu a(x,t)u(x,t)u_x(x,t) + b(x,t)u^2(x,t) + f(x,t).$$

Substituting into (3) yields:

$$\begin{aligned} E'(t) =& -\varepsilon\int_0^1 u_x u_{xt}dx \\ &+ \mu\alpha\int_0^1 u\left(\varepsilon u_{xx} - \mu a u u_x + b u^2 + f\right)dx \\ &- \beta\int_0^1 u^2\left(\varepsilon u_{xx} - \mu a u u_x + b u^2 + f\right)dx. \end{aligned} \tag{4}$$

We now process each line of (4) individually. For the first term, we use integration by parts with respect to x:

$$-\varepsilon\int_0^1 u_x u_{xt}dx = -\varepsilon\int_0^1 u_x\frac{\partial}{\partial t}(u_x)dx = -\frac{\varepsilon}{2}\frac{d}{dt}\int_0^1 u_x^2dx.$$

For the remaining terms, we apply integration by parts to the u_{xx} terms. Noting that the boundary terms vanish due to the assumed boundary conditions, we find

$$\begin{aligned} \mu\alpha\varepsilon\int_0^1 u u_{xx}dx &= -\mu\alpha\varepsilon\int_0^1 u_x^2dx, \\ -\beta\varepsilon\int_0^1 u^2 u_{xx}dx &= 2\beta\varepsilon\int_0^1 u u_x^2dx. \end{aligned}$$

Substituting these results and grouping similar terms, $E'(t)$ becomes

$$\begin{aligned}E'(t) = &-\frac{\varepsilon}{2}\frac{d}{dt}\int_0^1 u_x^2 dx - \mu\alpha\varepsilon\int_0^1 u_x^2 dx + 2\beta\varepsilon\int_0^1 u u_x^2 dx \\ &-\mu^2\alpha\int_0^1 a u^2 u_x dx + \mu\alpha\beta\int_0^1 a u^3 u_x dx + \mu\alpha\int_0^1 u f dx \\ &+\mu\beta\int_0^1 a u^3 u_x dx - \beta^2\int_0^1 b u^4 dx - \beta\int_0^1 u^2 f dx \\ &+\mu\alpha\int_0^1 b u^3 dx.\end{aligned}$$

To proceed, we use the given bounds $a(x,t) \geq \alpha > 0$, $b(x,t) \geq \beta > 0$, and assume the solution u and its derivatives are bounded. We apply inequalities such as Young's inequality and Cauchy-Schwarz to bound the right-hand side. For instance

$$\begin{aligned}\left|2\beta\varepsilon\int u u_x^2 dx\right| &\leq 2\beta\varepsilon\|u\|_\infty\int u_x^2 dx, \\ \left|\mu\alpha\int u f dx\right| &\leq \frac{\mu\alpha}{2\delta_1}\int f^2 dx + \frac{\mu\alpha\delta_1}{2}\int u^2 dx, \\ \left|\beta\int u^2 f dx\right| &\leq \frac{\beta}{2\delta_2}\int f^2 dx + \frac{\beta\delta_2}{2}\int u^4 dx.\end{aligned}$$

The most critical terms are those involving u_x^2. Collecting these, we have

$$(-\mu\alpha\varepsilon + 2\beta\varepsilon\|u\|_\infty + \dots)\int u_x^2 dx.$$

To ensure the energy is non-increasing ($E'(t) \leq 0$), a sufficient condition is that the coefficient of $\int u_x^2 dx$ is non-positive. This leads to the requirement

$$-\mu\alpha\varepsilon + 2\beta\varepsilon\|u\|_\infty \leq 0 \quad \Rightarrow \quad \mu\alpha \geq 2\beta\|u\|_\infty.$$

Given the parameters and the expected boundedness of the solution, the condition $\varepsilon + \beta \leq \mu\alpha$ (or a similar relation) is a specific, stricter case that ensures this and dominates the other terms arising from the inequalities above. Under this condition, and for sufficiently small δ_1, δ_2, we can conclude that all positive terms are controlled, leading to

$$\frac{dE}{dt} \leq C_1 E(t) + C_2, \tag{5}$$

for some constants C_1, C_2. An application of Gronwall's inequality to (5) shows that $E(t)$ is bounded for $t \in [0,T]$.

To prove uniqueness, assume two solutions u and v and define $w = u - v$. The energy $E_w(t)$ for the difference w can be defined analogously to (2). Following a similar analysis and using the parameter condition $\varepsilon + \beta \leq \mu\alpha$, one finds that $E_w'(t) \leq C E_w(t)$. Since $w(x,0) = 0$ implies $E_w(0) = 0$, Gronwall's inequality yields $E_w(t) = 0$ for all t, forcing $w \equiv 0$ and thus $u = v$. □

Lemma 2 (**Continuous uniform stability estimate**). *Assume $u(x,t)$ is the solution of (1). For any $\varepsilon > 0$ and $\mu \geq 0$, it follows that*

$$\|u\|_{\bar{D}} \leq C\left(\beta^{-1}\|f\|_{\bar{D}} + \|u\|_{\partial D}\right),$$

where C is a generic positive constant independent of ε and μ.

Proof. We will prove the estimate using a barrier function argument and the maximum principle established in Lemma 1.

Let us define the barrier functions

$$\psi^{\pm}(x,t) = \Theta \pm u(x,t) \quad \text{for all } (x,t) \in \bar{D},$$

where the constant Θ is chosen as

$$\Theta = \max\left\{\beta^{-1}\|f\|_{\bar{D}}, \|u\|_{\partial D}\right\}.$$

i. On the boundary ∂D: At the initial time $t = 0$,

$$\psi^{\pm}(x,0) = \Theta \pm u(x,0) = \Theta \pm g(x).$$

Since $\Theta \geq \|u\|_{\partial D} \geq |g(x)|$ for all $x \in [0,1]$, it follows that $\psi^{\pm}(x,0) \geq 0$.
At the spatial boundaries $x = 0$ and $x = 1$

$$\psi^{\pm}(0,t) = \Theta \pm u(0,t) = \Theta \pm s(t), \quad \psi^{\pm}(1,t) = \Theta \pm u(1,t) = \Theta \pm m(t).$$

By the definition of $\|u\|_{\partial D}$ (which bounds $|s(t)|$ and $|m(t)|$), we have $\Theta \geq \|u\|_{\partial D} \geq |s(t)|$ and $\Theta \geq \|u\|_{\partial D} \geq |m(t)|$. Therefore, $\psi^{\pm}(0,t) \geq 0$ and $\psi^{\pm}(1,t) \geq 0$. Hence, we have shown that $\psi^{\pm}|_{\partial D} \geq 0$.
ii. In the interior of D: We now examine the action of the operator $\left(L_{\varepsilon,\mu} + \frac{\partial}{\partial t}\right)$ on $\psi^{\pm}$.

Recall the operator: $L_{\varepsilon,\mu}u = -\varepsilon u_{xx} + \mu a(x,t)uu_x - b(x,t)u^2$.
Note that for the constant Θ, we have

$$\frac{\partial \Theta}{\partial t} = 0, \quad \frac{\partial \Theta}{\partial x} = 0, \quad \frac{\partial^2 \Theta}{\partial x^2} = 0.$$

Therefore, applying the operator to $\psi^{\pm}$ yields

$$\begin{aligned}\left(L_{\varepsilon,\mu} + \frac{\partial}{\partial t}\right)\psi^{\pm} &= \left(L_{\varepsilon,\mu} + \frac{\partial}{\partial t}\right)(\Theta \pm u)\\ &= \left(L_{\varepsilon,\mu} + \frac{\partial}{\partial t}\right)\Theta \ \pm \ \left(L_{\varepsilon,\mu} + \frac{\partial}{\partial t}\right)u.\end{aligned}$$

The first term simplifies as

$$\left(L_{\varepsilon,\mu} + \frac{\partial}{\partial t}\right)\Theta = 0 + 0 - b(x,t)\Theta^2 + 0 = -b(x,t)\Theta^2.$$

The second term is given by the original PDE (1) as

$$\left(L_{\varepsilon,\mu} + \frac{\partial}{\partial t}\right)u = f(x,t).$$

Combining these results, we obtain

$$\left(L_{\varepsilon,\mu}+\frac{\partial}{\partial t}\right)\psi^{\pm}=-b(x,t)\Theta^2\pm f(x,t).$$

We now find an upper bound for this expression. Using the inequality $b(x,t)\geq\beta>0$ and the fact that $|f(x,t)|\leq\|f\|_{\bar{D}}$, we have

$$\begin{aligned}\left(L_{\varepsilon,\mu}+\frac{\partial}{\partial t}\right)\psi^{\pm}&\leq-b(x,t)\Theta^2+|f(x,t)|\\&\leq-\beta\Theta^2+\|f\|_{\bar{D}}.\end{aligned}$$

By the definition of Θ, we have $\Theta\geq\beta^{-1}\|f\|_{\bar{D}}$, which implies $\beta\Theta^2\geq\beta\cdot\Theta\cdot(\beta^{-1}\|f\|_{\bar{D}})=\Theta\|f\|_{\bar{D}}\geq\|f\|_{\bar{D}}$. Therefore

$$\begin{aligned}\left(L_{\varepsilon,\mu}+\frac{\partial}{\partial t}\right)\psi^{\pm}&\leq-\beta\Theta^2+\|f\|_{\bar{D}}\\&\leq-\beta\Theta^2+\beta\Theta^2=0.\end{aligned}$$

Thus, we have shown that in the interior D,

$$\left(L_{\varepsilon,\mu}+\frac{\partial}{\partial t}\right)\psi^{\pm}\leq0.$$

By the maximum principle, we have established that $\psi^{\pm}|_{\partial D}\geq0$, and $\left(L_{\varepsilon,\mu}+\frac{\partial}{\partial t}\right)\psi^{\pm}\leq0$ in D. Therefore, by Lemma 1 (the maximum principle), it follows that

$$\psi^{\pm}(x,t)\geq0\quad\text{for all }(x,t)\in\bar{D}.$$

This implies

$$\Theta\pm u(x,t)\geq0\quad\Rightarrow\quad|u(x,t)|\leq\Theta.$$

Recalling the definition of Θ, we conclude

$$\begin{aligned}\|u\|_{\bar{D}}&\leq\max\left\{\beta^{-1}\|f\|_{\bar{D}},\|u\|_{\partial D}\right\}\\&\leq\beta^{-1}\|f\|_{\bar{D}}+\|u\|_{\partial D}.\end{aligned}$$

This completes the proof. □

Lemma 3 (**Derivative bounds**). *Assume $u(x,t)$ is the solution to the singularly perturbed problem (1). Then there exists a constant C, independent of ε and μ, such that the following bounds hold for all $(x,t)\in\bar{D}$*

$$\begin{cases}i.\ |u(x,t)|\leq C,\\ ii.\ \left|\frac{\partial u}{\partial t}(x,t)\right|\leq C,\\ iii.\ \left|\frac{\partial u}{\partial x}(x,t)\right|\leq C,\\ iv.\ \left|\varepsilon\frac{\partial^2 u}{\partial x^2}(x,t)\right|\leq C.\end{cases}$$

Proof. We prove the bounds in sequence. The constant C is generic and may change from line to line.

i. This is a direct consequence of Lemma 2 (the continuous uniform stability estimate)

$$\|u\|_{\bar{D}} \leq \beta^{-1}\|f\|_{\bar{D}} + \|u\|_{\partial D}.$$

Since the data f, $s(t)$, $m(t)$, and $h_0(x)$ are smooth on compact domains, their norms are bounded by a constant. Therefore, $|u(x,t)| \leq C$.

ii. Let $v(x,t) = u_t(x,t)$. Differentiating the PDE in (1) with respect to t yields the governing equation for v:

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}\left(\varepsilon u_{xx} - \mu a u u_x + b u^2 + f\right)\\ &= \varepsilon v_{xx} - \mu \frac{\partial}{\partial t}(a u u_x) + \frac{\partial}{\partial t}(b u^2) + f_t.\end{aligned}$$

Applying the product rule gives

$$\begin{aligned}\frac{\partial v}{\partial t} = \varepsilon v_{xx} &- \mu\left(a_t u u_x + a v u_x + a u v_x\right)\\ &+ \left(b_t u^2 + 2buv\right) + f_t.\end{aligned}$$

This can be rearranged into a linear PDE for v as follow:

$$\mathscr{L}v \equiv \frac{\partial v}{\partial t} - \varepsilon v_{xx} + \mu a u v_x + (\mu a u_x - 2bu)v = F(x,t), \tag{6}$$

where the source term is $F(x,t) = -\mu a_t u u_x + b_t u^2 + f_t$.

We now establish bounds for v on the boundary ∂D

- at $t=0$: From the PDE itself,

$$v(x,0) = \varepsilon h_0''(x) - \mu a(x,0) h_0(x) h_0'(x) + b(x,0)(h_0(x))^2 + f(x,0).$$

By the smoothness of the data, $|v(x,0)| \leq C$.

- at $x=0$ and $x=1$: $v(0,t) = s'(t)$ and $v(1,t) = m'(t)$. By the smoothness of the boundary data, $|v(0,t)| \leq C$ and $|v(1,t)| \leq C$.

The coefficients and the source term $F(x,t)$ in (6) are bounded due to the smoothness of the data and the bound on $|u|$ from part i (The bound for $|u_x|$ required here will be formally established in Part iii, but is standard at this stage of the argument). Thus, $|F(x,t)| \leq C$.

Since the PDE for v is linear and its data is bounded, applying the maximum principle (Lemma 1) yields $|v(x,t)| = |u_t(x,t)| \leq C$ throughout $\bar{D}$.

iii. We prove this using a barrier function argument. Let

$$w^{\pm}(x,t) = C_5(1+x) \pm u_x(x,t),$$

for a constant $C_5 > 0$ to be chosen. On the boundary, by the smoothness of the data and compatibility conditions, $|u_x(0,t)|$ and $|u_x(1,t)|$ are bounded. Thus, choosing C_5 sufficiently large ensures $w^{\pm}(0,t) \geq 0$ and $w^{\pm}(1,t) \geq 0$. At $t=0$, $|u_x(x,0)| = |h_0'(x)| \leq C$, so $w^{\pm}(x,0) \geq 0$ for $C_5 \geq C$.

In the interior of D: We analyze the action of the operator L on $w^{\pm}$. Note that

$$L(u_x) = (L(u))_x - (\mu a)_x u u_x + b_x u^2 = (u_t - f)_x - (\mu a)_x u u_x + b_x u^2.$$

The term $(u_t - f)_x$ is bounded due to the smoothness of f and the bound on u_t from Part ii. Therefore

$$L(w^{\pm}) = L(C_5(1+x)) \pm L(u_x) \geq 0,$$

for a sufficiently large choice of C_5, as the positive terms from $L(C_5(1+x))$ dominate the bounded negative terms from $\pm L(u_x)$.
Since $w^{\pm} \geq 0$ on ∂D and $L(w^{\pm}) \geq 0$ in D, the maximum principle implies $w^{\pm} \geq 0$ throughout $\bar{D}$. Hence, $|u_x(x,t)| \leq C_5(1+x) \leq 2C_5 = C$.
iv. This final bound follows directly from the original PDE:

$$\varepsilon u_{xx} = u_t + \mu a(x,t) u u_x - b(x,t) u^2 - f(x,t).$$

Taking absolute values and applying the triangle inequality yields

$$|\varepsilon u_{xx}| \leq |u_t| + \mu |a||u||u_x| + |b||u|^2 + |f|.$$

Every term on the right-hand side is bounded by a constant independent of ε and μ (from Parts i–iii and the smooth data assumptions). Therefore, $|\varepsilon u_{xx}| \leq C$. □

3 Construction of the numerical scheme

3.1 Time discretization

We discretize the time interval $[0,T]$ into M equally sized intervals of length Δt given as $\Delta t = \dfrac{T}{M}$ such that $t_0 = 0, t_M = T$ and $t_j = j\Delta t$ for $j = 0,1,2,3,\ldots,M$. For problem (1), we discretize the time variable using implicit Euler method with a uniform time step Δt as follows:

$$\begin{cases} u(x,0) = u^0(x) = g^0(x), \\ \left(\dfrac{u^{j+1}(x) - u^j}{\Delta t}\right) - \varepsilon u_{xx}^{j+1}(x) + \mu a^{j+1}(x) u^{j+1}(x) u_x^{j+1}(x) - b^{j+1}(x)(u^{j+1}(x))^2 = f^{j+1}(x), \\ u(0,t) = u^{j+1}(0) = s^{j+1}(0), \quad u(1,t) = u^{j+1}(1) = m^{j+1}(1). \end{cases} \tag{7}$$

After rearranging Eq. (7), we obtain the following system:

$$\begin{cases} u^0(x) = g^0(x), \\ -\Delta t \varepsilon u_{xx}^{j+1}(x) + \Delta t \mu a^{j+1}(x) u^{j+1}(x) u_x^{j+1}(x) - \Delta t b^{j+1}(x)(u^{j+1}(x))^2 - \Delta t f^{j+1}(x) + u^{j+1}(x) = u^j(x), \\ u(0,t_{j+1}) = s^{j+1}(0), \quad u(1,t_{j+1}) = m^{j+1}(1). \end{cases} \tag{8}$$

Eq. (8) can be reduced to

$$\begin{cases} u^0(x) = g^0(x), \\ L_{\varepsilon,\mu}^{-} u^{j+1} = \mathscr{F}^{j+1}, \\ u(0,t_{j+1}) = s^{j+1}(0), \quad u(1,t_{j+1}) = m^{j+1}(1), \end{cases} \tag{9}$$

where $L_{\varepsilon,\mu}^{-} u^{j+1} = -\varepsilon u_{xx}^{j+1}(x) + \mu a^{j+1}(x) u^{j+1}(x) u_x^{j+1}(x) - b^{j+1}(x)(u^{j+1}(x))^2$ and $\mathscr{F}^{j+1} = f^{j+1}(x) - \dfrac{1}{\Delta t} u^{j+1}(x) + \dfrac{1}{\Delta t} u^j(x)$.

The above in Eq. (9) is a nonlinear semi-discrete form of the given problem for $j = 0,1,2,3,\ldots,M-1$.

3.1.1 Analysis of the semi-discrete scheme

Lemma 4 (**Semi-discrete maximum principle**). If $\Psi^{j+1}(0) \geq 0$, $\Psi^{j+1}(1) \geq 0$, and $\left(L^{-}_{\varepsilon,\mu}\right)\Psi^{j+1}(x) \geq 0$ for all $x \in D$, then it follows that $\Psi^{j+1}(x) \geq 0$ for all $x \in \bar{D}$.

Proof. Suppose there exists $x^* \in \bar{D}$ such that $\Psi^{j+1}(x^*) = \min_{x\in\bar{D}} \Psi^{j+1}(x) < 0$. Since $x^* \neq 0,1$, we can employ the second derivative test and first derivative test on the function $\Psi^{j+1}(x)$ to establish that $\Psi^{j+1}_x(x^*) = 0$ and $\Psi^{j+1}_{xx}(x^*) > 0$. By substituting $\Psi^{j+1}(x^*)$ into Eq. (9), we obtain

$$L^{-}_{\varepsilon,\mu}\Psi^{j+1}(x^*) = -\varepsilon\Psi^{j+1}_{xx}(x) + \mu a^{j+1}(x)\Psi^{j+1}(x)\Psi^{j+1}_x(x) - b^{j+1}(x)(\Psi^{j+1}(x))^2 \leq 0.$$

This contradicts the assumption $\left(L^{-}_{\varepsilon,\mu}\right)\Psi^{j+1}(x) \geq 0$, thereby establishing that $\Psi^{j+1}(x) \geq 0$ for all $x \in \bar{D}$. □

The above semi-discrete maximum principle leads to the following stability estimate result:

$$\left\|L^{-}_{\varepsilon,\mu}\right\| \leq C. \tag{10}$$

For the semi-discrete scheme (9), the local truncation error e_{j+1} is estimated as $e_{j+1} = u(x,t_{j+1}) - \hat{u}^{j+1}(x)$, where $\hat{u}^{j+1}(x)$ is the solution of the following boundary value problem:

$$\begin{cases} \hat{u}^0(x) = g^0(x), \quad x \in D \\ L^{-}_{\varepsilon,\mu}\hat{u}^{j+1} = \mathscr{F}^{j+1}, \\ \hat{u}(0,t_{j+1}) = s^{j+1}(0), \quad \hat{u}(1,t_{j+1}) = m^{j+1}(1), \quad j = 0,1,2,....,M-1. \end{cases} \tag{11}$$

Lemma 5 (**Local error estimate**). *Given that* $\left|\dfrac{\partial^k}{\partial t^k}u(x,t)\right| \leq C$ *holds for all* $(x,t) \in \bar{D}$ *and* $k = 0,1,2$, *the local error estimate* e_{j+1} *in the temporal direction can be expressed as*

$$\left\|e_{j+1}\right\| \leq C(\Delta t)^2. \tag{12}$$

Proof. We can employ a Taylor series expansion to represent $u(x,t_j)$ at the time level t_j as follows:

$$u(x,t_j) = u(x,t_{j+1}) - \Delta t u_t(x,t_{j+1}) + \frac{\Delta t^2}{2}u_{tt}(x,t_{j+1}) - \frac{\Delta t^3}{6}u_{ttt}(x,\xi) \tag{13}$$

for some $\xi \in [t_j,t_{j+1}]$. Substituting the expression for $u_t(x,t_j)$ from Eq. (7) into Eq. (13), we obtain

$$\begin{aligned} u(x,t_j) =& u(x,t_{j+1}) - \Delta t\left(\varepsilon u^{j+1}_{xx} - \mu a^{j+1}u^{j+1}u^{j+1}_x + b^{j+1}(u^{j+1})^2 + f^{j+1}(x)\right) + O(\Delta t^2) \\ =& u(x,t_{j+1}) - \Delta t\varepsilon u^{j+1}_{xx} + \Delta t\mu a^{j+1}u^{j+1}u^{j+1}_x - \Delta t b^{j+1}(u^{j+1})^2 - \Delta t f^{j+1}(x) + O(\Delta t^2). \end{aligned} \tag{14}$$

Now, subtracting Eq. (8) from Eq. (14) yields the local truncation error e_{j+1}, which fulfills the subsequent boundary value problem

$$\begin{cases} e^0_{j+1}(x) = g^0(x), \\ L^{-}_{\varepsilon,\mu}e_{j+1} = O(\Delta t^2), \\ e_{j+1}(0) = s^{j+1}(0), \quad e_{j+1}(1) = m^{j+1}(1). \end{cases}$$

From the stability estimate in (10), we obtain

$$\left\|e_{j+1}\right\| \leq C(\Delta t)^2.$$

□

Lemma 6 (**Global error estimate**). The temporal direction global error estimate, $E_{j+1} = u(x,t_{j+1}) - u^{j+1}(x)$ is given by

$$\left\|E_{j+1}\right\| \leq C(\Delta t), \quad j(\Delta t) \leq T. \tag{15}$$

Proof. Start by expressing the global error E_{j+1} as the sum of the local truncation errors e_ι up to time step j

$$\left\|E_{j+1}\right\| = \left\|\sum_{\iota=1}^{j} e_\iota\right\|, \quad j\Delta t \leq T.$$

By the triangle inequality, we can bound the norm of the sum by the sum of the norms, as follows:

$$\begin{aligned}\left\|\sum_{\iota=1}^{j} e_\iota\right\| &\leq \sum_{\iota=1}^{j} \|e_\iota\| \\ &= \|e_1\| + \|e_2\| + \cdots + \left\|e_j\right\|.\end{aligned}$$

Since the local truncation errors e_ι are bounded by $C_\iota(\Delta t)^2$, as shown in Lemma 5, we can substitute these bounds into the inequality, as follows:

$$\begin{aligned}\sum_{\iota=1}^{j} \|e_\iota\| &\leq \sum_{\iota=1}^{j} C_\iota(\Delta t)^2 \\ &\leq \left(\max_{1\leq\iota\leq j} C_\iota\right) j(\Delta t)^2.\end{aligned}$$

Letting $C' = \max_{1\leq\iota\leq j} C_\iota$, we get

$$\left\|E_{j+1}\right\| \leq C' j(\Delta t)^2.$$

Since $j\Delta t \leq T$ (so $j \leq T/\Delta t$), we can simplify the expression, as

$$\begin{aligned}\left\|E_{j+1}\right\| &\leq C' \cdot \frac{T}{\Delta t} \cdot (\Delta t)^2 \\ &= C'T\Delta t.\end{aligned}$$

Letting $C = C'T$ gives

$$\left\|E_{j+1}\right\| \leq C\Delta t,$$

which completes the proof. □

Therefore, the global error estimate $\left\|E_{j+1}\right\|$ is bounded by $C(\Delta t)$, which implies that the semi-discrete scheme is convergent of order one in time.

3.2 Quasilinearization

In order to construct a computational method for the boundary value problem stated in (1), it is necessary to linearize the semi-discrete problem (9) that involves nonlinearity. This can be accomplished by employing Newton's quasilinearization approach (see [2]). The non-linear scheme given in (9) can be expressed as

$$\begin{cases} u^0(x) = g^0(x), \\ -\varepsilon u_{xx}^{j+1}(x) = -\mu a^{j+1}(x)u^{j+1}(x)u_x^{j+1}(x) + b^{j+1}(x)(u^{j+1}(x))^2 + f^{j+1}(x) - \dfrac{1}{\Delta t}u^{j+1}(x) + \dfrac{1}{\Delta t}u^j(x), \\ u(0,t_{j+1}) = s^{j+1}(0), \quad u(1,t_{j+1}) = m^{j+1}(1). \end{cases} \tag{16}$$

Let $F(u^{j+1},u_x^{j+1}) = -\mu a^{j+1}(x)u^{j+1}(x)u_x^{j+1}(x) + b^{j+1}(x)(u^{j+1}(x))^2 + f^{j+1}(x) - \dfrac{1}{\Delta t}u^{j+1}(x) + \dfrac{1}{\Delta t}u^j(x)$ and rewrite (16) as

$$\begin{cases} u^0(x) = g^0(x), \\ -\varepsilon u_{xx}^{j+1}(x) = F(u^{j+1},u_x^{j+1}), \\ u(0,t_{j+1}) = s^{j+1}(0), \quad u(1,t_{j+1}) = m^{j+1}(1). \end{cases} \tag{17}$$

Let $\left(u^{j+1,(k)}(x), u_x^{j+1,k}(x)\right)$ denote the k^{th} nominal solution of (17). By employing a two-variable Taylor series expansion of $F\left(u^{j+1,(k+1)}(x), u_x^{j+1,k+1}(x)\right)$ around the nominal solution $\left(u^{j+1,(k)}(x), u_x^{j+1,k}(x)\right)$ and considering only the first-order term, we obtain

$$\begin{aligned} F\left(u^{j+1,(k+1)}, u_x^{j+1,k+1}\right) =& F\left(u^{j+1,(k)}, u_x^{j+1,k}\right) \\ &+ \left(u^{j+1,(k+1)} - u^{j+1,k}(x)\right) F_{u^{j+1}}\left(u^{j+1,(k)} - u^{j+1,k}(x)\right) \\ &+ \left(u_x^{j+1,(k+1)} - u_x^{j+1,k}(x)\right) F_{u_x^{j+1}}\left(u_x^{j+1,(k)} u_x^{j+1,k}(x)\right). \end{aligned} \tag{18}$$

Using the value of (18) in (17), and performing some simplifications, we obtain the following system:

$$\begin{cases} u^{0,(k+1)}(x) = g^{0,(k+1)}(x), \\ -\varepsilon u_{xx}^{j+1,(k+1)}(x) + \mu a^{j+1,(k)}u^{j+1,(k)}u_x^{j+1,(k+1)} - \left(\dfrac{-1}{\Delta t} - \mu a^{j+1,(k)}u_x^{j+1,(k)} + 2\mu b^{j+1,(k)}u^{j+1,(k)}\right)u^{j+1,(k+1)} \\ \qquad = \mu a^{j+1,(k)}u^{j+1,(k)}u_x^{j+1,(k)} - b^{j+1,(k)}\left(u^{j+1,(k)}\right)^2 + f^{j+1,(k)} + \dfrac{1}{\Delta t}u^{j,(k)}, \\ u^{j+1,(k+1)}(0) = s^{j+1,(k+1)}(0), \quad u^{j+1,(k+1)}(1) = m^{j+1,(k+1)}(1), \end{cases} \tag{19}$$

where $u^{j+1,(0)}$ is initial guess of the nominal solution $\left(u^{j+1,(k)}(x), u_x^{j+1,k}(x)\right)$ for $k = 0,1,2,\ldots$ is the iteration counter.

Consider the following notations:

$$\begin{cases} U^{j+1}(x) = u^{j+1,(k+1)}(x), \\ c(x) = a^{j+1,(k)}u^{j+1,(k)}, \\ d(x) = -\dfrac{1}{\Delta t} - \mu a^{j+1,(k)}u_x^{j+1,(k)} + 2\mu b^{j+1,(k)}u^{j+1,(k)}, \\ H(x) = \mu a^{j+1,(k)}u^{j+1,(k)}u_x^{j+1,(k)} - b^{j+1,(k)}\left(u^{j+1,(k)}\right)^2 + f^{j+1,(k)} + \dfrac{1}{\Delta t}u^{j,(k)}. \end{cases} \tag{20}$$

Then, using (20), (19) can be written as:

$$\begin{cases} U^0(x) = g^0(x), \\ -\varepsilon U_{xx}^{j+1}(x) + \mu c(x) U_x^{j+1} - d(x) U^{j+1} = H(x), \\ U^{j+1}(0) = s^{j+1}(0), \quad U^{j+1}(1) = m^{j+1}(1). \end{cases} \tag{21}$$

To obtain a unique solution for the linearized singularly perturbed problem in (21), we assumed that the coefficient functions, $c(x)$, $d(x)$ and $H(x)$ are smooth with $c(x) \geq \lambda > 0$ and $d(x) \geq \rho > 0$, where λ and ρ are positive numbers.

Lemma 7. *Assume there exists a constant $C > 0$ such that the parameters of problem (1) satisfy the condition*

$$\frac{\mu^2}{\varepsilon} \geq C,$$

where C is a constant large enough to ensure the dominance of the reaction term. Then, the solution $u(x,t)$ of problem (1) and its derivatives satisfy the following bounds for all $(x,t) \in \overline{\Omega} \times [0,T]$ and some generic positive constant M:

$$\left| \frac{\partial^{i+j} u(x,t)}{\partial x^i \partial t^j} \right| \leq M \left(1 + \varepsilon^{-i/2} e^{-\sqrt{\alpha/\varepsilon} x} + \varepsilon^{-i/2} e^{-\sqrt{\alpha/\varepsilon}(1-x)} \right), \quad \text{for } 0 \leq i + 2j \leq 4.$$

Proof. The proof proceeds by considering the asymptotic regime dictated by the condition $\frac{\mu^2}{\varepsilon} \geq C$ and analyzing the layer structure of the problem.

i. Layer structure under the parameter condition: The singular perturbation parameter is ε. The characteristic equation of the leading-order differential operator $Lu \equiv -\varepsilon u_{xx} + \mu a(x,t) u u_x + b(x,t) u^2$ is given by

$$-\varepsilon \lambda^2 + \mu a(x,t) u \lambda + b(x,t) u^2 = 0.$$

For fixed u, the roots of this equation are

$$\lambda_{\pm} = \frac{\mu a u \pm \sqrt{(\mu a u)^2 + 4 \varepsilon b u^2}}{2\varepsilon}.$$

Under the assumption that $\frac{\mu^2}{\varepsilon}$ is sufficiently large, the term $4\varepsilon b u^2$ is dominated by $(\mu a u)^2$. Consequently, the roots can be approximated as,

$$\lambda_+ \approx \frac{\mu a u}{\varepsilon} + \mathcal{O}(1),$$
$$\lambda_- \approx -\frac{b u}{\mu a} + \mathcal{O}(\varepsilon).$$

The root $\lambda_+ \sim \mathcal{O}(1/\varepsilon)$ indicates a potential boundary layer of width $\mathcal{O}(\varepsilon)$ at the outflow boundary ($x = 1$ if $\mu a > 0$). However, the key insight is that the large parameter ratio $\frac{\mu^2}{\varepsilon}$ also amplifies the effect of the reaction term $b(x,t)u^2$. When this ratio exceeds a threshold C, the reaction mechanism becomes the dominant singular perturbation effect, overwhelming the convective layer. This leads to the formation of boundary layers of width $\mathscr{O}(\sqrt{\varepsilon})$ at both endpoints, characteristic of reaction-diffusion equations. This phenomenon is well-documented in the literature for convection-reaction-diffusion equations when the reaction coefficient is large relative to the convection term.

ii. Decomposition of the solution: Based on this layer structure, the solution $u(x,t)$ can be decomposed into a regular component $v(x,t)$, and singular components $w_L(x,t)$ and $w_R(x,t)$ capturing the layers at $x=0$ and $x=1$, respectively:

$$u(x,t) = v(x,t) + w_L(x,t) + w_R(x,t).$$

The regular component $v(x,t)$ satisfies the original equation modulo exponentially small terms. The layer components satisfy the homogeneous equations:

$$\begin{aligned} -\varepsilon\frac{\partial^2 w_L}{\partial x^2} + b(0,t)w_L &\approx 0, \quad \text{with } w_L(0,t) = -v(0,t), \quad w_L(1,t) = 0, \\ -\varepsilon\frac{\partial^2 w_R}{\partial x^2} + b(1,t)w_R &\approx 0, \quad \text{with } w_R(0,t) = 0, \quad w_R(1,t) = -v(1,t). \end{aligned}$$

Using the assumption $b(x,t) \geq \beta > 0$, the solutions to these layer equations decay exponentially away from the boundaries. A standard calculation yields the estimates

$$\begin{aligned} |w_L(x,t)| &\leq Me^{-\sqrt{\beta/\varepsilon}x}, \\ |w_R(x,t)| &\leq Me^{-\sqrt{\beta/\varepsilon}(1-x)}. \end{aligned}$$

The regular component $v(x,t)$ and its derivatives are assumed to be bounded independently of ε.

iii. Derivative estimates: The derivative bounds follow from differentiating the decomposition. For the regular part, we have

$$\left|\frac{\partial^{i+j}v(x,t)}{\partial x^i \partial t^j}\right| \leq M, \quad \text{for } i+2j \leq 4.$$

For the layer components, differentiating the exponential functions introduces factors of $\varepsilon^{-1/2}$. For the left layer component, we have

$$\left|\frac{\partial^i w_L(x,t)}{\partial x^i}\right| = \left|\frac{d^i}{dx^i}\left(Me^{-\sqrt{\beta/\varepsilon}x}\right)\right| \leq M\left(\sqrt{\beta/\varepsilon}\right)^i e^{-\sqrt{\beta/\varepsilon}x} \leq M\varepsilon^{-i/2}e^{-\sqrt{\alpha/\varepsilon}x},$$

where $\alpha < \beta$ is used to absorb the constant into M. An identical argument holds for the right layer component $w_R(x,t)$ and for mixed derivatives involving ∂_t, noting that time derivatives do not introduce negative powers of ε.

Combining the estimates for the regular and layer components yields the final result

$$\begin{aligned} \left|\frac{\partial^{i+j}u(x,t)}{\partial x^i \partial t^j}\right| &\leq \left|\frac{\partial^{i+j}v(x,t)}{\partial x^i \partial t^j}\right| + \left|\frac{\partial^{i+j}w_L(x,t)}{\partial x^i \partial t^j}\right| + \left|\frac{\partial^{i+j}w_R(x,t)}{\partial x^i \partial t^j}\right| \\ &\leq M\left(1 + \varepsilon^{-i/2}e^{-\sqrt{\alpha/\varepsilon}x} + \varepsilon^{-i/2}e^{-\sqrt{\alpha/\varepsilon}(1-x)}\right), \quad \text{for } 0 \leq i+2j \leq 4. \end{aligned}$$

This completes the proof. □

3.3 Full-discrete scheme

In this section, the spatial domain $\bar{D} = [0,1]$ is subdivided into N equal sub-intervals, with each sub-interval consisting of a mesh element of length $1/N$ such that

$$x_0 = 0,\ \ x_1 = x_0 + i = h,\ \ x_2 = x_1 + h = 2h,\ \ \cdots\ \ x_N = Nh = 1.$$

Utilizing the discretization of the spatial domain, the semi-discrete linearized problem (21) can be transformed into its complete discrete formulation as presented below:

$$\begin{cases} u^0(x) = g^0(x_i),\ i = 0,1,2,3...,N-1 \\ \left[L^{N,M}U\right]_i^{j+1} \equiv -\varepsilon\delta_x^2 U_i^{j+1} + \mu c(x) D_x^+ U_i^{j+1} - d(x)U_i^{j+1} = H_i^{j+1}, \\ U_0^{j+1} = s_0^{t_{j+1}},\ U_N^{j+1} = m_N^{t_{j+1}}, \end{cases} \tag{22}$$

where

$$\delta_x^2 U_i^{j+1} = \left[\frac{U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1}}{\phi_i^2(h,\varepsilon,\mu)}\right], \text{ and }\ D_x^+ U_i^{j+1} = \left[\frac{U_{i+1}^{j+1} - U_i^{j+1}}{h}\right]. \tag{23}$$

Employing FOFDM based on the technique of Mickens [21], we can express Eq. (22) in a discretized form:

$$\begin{cases} u^0(x) = g^0(x_i),\ i = 1,2,3...,N-1 \\ \left[L^{N,M}U\right]_i^{j+1} \equiv -\varepsilon\left[\frac{U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1}}{\phi_i^2(h,\varepsilon,\mu)}\right] + \mu c(x)\left[\frac{U_{i+1}^{j+1} - U_i^{j+1}}{h}\right] - d(x)U_i^{j+1} = H_i^{j+1}, \\ U_0^{j+1} = s_0^{t_{j+1}},\ U_N^{j+1} = m_N^{t_{j+1}},\ \ j = 1,2,3,...,M-1, \end{cases} \tag{24}$$

where the denominator function $\phi_i^2(h,\varepsilon,\mu)$ is written as

$$\phi_i^2(h,\varepsilon,\mu) \equiv \phi_i^2 = \frac{h\varepsilon}{\mu c(x)}\left(\exp\left(\frac{\mu c(x)h}{\varepsilon}\right) - 1\right).$$

Now, Eq. (24) can be expressed as

$$\begin{cases} u^0(x_i) = g^0(x_i), & 1 \le i \le N-1, \\ E_i^{j+1}U_{i+1}^{j+1} + F_i^{j+1}U_i^{j+1} + G_i^{j+1}U_{i-1}^{j+1} = H_i^{j+1}, & \\ U_0^{j+1} = s_0^{t_{j+1}},\quad U_N^{j+1} = m_N^{t_{j+1}}, & 1 \le j \le M-1, \end{cases} \tag{25}$$

with

$$E_i^{j+1} = \left(-\frac{\varepsilon}{\phi_i^2} + \frac{\mu c(x)}{h}\right),\quad F_i^{j+1} = \left(\frac{2\varepsilon}{\phi_i^2} - \frac{\mu c(x)}{h} - d(x)\right),$$

$$G_i^{j+1} = \left(-\frac{\varepsilon}{\phi_i^2}\right),\qquad H_i^{j+1} = \mu a^{j+1,(k)}u^{j+1,(k)}u_x^{j+1,(k)} - b^{j+1,(k)}\left(u^{j+1,(k)}\right)^2 + f^{j+1,(k)} + \frac{1}{\Delta t}u^{j,(k)}.$$

4 Analysis of the developed numerical scheme

Lemma 8. Let Ψ_i^{j+1} be any mesh function for $i=0,1,2,\ldots,N$. If $\Psi_0^{j+1}\geq 0$, $\Psi_N^{j+1}\geq 0$ and $\left[L_{\varepsilon,\mu}^{N,M}\Psi\right]_i^{j+1}\geq 0$ for $1\leq i\leq N$, then $\Psi_i^{j+1}\geq 0$, for $i\in[0,N]$.

Proof. The proof is by contradiction. Assume there exists an index $i^*\in\{1,2,\ldots,N-1\}$ such that $\Psi_{i^*}^{j+1}<0$ and that this is the point where the mesh function attains its minimum value, i.e.,

$$\Psi_{i^*}^{j+1}=\min_{0\leq i\leq N}\Psi_i^{j+1}.$$

By the hypothesis, $\Psi_0^{j+1}\geq 0$ and $\Psi_N^{j+1}\geq 0$, so the minimum must occur at an interior point, confirming $i^*\notin\{0,N\}$.

Since i^* is an interior point where the minimum occurs, the following discrete conditions hold:

$$\Psi_{i^*+1}^{j+1}-\Psi_{i^*}^{j+1}\geq 0,\quad \Psi_{i^*}^{j+1}-\Psi_{i^*-1}^{j+1}\leq 0. \tag{26}$$

Now, consider the discrete operator applied at this point $i=i^*$. Recall the full definition of the operator from the scheme:

$$\left[L_{\varepsilon,\mu}^{N,M}\Psi\right]_i^{j+1}=-\varepsilon\delta_x^2\Psi_i^{j+1}+\mu c_i^{j+1}D_x^+\Psi_i^{j+1}-d_i^{j+1}\Psi_i^{j+1},$$

where

$$\delta_x^2\Psi_i^{j+1}=\frac{\Psi_{i+1}^{j+1}-2\Psi_i^{j+1}+\Psi_{i-1}^{j+1}}{\phi_i^2},\quad D_x^+\Psi_i^{j+1}=\frac{\Psi_{i+1}^{j+1}-\Psi_i^{j+1}}{h}.$$

We evaluate the linear operator, $\left[L_{\varepsilon,\mu}^{N,M}\Psi\right]_i^{j+1}$, at $i=i$, and by performing some simplifications, we get

$$\left[L_{\varepsilon,\mu}^{N,M}\Psi\right]_{i^*}^{j+1}=-\varepsilon\left(\frac{\Psi_{i^*+1}^{j+1}-2\Psi_{i^*}^{j+1}+\Psi_{i^*-1}^{j+1}}{\phi_{i^*}^2}\right)+\mu c_{i^*}^{j+1}\left(\frac{\Psi_{i^*+1}^{j+1}-\Psi_{i^*}^{j+1}}{h}\right)-d_{i^*}^{j+1}\Psi_{i^*}^{j+1}\leq 0.$$

This contradicts the assumption that $\left[L_{\varepsilon,\mu}^{N,M}\Psi\right]_i^{j+1}\geq 0$ for $i\in[1,N-1]$. Hence, we conclude that $\Psi_i^{j+1}\geq 0$ for $i\in[0,N]$. □

From this lemma, the full-discrete maximum principle ensures that if the given conditions are satisfied, the mesh function Ψ_i^{j+1} remains non-negative for all grid points in the discrete domain.

Lemma 9 (Discrete uniform stability). Let $\Psi_i^{j+1}, i=0,1,\ldots,N$ be any mesh functions such that $\Psi_0^{j+1}=\Psi_N^{j+1}=0$, then

$$\left|\Psi_i^{j+1}\right|\leq\frac{1}{\rho}\max_{1\leq i\leq N-1}\left|L_\varepsilon^{N,M}\Psi_i^{j+1}\right|,\quad 0\leq i\leq N,$$

where $d(x_i)\geq\rho>0$.

Proof. Define

$$\left[\Psi^{\pm}\right]_{m}^{n+1}=\frac{1}{\rho}\max_{1\leq i\leq N-1}\left|L_{\varepsilon}^{N,M}\Psi_{i}^{n+1}\right|\pm\Psi_{m}^{n+1}.$$

Now, we can show that these comparison functions satisfy the required properties. For $i=0$.

$$\left[\Psi^{\pm}\right]_{0}^{j+1}=\frac{1}{\rho}\max_{1\leq i\leq N-1}\left|L_{\varepsilon}^{N,M}\Psi_{i}^{j+1}\right|\pm\Psi_{0}^{j+1}=\frac{1}{\rho}\max_{1\leq i\leq N-1}\left|L_{\varepsilon}^{N,M}\Psi_{i}^{j+1}\right|\pm s_{0}^{j+1}\geq 0,$$

and for $i=N$

$$\left[\Psi^{\pm}\right]_{N}^{j+1}=\frac{1}{\rho}\max_{1\leq i\leq N-1}\left|L_{\varepsilon}^{N,M}\Psi_{i}^{j+1}\right|\pm\Psi_{N}^{j+1}=\frac{1}{\rho}\max_{1\leq i\leq N-1}\left|L_{\varepsilon}^{N,M}\Psi_{i}^{j+1}\right|\pm m_{N}^{j+1}\geq 0.$$

For $i=1,2,\ldots,N-1$, we have

$$\begin{aligned}L_{\varepsilon}^{N,M}\left[\Psi^{\pm}\right]_{i}^{j+1}&=L_{\varepsilon}^{N,M}\left(\frac{1}{\rho}\max_{1\leq i\leq N-1}\left|L_{\varepsilon}^{N,M}\Psi_{i}^{j+1}\right|\pm\Psi_{i}^{j+1}\right)\\&=\frac{d\left(x_{i}\right)}{\rho}\max_{0\leq i\leq N}\left|L_{\varepsilon}^{N,M}\Psi_{i}^{j+1}\right|\pm L_{\varepsilon}^{N,M}\Psi_{i}^{j+1}\\&\geq\max_{0\leq i\leq N}\left|L_{\varepsilon}^{N,M}\Psi_{i}^{j+1}\right|\pm L_{\varepsilon}^{N,M}\Psi_{i}^{j+1}\geq 0,\end{aligned}$$

where the inequality holds because $d\left(x_{i}\right)\geq\rho$. Therefore, $[\Psi^{\pm}]_{i}^{j+1}$ satisfy the conditions of the full-discrete maximum principle (Lemma 8), which implies that $[\Psi^{\pm}]_{i}^{j+1}\geq 0$ for $0\leq i\leq N$. Consequently, we have
$\left|\Psi_{i}^{j+1}\right|\leq\frac{1}{\rho}\max_{1\leq i\leq N-1}\left|L_{\varepsilon}^{N,M}\Psi_{i}^{j+1}\right|,\quad 0\leq i\leq N.$ □

Theorem 2 (Error estimate). Assume $U^{j+1}(x)$ represents the solution of the semi-discrete problem (21), and U_{i}^{j+1} denotes the solution of the fully-discrete problem (24). If N represents the number of grid points and C denotes a suitably large constant, the following error bound is fulfilled:

$$\left|L_{\varepsilon,\mu}^{N,M}\left(U^{j+1}(x_{i})-U_{i}^{j+1}\right)\right|\leq CN^{-1}.\tag{27}$$

Proof. We proceed by considering the linearized form of the semi-discrete problem, given in (21):

$$-\varepsilon U_{xx}^{j+1}+\mu c(x)U_{x}^{j+1}-d(x)U^{j+1}=H(x),$$

where the coefficients $c(x)$, $d(x)$, and $H(x)$ are defined in (21). The dependence on ε and μ is now explicit in the differential operator.

To establish stability, we construct a barrier function. Let

$$\Psi^{\pm}(x)=\Theta\pm U^{j+1}(x),$$

where the constant Θ is chosen as

$$\Theta=\max\left\{\frac{1}{\rho}\|H\|_{\infty},\|U^{j+1}\|_{\partial D}\right\}.$$

Here, ρ is a lower bound for $-d(x)$ over $[0,1]$. Given the definition of $d(x)$ in (10a):

$$d(x) = -\frac{1}{\Delta t} - \mu a(x,t)u_x + 2b(x,t)u,$$

its sign and lower bound depend critically on μ and the behavior of the solution. However, for a fixed μ and fixed Δt, and assuming the solution derivatives are bounded, we can define $\rho = \inf_{x\in[0,1]}(-d(x)) > 0$.

Now, we verify the conditions for the maximum principle: On the boundary ∂D, we have

$$\Psi^{\pm}(0) = \Theta \pm U^{j+1}(0) \geq 0, \quad \Psi^{\pm}(1) = \Theta \pm U^{j+1}(1) \geq 0,$$

by the definition of Θ.
In the interior points, we apply the linearized operator $L_{\varepsilon,\mu} = -\varepsilon\frac{d^2}{dx^2} + \mu c(x)\frac{d}{dx} - d(x)$ to $\Psi^{\pm}$:

$$L_{\varepsilon,\mu}\Psi^{\pm} = L_{\varepsilon,\mu}(\Theta) \pm L_{\varepsilon,\mu}(U^{j+1}) = -d(x)\Theta \pm H(x).$$

Using the definition of Θ and the fact that $-d(x) \geq \rho > 0$, we obtain

$$L_{\varepsilon,\mu}\Psi^{\pm} \leq -d(x)\Theta + |H(x)| \leq -\rho\Theta + \|H\|_\infty \leq 0.$$

Since $\Psi^{\pm}$ satisfies $L_{\varepsilon,\mu}\Psi^{\pm} \leq 0$ in $(0,1)$ and $\Psi^{\pm} \geq 0$ on ∂D, the maximum principle for linear elliptic equations implies that $\Psi^{\pm}(x) \geq 0$ for all $x \in [0,1]$. Consequently,

$$|U^{j+1}(x)| \leq \Theta = \max\left\{\frac{1}{\rho}\|H\|_\infty, \|U^{j+1}\|_{\partial D}\right\}.$$

Recall that $H(x)$ is defined in (10a) and involves terms like μauu_x, bu^2, f, and $\frac{1}{\Delta t}u^j$. Therefore, $\|H\|_\infty$ can be bounded by

$$\|H\|_\infty \leq C_1(\mu) + \frac{1}{\Delta t}\|u^j\|_\infty + \|f^{j+1}\|_\infty,$$

where $C_1(\mu)$ depends on μ and the bounds of a, b, and the solution u.

Combining these estimates, we obtain

$$\|U^{j+1}\|_\infty \leq \max\left\{\frac{1}{\rho}\left(C_1(\mu) + \frac{1}{\Delta t}\|u^j\|_\infty + \|f^{j+1}\|_\infty\right), \|U^{j+1}\|_{\partial D}\right\}.$$

Since ρ depends on μ, Δt, and the coefficients, we can write

$$\|U^{j+1}\|_\infty \leq C_2(\varepsilon,\mu)\left(\|u^j\|_\infty + \Delta t\|f^{j+1}\|_\infty + \|U^{j+1}\|_{\partial D}\right),$$

where $C_2(\varepsilon,\mu)$ absorbs the dependence on ε (through the linearization process) and μ (through the coefficients $c(x)$ and $d(x)$). This completes the proof. □

By combining Lemma 6 and Theorem 2, we arrive at Theorem 3 below.

Theorem 3 (The main result). If $u(x,t)$ represents the exact solution of problem (1), and U_i^j denotes the solution of the numerical scheme (24), the error estimate for the fully discrete scheme can be expressed as

$$\sup_{0<\varepsilon,\mu\leq 1}\max_{\substack{0\leq i\leq N\\0\leq j\leq M}}\left|u(x_i,t_j) - U_i^j\right| \leq C\left(\Delta t + N^{-1}\right), \tag{28}$$

where C is a constant independent of ε, μ, the time step Δt, and the number of spatial mesh intervals N.

Proof. Let $e_i^j = u(x_i,t_j) - U_i^j$ be the error at the mesh point (x_i,t_j). This error can be decomposed using the solutions of the intermediate problems defined in the analysis, as below

$$e_i^j = \underbrace{\left(u(x_i,t_j) - \hat{u}^j(x_i)\right)}_{\text{Temporal Error, } E^j} + \underbrace{\left(\hat{u}^j(x_i) - U(x_i,t_j)\right)}_{\text{Linearization Error, } L^j} + \underbrace{\left(U(x_i,t_j) - U_i^j\right)}_{\text{Spatial Error, } S_i^j},$$

where $\hat{u}^j(x)$ is the solution of the semi-discrete nonlinear problem, $U(x,t_j)$ is the solution of the linearized semi-discrete problem and U_i^j is the solution of the fully discrete problem.
The linearization error L^j is introduced by Newton's quasilinearization process. Under standard assumptions for Newton's method (sufficient smoothness and a good initial guess), this error can be made negligible after a few iterations, converging quadratically to zero. Therefore, for the purpose of an asymptotic error estimate, we have $\|L^j\| = \mathscr{O}((\Delta t)^2 + N^{-2})$, which is of higher order than the other terms.

The primary errors are thus the temporal discretization error and the spatial discretization error. From Lemma 6, the global error due to the implicit Euler time discretization is bounded as

$$\|E^j\| = \|u(\cdot,t_j) - \hat{u}^j\| \le C_1 \Delta t, \quad \text{for } 0 \le t_j \le T. \tag{29}$$

From Theorem 2, the error due to the spatial discretization using the fitted operator finite difference method is bounded as

$$\max_{0\le i\le N} \left|S_i^j\right| = \max_{0\le i\le N} \left|U(x_i,t_j) - U_i^j\right| \le C_2 N^{-1}. \tag{30}$$

Combining these bounds and using the triangle inequality, the total error satisfies

$$\begin{aligned}\left|e_i^j\right| &\le \left|u(x_i,t_j) - \hat{u}^j(x_i)\right| + \left|\hat{u}^j(x_i) - U(x_i,t_j)\right| + \left|U(x_i,t_j) - U_i^j\right| \\ &\le C_1\Delta t + \mathscr{O}((\Delta t)^2 + N^{-2}) + C_2 N^{-1}.\end{aligned}$$

Neglecting the higher-order terms, we obtain the desired uniform error estimate

$$\left|u(x_i,t_j) - U_i^j\right| \le C\left(\Delta t + N^{-1}\right),$$

where $C = \max\{C_1, C_2\}$ is a constant independent of the parameters ε, μ, Δt, and N. □

5 Numerical examples and discussions

The following examples are implemented to demonstrate the applicability of the proposed scheme in (25). Here, maximum absolute errors (point-wise error), $E_{rr}^{N,M}$ and numerical rate of convergence, $Roc^{N,M}$ are calculated using the double mesh principle given in [6] as follows:

$$E_{rr}^{N,M} = \max_{0\le i,j\le N,M} \left|U^{N,M}(x_i,\ t_j) - U^{2N,2M}(x_{2i},\ t_{2j})\right|,$$

$$Roc^{N,M} = \log_2\left(\frac{E_{rr}^{N,M}}{E_{rr}^{2N,2M}}\right).$$

Example 1. Consider the following nonlinear (nonlinearity in both convection and reaction terms) singularly perturbed problem:

$$\begin{cases} \varepsilon u_{xx}(x,t)+\mu(1+x)u(x,t)u_x(x,t)-(u(x,t))^2-u_t(x,t)=16x^2(1-x)^2, \\ \qquad (x,t)\in(0,1)\times(0,1], \\ u(x,0)=0, \quad x\in[0,1], \\ u(0,t)=0, \quad u(1,t)=0, \quad t\in[0,1]. \end{cases}$$

Example 2. Consider the following nonlinear singularly perturbed problem:

$$\begin{cases} \varepsilon u_{xx}(x,t)+\mu(1+xt)u(x,t)u_x(x,t)-(u(x,t))^2-u_t(x,t)=|xt|x^2t^2(1-xt)^2, & \text{for } (x,t)\in(0,1)\times(0,1], \\ u(x,0)=x(1-x^2)\sin x, & \text{for } x\in[0,1], \\ u(0,t)=0, \quad u(1,t)=0, & \text{for } t\in[0,1]. \end{cases}$$

In this section, we present the results of our numerical experiments to validate the proposed scheme outlined in Eq. (25). The focus is on two nonlinear singularly perturbed problems, where we compute the maximum absolute errors $E_{rr}^{N,M}$ and the numerical rate of convergence $Roc^{N,M}$ using the double mesh principle.

For Example 1, Table 1 illustrates the computed errors and convergence rates with $\mu = 10^{-5}$ and various values of ε. It is evident that as N and M increase, the errors decrease significantly, demonstrating the effectiveness of the numerical scheme. The computed convergence rates indicate that the method achieves a first-order uniform convergence with respect to both time and space variables. In Table 2, we observe the impact of varying the parameter μ while keeping ε constant at 10^{-5}. The results further affirm the stability of the method, as the convergence rates remain consistent across different values of μ.

For Example 2, Table 3 shows the errors and convergence rates when $\mu = 10^{-5}$ is employed. The results indicate a similar trend, with decreasing errors as N and M are increased. The convergence rates remain stable, confirming the robustness of our approach across different problem parameters. In Table 4, we explore the effects of different values of μ for Example 2. Again, the results reveal that the numerical scheme retains its first-order convergence properties regardless of the perturbation parameters.

The solutions plotted in Figures 1 and 3 confirm the method's capability to accurately resolve sharp boundary layers without non-physical oscillations, effectively capturing both the right-layer phenomenon that occurs when μ dominates and the left-layer phenomenon that forms when ε is the dominant parameter. A closer examination of the layer interaction in Figures 2 and 4 further highlights the scheme's stability, providing a monotonic and sharp resolution of the solution's behavior regardless of which perturbation parameter dictates the layer's location. Figure 5 presents the log-log plots of both examples, showing that the convergence is uniform.

Overall, the numerical results from both examples demonstrate that the proposed scheme effectively captures the behavior of the nonlinear singularly perturbed problems while achieving uniform convergence. The method's adaptability to varying parameters, as shown in the tables and figures, underscores its applicability in solving complex PDEs.

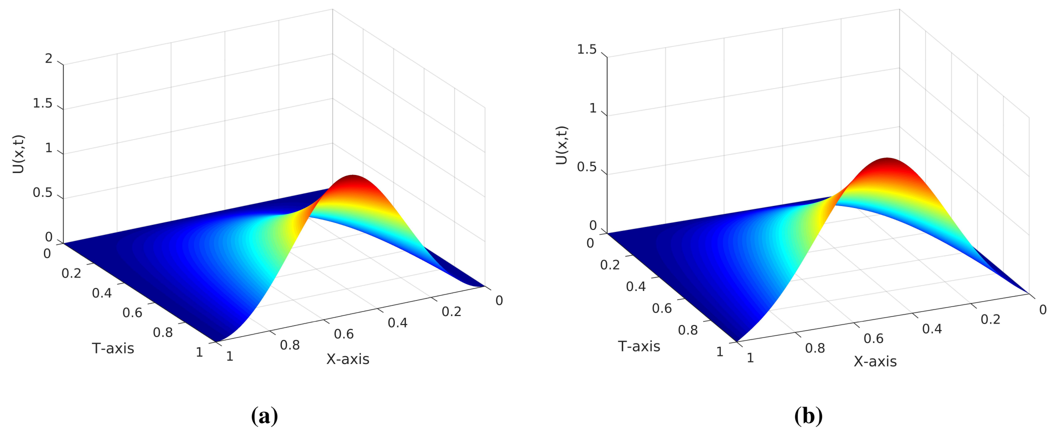

Figure 1: Numerical solution of Example 1 using scheme (25) with (a) $N = 1280,\ M = N/2,\ \varepsilon = 10^{-12},\ \mu = 10^{-2}$ and (b) $N = 1280,\ M = N/2,\ \varepsilon = 10^{-2},\ \mu = 10^{-12}$

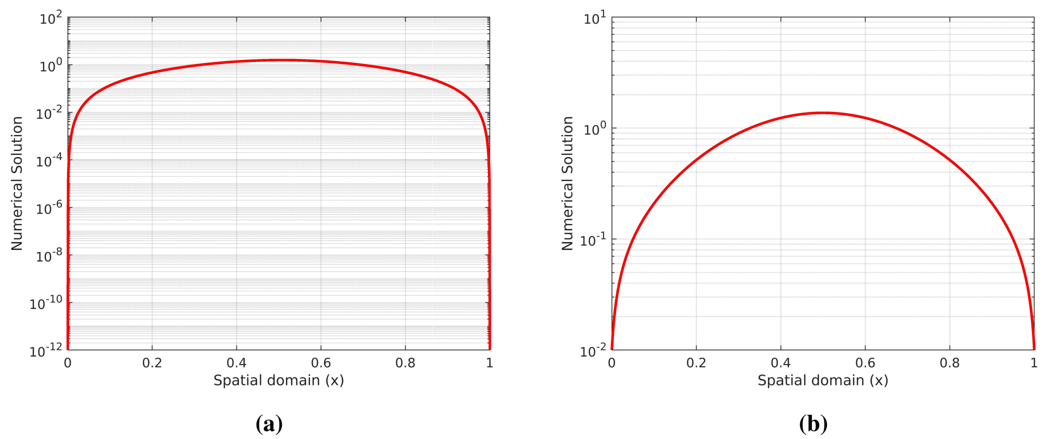

Figure 2: Effect of perturbation parameter on the formation of boundary layers for Example 1 using scheme (25) with (a) $N = 1280,\ M = N/2,\ \varepsilon = 10^{-12},\ \mu = 10^{-2}$ and (b) $N = 1280, M = N/2,\ \varepsilon = 10^{-2},\ \mu = 10^{-12}$

6 Conclusion

This work presents a fitted operator scheme for solving two-parameter nonlinear singularly perturbed initial boundary value problems. The scheme combines Euler's explicit method for temporal discretization on a uniform mesh with a fitted operator central difference scheme for spatial discretization on a uniform mesh. The nonlinearity is addressed using Newton's quasilinearization. Analysis shows that the proposed method is stable and uniformly convergent with first-order accuracy in both the temporal and spatial variables. The theoretical findings are validated through two numerical examples, with results detailed in tables and graphs. Future work could focus on developing higher-order methods to improve computational efficiency.

Table 1: $Err_{\varepsilon,\mu}^{N,M}$ and $Roc_{\varepsilon,\mu}^{N,M}$ using scheme (25) for Example 1 with $\mu = 10^{-5}$ and different values of ε

N→ $\varepsilon \downarrow$ M→	20 10	40 20	80 40	160 80	320 160	640 320
10^{-4}	2.1214e-01 1.0543	1.0215e-01 1.1020	4.7590e-02 1.1957	2.0776e-02 1.4667	7.5168e-03 2.4389	1.3863e-03 -
10^{-6}	2.1624e-01 1.0388	1.0525e-01 1.0661	5.0267e-02 1.1115	2.3265e-02 1.2303	9.9157e-03 1.5949	3.2825e-03 -
10^{-8}	2.1627e-01 1.0387	1.0527e-01 1.0657	5.0292e-02 1.1107	2.3289e-02 1.2284	9.9394e-03 1.5881	3.3060e-03 -
10^{-10}	2.1627e-01 1.0387	1.0527e-01 1.0657	5.0292e-02 1.1107	2.3289e-02 1.2284	9.9394e-03 1.5881	3.3060e-03 -
10^{-12}	2.1627e-01 1.0387	1.0527e-01 1.0657	5.0292e-02 1.1107	2.3289e-02 1.2284	9.9394e-03 1.5881	3.3060e-03 -
⋮	⋮	⋮	⋮	⋮	⋮	⋮
10^{-40}	2.1627e-01 1.0387	1.0527e-01 1.0657	5.0292e-02 1.1107	2.3289e-02 1.2284	9.9394e-03 1.5881	3.3060e-03 -
$Err_{\varepsilon,\mu}^{N,M}$	2.1627e-01	1.0527e-01	5.0292e-02	2.3289e-02	9.9394e-03	3.3060e-03
$Roc_{\varepsilon,\mu}^{N,M}$	1.0387	1.0657	1.1107	1.2284	1.5881	-

Table 2: $Err_{\varepsilon,\mu}^{N,M}$ and $Roc_{\varepsilon,\mu}^{N,M}$ using (25) for Example 1 with $\varepsilon = 10^{-5}$ and different values of μ

(N →) $\mu \downarrow$ (M →)	20 10	40 20	80 40	160 80	320 160	640 320
10^{-4}	2.1633e-01 1.0388	1.0529e-01 1.0658	5.0300e-02 1.1107	2.3293e-02 1.2272	9.9496e-03 1.5790	3.3303e-03 -
10^{-6}	2.1608e-01 1.0385	1.0519e-01 1.0657	5.0256e-02 1.1107	2.3272e-02 1.2284	9.9326e-03 1.5880	3.3038e-03 -
10^{-8}	2.1607e-01 1.0385	1.0519e-01 1.0657	5.0255e-02 1.1107	2.3272e-02 1.2284	9.9325e-03 1.5880	3.3037e-03 -
10^{-10}	2.1607e-01 1.0385	1.0519e-01 1.0657	5.0255e-02 1.1107	2.3272e-02 1.2284	9.9325e-03 1.5880	3.3037e-03 -
10^{-12}	2.1607e-01 1.0385	1.0519e-01 1.0657	5.0255e-02 1.1107	2.3272e-02 1.2284	9.9325e-03 1.5880	3.3037e-03 -
⋮	⋮	⋮	⋮	⋮	⋮	⋮
10^{-40}	2.1607e-01 1.0385	1.0519e-01 1.0657	5.0255e-02 1.1107	2.3272e-02 1.2284	9.9325e-03 1.5880	3.3037e-03 -
$Err_{\varepsilon,\mu}^{N,M}$ $Roc_{\varepsilon,\mu}^{N,M}$	2.1607e-01 1.0385	1.0519e-01 1.0657	5.0255e-02 1.1107	2.3272e-02 1.2284	9.9325e-03 1.5880	3.3037e-03 -

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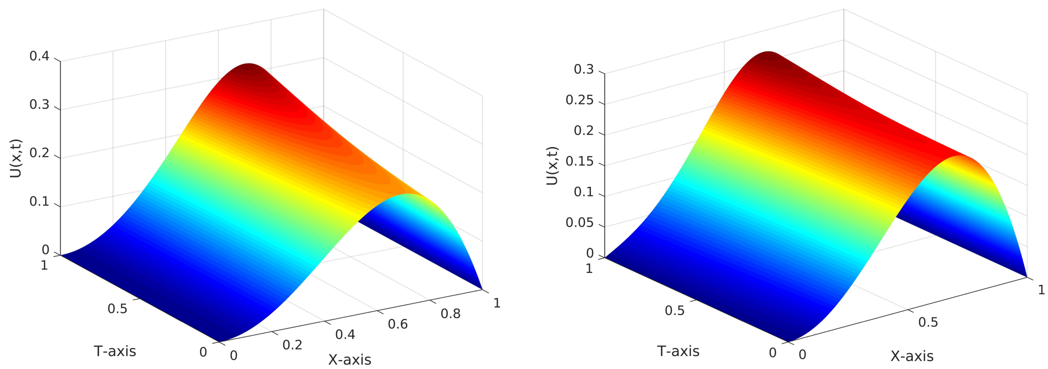
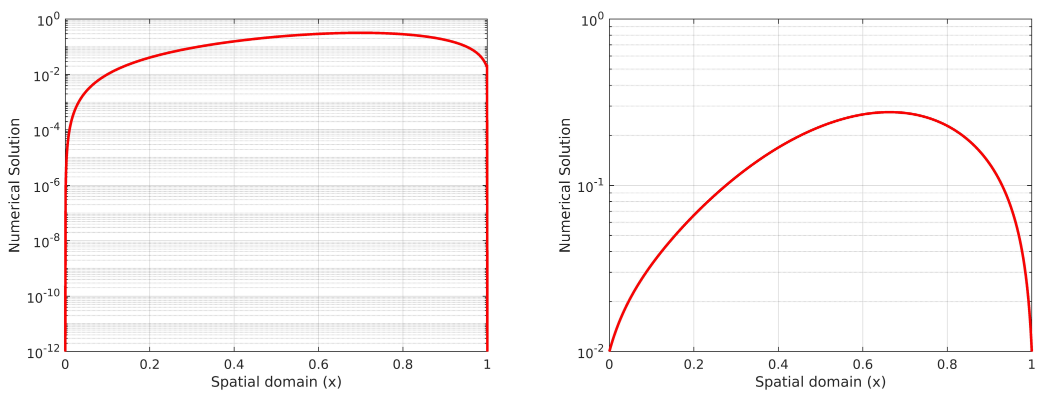


(a) **(b)**

Figure 3: Numerical solution of Example 1 using scheme (25) with (a) $N = 1280,\ M = N/2,\ \ \varepsilon = 10^{-12},\ \mu = 10^{-2}$ and (b) $N = 1280,\ M = N/2,\ \varepsilon = 10^{-2},\ \mu = 10^{-12}$



(a) **(b)**

Figure 4: Effect of perturbation parameter on the formation of boundary layers for Example 1 using scheme (25) with (a) $N = 1280,\ M = N/2,\ \ \varepsilon = 10^{-12},\ \mu = 10^{-2}$ and (b) $N = 1280, M = N/2,\ \varepsilon = 10^{-2},\ \mu = 10^{-12}$

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Table 3: $Err^{N,M}_{\varepsilon,\mu}$ and $Roc^{N,M}_{\varepsilon,\mu}$ using (25) for Example 2 with $\mu = 10^{-5}$ and different values of ε

(N →) μ ↓ (M →)	20 10	40 20	80 40	160 80	320 160	640 320
10^{-8}	5.3701e-03 1.0264	2.6364e-03 1.0321	1.2892e-03 1.0527	6.2147e-04 1.1021	2.8951e-04 1.2240	1.2394e-04 -
10^{-10}	5.3701e-03 1.0264	2.6364e-03 1.0321	1.2892e-03 1.0527	6.2147e-04 1.1020	2.8952e-04 1.2238	1.2396e-04 -
10^{-12}	5.3701e-03 1.0264	2.6364e-03 1.0321	1.2892e-03 1.0527	6.2147e-04 1.1020	2.8952e-04 1.2238	1.2396e-04 -
10^{-14}	5.3701e-03 1.0264	2.6364e-03 1.0321	1.2892e-03 1.0527	6.2147e-04 1.1020	2.8952e-04 1.2238	1.2396e-04 -
10^{-16}	5.3701e-03 1.0264	2.6364e-03 1.0321	1.2892e-03 1.0527	6.2147e-04 1.1020	2.8952e-04 1.2238	1.2396e-04 -
⋮	⋮	⋮	⋮	⋮	⋮	⋮
10^{-40}	5.3701e-03 1.0264	2.6364e-03 1.0321	1.2892e-03 1.0527	6.2147e-04 1.1020	2.8952e-04 1.2238	1.2396e-04 -
$E^{N,M}_{\varepsilon,\mu}$ $Roc^{N,M}_{\varepsilon,\mu}$	5.3701e-03 1.0264	2.6364e-03 1.0321	1.2892e-03 1.0527	6.2147e-04 1.1020	2.8952e-04 1.2238	1.2396e-04 -

Table 4: $Err^{N,M}_{\varepsilon,\mu}$ and $Roc^{N,M}_{\varepsilon,\mu}$ using (25) for Example 1 with $\varepsilon = 10^{-5}$ and different values of μ

(N →) μ ↓ (M →)	20 10	40 20	80 40	160 80	320 160	640 320
10^{-4}	5.3929e-03 1.0297	2.6415e-03 1.0311	1.2926e-03 0.5187	9.0226e-04 1.6331	2.9089e-04 1.1845	1.2799e-04 -
10^{-6}	5.3644e-03 1.0262	2.6339e-03 1.0322	1.2879e-03 0.3773	9.9154e-04 1.5054	3.4926e-04 1.4961	1.2382e-04 -
10^{-8}	5.3640e-03 1.0262	2.6338e-03 1.0322	1.2878e-03 0.3759	9.9245e-04 1.5028	3.5020e-04 1.4997	1.2384e-04 -
10^{-10}	5.3640e-03 1.0262	2.6338e-03 1.0322	1.2878e-03 0.3759	9.9246e-04 1.5028	3.5020e-04 1.4998	1.2384e-04 -
10^{-12}	5.3640e-03 1.0262	2.6338e-03 1.0322	1.2878e-03 0.3759	9.9246e-04 1.5028	3.5020e-04 1.4998	1.2384e-04 -
⋮	⋮	⋮	⋮	⋮	⋮	⋮
10^{-40}	5.3640e-03 1.0262	2.6338e-03 1.0322	1.2878e-03 0.3759	9.9246e-04 1.5028	3.5021e-04 1.4998	1.2384e-04 -
$Err^{N,M}_{\varepsilon,\mu}$ $Roc^{N,M}_{\varepsilon,\mu}$	5.3640e-03 1.0262	2.6338e-03 1.0322	1.2878e-03 0.3759	9.9246e-04 1.5028	3.5021e-04 1.4998	1.2384e-04 -

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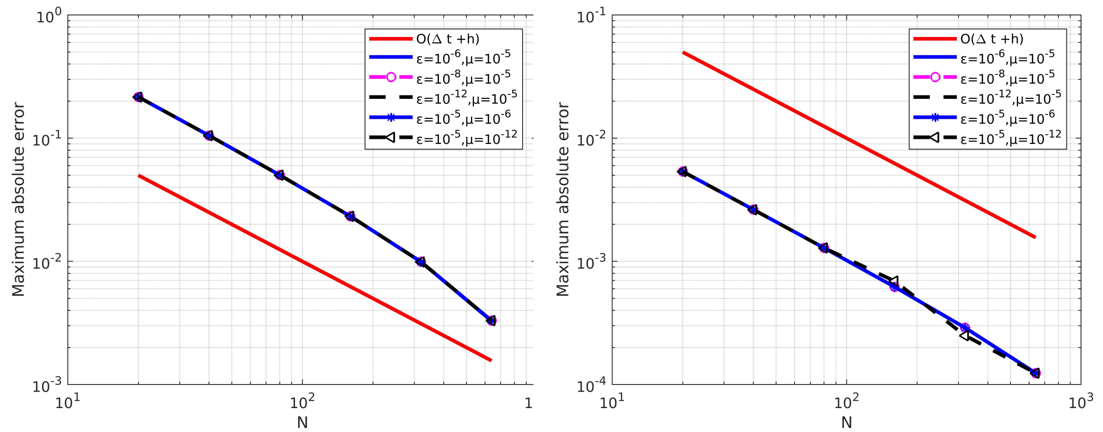

(a) **(b)**

Figure 5: Log Log plots of (a) Example 1 and (b) Example 2

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