



University of Guilan



On a best proximity point theorems for (α, d_G) –regular contractive type mappings in G-metric spaces

Sami H. Jasim^a, Sayyed Hashem Rasouli^a, Azizollah Babakhani^{a,*}

^a Department of Mathematics, Faculty of Basic Sciences, Babol Noshirvani University of Technology, Babol, Iran

ARTICLE INFO

Article history:

Received 14 January 2024

Received in revised form 16 March 2024

Accepted 21 March 2024

Available online 14 November 2025

Keywords:

Best proximity point

 α - ψ -proximal contractive

G-metric space

ABSTRACT

We consider the G-metric space with including the P-property which are introduced by Z. Mustafa and B. Sims (Nonlinear Convex Anal. 7 (2006) 289-297) and presented by B. Samet and et al. in a metric space (Nonlinear Anal. 4 (75) (2012) 2154-2165) respectively. In the present work we define P-property in a G-metric space and proved that under which various conditions there exist a best proximity point for non-self-mapping in G-metric space. Also, we introduced for a certain such mappings which its best proximity point is unique under further conditions.

1. Introduction

Let A and B be two nonempty subsets of a metric space (X, d) . An element $x \in A$ is said to be a fixed point of a given map $T: A \rightarrow B$ if $T(x) = x$. Clearly, $T(A) \cap A \neq \emptyset$ is a necessary (but not sufficient) condition for the existence of a fixed point of T . If $T(A) \cap A = \emptyset$, then $d(x, T(x)) > 0$ for all $x \in A$, that is, the set of fixed points of T is empty. In a such situation, one often attempts to find an element x which is in some sense closest to $T(x)$. Best approximation theory and best proximity point analysis have been developed in this direction.

An element $a \in A$ is called a best approximate point of T if

$$d(a, T(a)) = \inf\{d(T(a), x) : x \in A\}$$

An element $a \in A$ is called a best proximity point of T if

$$d(a, T(a)) = d(A, B)$$

* Corresponding author.

E-mail addresses: babakhani@nit.ac.ir (A. Babakhani)

where

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$$

Because of the fact that $d(x, T(x)) \geq d(A, B)$ for all $x \in A$, the global minimum of the mapping $x \rightarrow d(x, T(x))$ is attained at a best proximity point. Clearly, if the underlying mapping is a self-mapping, then it can be observed that a best proximity point is essentially a fixed point. The goal of best proximity point theory is to furnish sufficient conditions that assure the existence of such points. For more details on this approach, we refer the reader to De la Sen and Ibeas [5], Di Bari, Suzuki and Vetro [6], Suzuki, Eldred and P Veeramani [7], Al-Thagafi and Shahzad [2,3], Sadiq Basha [14,16], and Abkar and Gabeleh [1].

First, we recollect some necessary definitions and results in this direction. The notion of G-metric spaces is defined as follows.

Definition 1. [19] Let X be a non-empty set, $G: X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying the following properties:

- (G1) $G(x, y, z) = 0$ if $x = y = z$,
- (G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
- (G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
- (G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables),
- (G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specially, a G-metric on X , and the pair (X, G) is called a G-metric space.

It should be noted that every G-metric on X induces a metric d_G on X defined by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \text{ for all } x, y \in X.$$

Definition 2. [19] Let (X, G) be a G-metric space, and let (x_n) be a sequence of points of X . We say that (x_n) is G-convergent to $x \in X$ if

$$\lim_{n, m \rightarrow \infty} G(x, x_n, x_m) = 0$$

that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \rightarrow x$ or $\lim x_n = x$.

Proposition 1. [19] Let (X, G) be a G-metric space. The following are equivalent:

- (i) (x_n) is G-convergent to x ,
- (ii) $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow +\infty$,
- (iii) $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow +\infty$,
- (iv) $G(x_n, x_m, x) \rightarrow 0$ as $n, m \rightarrow +\infty$.

Definition 3. [19] Let (X, G) be a G – *metric* space. A sequence (x_n) is called a G – *Cauchy* sequence if, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $m, n, l \geq N$, that is, $G(x_n, x_m, x_l) \rightarrow 0$ as $n, m, l \rightarrow +\infty$.

Proposition 2. [19] Let (X, G) be a G –*metric* space. Then the following statements are equivalent:

- (i) The sequence (x_n) is G -Cauchy,
- (ii) For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $m, n \geq N$.

Proposition 3. [19] Let (X, G) be a G -metric space. A mapping $f: X \rightarrow X$ is G – *continuous* at

$x \in X$ if and only if it is G – *sequentially* continuous at x , that is, whenever (x_n) is G -convergent to x , $(f(x_n))$ is G – *convergent to* $f(x)$.

Proposition 4. [19] Let (X, G) be a G -metric space. Then, the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Proposition 5. [19] Let (X, G) be a G -metric space, then for any $x, y, z, a \in X$ it follows:

- (i) If $G(x, y, z) = 0$ then $x = y = z$,
- (ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
- (iii) $G(x, y, y) \leq 2G(y, x, x)$,
- (iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$.

Definition 4. [19] A G -metric space (X, G) is called G -complete if every G – *Cauchy* sequence is G -convergent in (X, G)

Definition 5. [19] Let (X, G) be a G -metric space. A mapping $F: X \times X \rightarrow X$ is said to be continuous if for any two G – *convergent* sequences (x_n) and (y_n) converging to x and y respectively, $\{F(x_n, y_n)\}$ is G -convergent to $F(x, y)$.

Samet, Vetro and Vetro [18] introduced a new class of contractive mappings called α – ψ – *contractive* type mappings. Let (X, d) be a metric space. Let Ψ be the set of nondecreasing functions $\psi: [0, \infty) \rightarrow [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the n -th iterate of ψ .

Definition 6. [10] A self-mapping $T: X \rightarrow X$ is said to be an α - ψ -contraction, where

$\alpha: X \times X \rightarrow [0, \infty)$ and $\psi \in \Psi$, if

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y)), \text{ for all } x, y \in X.$$

Definition 7. [10] A self-mapping $T: X \rightarrow X$ is said to be α -admissible, where

$\alpha: A \times A \rightarrow [0, \infty)$ and $\psi \in \Psi$, if

$$x, y \in X, \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

In [19] Jleli and Samet used following notions:

$$d(A, B) := \inf \{d(a, b) : a \in A, b \in B\},$$

$$A_0 := \{a \in A : d(a, b) = d(A, B), \text{ for some } b \in B\},$$

$$B_0 := \{b \in B : d(a, b) = d(A, B), \text{ for some } a \in A\}.$$

Definition 8. An element $x^* \in A$ is said to be a best proximity point of the non-self-mapping $T: A \rightarrow B$ if it satisfies the condition that

$$d(x^*, T(x^*)) = d(A, B).$$

Definition 9. [18] Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P – property if and only if :

If $d(x_1, y_1) = d(A, B)$ and $d(x_2, y_2) = d(A, B)$ then $d(x_1, x_2) = d(y_1, y_2)$,

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Definition 10. [18] Let $T: A \rightarrow B$ and $\alpha: A \times A \rightarrow [0, \infty)$. We say that T is α – proximal admissible if:

$$\alpha(x_1, x_2) \geq 1 \text{ and } d(u_1, T(x_1)) = d(A, B) \text{ and } d(u_2, T(x_2)) = d(A, B)$$

$$\text{then } \alpha(u_1, u_2) \geq 1,$$

for all $x_1, x_2, u_1, u_2 \in A$.

Definition 11. [18] A non-self-mapping $T: A \rightarrow B$ is said to be an α - ψ -proximal contraction, where $\alpha: A \times A \rightarrow [0, \infty)$ and $\psi \in \Psi$, if

$$\alpha(x, y) d(T(x), T(y)) \leq \psi(d(x, y)) \text{ for all } x, y \in A.$$

Let (X, G) be a G -metric space. Suppose that A and B are nonempty subsets of a G – metric space (X, G) . Hussain, Latif and Salimi in [20] defined the following sets:

$$A_0 := \{x \in A : d_G(x, y) = d_G(A, B), \text{ for some } y \in B\}$$

$$B_0 := \{y \in B : d_G(x, y) = d_G(A, B), \text{ for some } x \in A\}$$

$$\text{where } d_G(A, B) = \inf \{d_G(x, y) : x \in A, y \in B\}.$$

Definition 12. Let (X, G) be a G – metric space and let A and B be two nonempty subsets of X . Then B is said to be approximatively compact with respect to A if every sequence $\{y_n\}$ in B , satisfying the condition $d_G(x, y_n) \rightarrow d_G(x, B)$ for some $x \in A$, has a convergent subsequence.

2. Main results

The first result of this section is under which conditions there exist a best proximity point for non-self-mapping T in G – metric space.

Before proving this idea, we provide the following definition.

Definition 13. Let (A, B) be a pair of nonempty subsets of a G – metric space (X, G) with

$A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P – property if and only if:

If $d_G(x_1, y_1) = d(A, B)$ and $d_G(x_2, y_2) = d(A, B)$ then

$d_G(x_1, y_1) = d_G(x_2, y_2)$, where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Theorem 1. Let A and B be two subsets of a G – metric space (X, G) such that (A, G) is a complete G – metric space, A_0 is nonempty. Let $\alpha: A \times A \rightarrow [0, \infty)$ and $\psi \in \Psi$. Suppose that $T: A \rightarrow B$ is a non-self mapping satisfying the following conditions:

(i) $T(A_0) \subseteq B_0$ and (A, B) satisfies the P – property;

(ii) T is α – proximal admissible;

(iii) there exist elements x_0 and x_1 in A_0 such that

$$d_G(x_1, T(x_0)) = d_G(A, B) \text{ and } \alpha(x_0, x_1) \geq 1;$$

(iv) T is a G – continuous α – ψ proximal contraction.

Then, there exists an element $x^* \in A_0$ such that $d_G(x^*, T(x^*)) = d_G(A, B)$.

Proof: From condition (iii), there exist elements x_0 and x_1 in A_0 such that

$$d_G(x_1, T(x_0)) = d_G(A, B) \text{ and } \alpha(x_0, x_1) \geq 1. \quad (1)$$

Since $T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d_G(x_2, T(x_1)) = d_G(A, B).$$

Now we have

$$\alpha(x_0, x_1) \geq 1, d_G(x_1, T(x_0)) = d_G(A, B) \text{ and } d_G(x_2, T(x_1)) = d_G(A, B). \quad (2)$$

Since T is proximal admissible, this implies $\alpha(x_1, x_2) \geq 1$. Thus, we have

$$d_G(x_2, T(x_1)) = d_G(A, B) \text{ and } \alpha(x_1, x_2) \geq 1. \quad (3)$$

Again, since $T(A_0) \subseteq B_0$ and continuing this process by induction we obtain a sequence

$\{x_n\} \subseteq A_0$ such that

$$d_G(x_{n+1}, T(x_n)) = d_G(A, B) \text{ and } \alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}. \quad (4)$$

Since (A, B) satisfies P – property, we conclude from (1) that

$$d_G(x_n, T(x_n)) = d_G(A, B), \text{ for all } n \in \mathbb{N}. \quad (5)$$

From the condition (iv), that is, T is α – ψ -proximal contraction for all $n \in \mathbb{N}$, we have

$$\alpha(x_n, x_{n+1}) d_G(T(x_{n-1}), T(x_n)) \leq \psi(d_G(x_{n-1}, x_n)).$$

On the other hand, from (1) we have $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Which implies the above inequality that

$$d_G(T(x_{n-1}), T(x_n)) \leq \psi(d_G(x_{n-1}, x_n)), \text{ for all } n \in \mathbb{N}. \quad (6)$$

Combining (2) and (3) yields

$$d_G(x_n, x_{n+1}) \leq \psi(d_G(x_{n-1}, x_n)), \text{ for all } n \in \mathbb{N}. \quad (7)$$

Suppose that for some positive integer k , we have $x_k = x_{k+1}$. This implies immediately from (1) that,

$$d_G(x_k, T(x_k)) = d_G(x_{k+1}, T(x_k)) = d_G(A, B),$$

that is x_k is a best proximity point of T . So, we can suppose that

$$d_G(x_n, x_{n+1}) > 0 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Using the monotony of ψ , by induction, it follows from (4) that

$$d_G(x_n, x_{n+1}) \leq \psi^n(d_G(x_1, x_0)), \text{ for all } n \in \mathbb{N} \cup \{0\}. \quad (8)$$

Now we will show that $\{x_n\}$ is a Cauchy sequence. Let $\epsilon > 0$ be fixed. Since

$$\sum_{n=1}^{\infty} \psi^n(d_G(x_1, x_0)) < \infty, \text{ there exists some positive integer } h = h(\epsilon) \text{ such that}$$

$$\sum \psi^k(d_G(x_1, x_0)) < \epsilon. \quad (9)$$

Let $m > n > k$, using the rectangle inequality, (5) and (6) we have

$$d_G(x_n, x_m) \leq \sum_{k=n}^{m-1} (d_G(x_k, x_{k+1})) \leq \sum_{k=n}^{m-1} \psi^k(d_G(x_1, x_0)) \leq \sum_{k>n} \psi^k(d_G(x_1, x_0)) < \epsilon.$$

Thus, we prove that $\{x_n\}$ is a Cauchy sequence in G -metric space (X, G) . Since (X, G) is G -complete and A is closed, there exists some $x^* \in A$ such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$.

On the other hand, T is a continuous mapping. Then we have $T(x_n) \rightarrow T(x^*)$ as $n \rightarrow \infty$. The continuity of G -metric function d_G implies that

$$d_G(A, B) = d_G(x_{n+1}, T(x_n)) \rightarrow d_G(x^*, T(x^*)) \text{ as } n \rightarrow \infty.$$

Therefore $d_G(x^*, T(x^*)) = d_G(A, B)$. This completes the proof of the theorem. $\square$

In the next result, we remove the continuity hypothesis, assuming the following condition in A :

(H) If $\{x_n\}$ be a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow x \in A$ as $n \rightarrow \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k .

Theorem 2. Let A and B be two subsets of a G -metric space (X, G) such that (A, G) is a complete G -metric space, A_0 is nonempty. Let $\alpha: A \times A \rightarrow [0, \infty)$ and $\psi \in \Psi$. Suppose that $T: A \rightarrow B$ is a non-self mapping satisfying the following conditions:

- (i) $T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property;
- (ii) T is α -proximal admissible;

(iii) there exist elements x_0 and x_1 in A_0 such that

$$d_G(x_1, T(x_0)) = d_G(A, B) \text{ and } \alpha(x_0, x_1) \geq 1;$$

(iv) (H) holds and T is an $\alpha - \psi$ proximal contraction.

Then, there exists an element $x^* \in A_0$ such that $d(x^*, T(x^*)) = d(A, B)$.

Proof: Following the proof of Theorem 1, there exist a Cauchy sequence $\{x_n\} \subset A$ such that (1) holds and $x_n \rightarrow x^* \in A$ as $n \rightarrow \infty$. From the condition (H), there exists a subsequence

$\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all k . We claim that

$$T(x_{n(k)}) \rightarrow T(x^*), \text{ as } k \rightarrow \infty. \quad (10)$$

Since T is an $\alpha - \psi$ proximal contraction, we have

$$d_G(T(x_{n(k)}), T(x^*)) \leq \alpha(x_{n(k)}, x^*) d_G(T(x_{n(k)}), T(x^*)) \leq \psi(d_G(x_{n(k)}, x^*)), \text{ for all } k.$$

Now our claim (10) follows immediately by letting $k \rightarrow \infty$ in the above inequality. The continuity of the G -metric function d_G implies that

$$d_G(A, B) = d_G((x_{n(k)+1}), T(x_{n(k)})) \rightarrow d_G(x^*, T(x^*)), \text{ as } n \rightarrow \infty.$$

Therefore, $d_G(x^*, T(x^*)) = d(A, B)$. This completes the proof. $\square$

The next result gives us a sufficient condition that assures the uniqueness of the best proximity point. At first, we need the following definition.

Definition 14. Let $T: A \rightarrow B$ be a non-self-mapping and $\alpha: A \times A \rightarrow [0, \infty)$. We say that T is (α, d_G) -regular if for all $(x, y) \in \alpha^{-1}([0, 1])$, there exists $z \in A_0$ such that

$$\alpha(x, z) \geq 1 \text{ and } \alpha(y, z) \geq 1.$$

Theorem 3. In addition to the hypothesis of Theorem 1 (resp. Theorem 2), suppose that T is (α, d_G) -regular. Then T has a unique best proximity point.

Proof: From the proof of Theorem 1, we know that the set of best proximity points of T is

nonempty ($x^* \in A_0$ is a best proximity point). Suppose that $y^* \in A_0$ is another best proximity point of T , that is,

$$d_G(T(x^*), x^*) = d_G(T(y^*), y^*) = d_G(A, B). \quad (11)$$

Using the P-property and (11), we get

$$d_G(T(x^*), T(y^*)) = d_G(x^*, y^*). \quad (12)$$

We distinguish two cases:

Case 1. If $\alpha(x^*, y^*) \geq 1$.

Since T is an α - ψ proximal contraction, using (12), we obtain that

$$d_G(x^*, y^*) = d_G(T(x^*), T(y^*)) \leq \alpha(x^*, y^*) d_G(T(x^*), T(y^*)) \leq \psi(d_G(x^*, y^*)).$$

Since $\psi(t) < t$ for all $t > 0$, the above inequality holds only if $d_G(x^*, y^*) = 0$, that is

$$x^* = y^*.$$

Case 2. If $\alpha(x^*, y^*) < 1$.

By hypothesis, there exists $z_0 \in A_0$ such that $\alpha(x^*, z_0) \geq 1$ and $\alpha(y^*, z_0) \geq 1$. Since $T(A_0) \subseteq B_0$, there exists $z_1 \in A_0$ such that

$$d_G(z_1, T(z_0)) = d_G(A, B).$$

Now we have

$$\alpha(x^*, z_0) \geq 1, d_G(x^*, T(x^*)) = d_G(A, B) \text{ and } d_G(z_1, T(z_0)) = d_G(A, B).$$

Since T is α -proximal admissible, we get that $\alpha(x^*, z_1) \geq 1$. Thus, we have

$$\alpha(x^*, z_0) \geq 1,$$

$$d_G(z_1, T(z_0)) = d_G(A, B) \text{ and } \alpha(x^*, z_1) \geq 1.$$

Continuing this process, by induction, we can construct a sequence $\{z_n\}$ in A_0 such that

$$d_G(z_{n+1}, T(z_n)) = d_G(A, B) \text{ and } \alpha(x^*, z_n) \geq 1, \text{ for all } n \in \mathbb{N} \cup \{0\}. \quad (13)$$

Using the P -property and (13),

$$d_G(z_{n+1}, x^*) = d_G(T(z_n), T(x^*)), \text{ for all } n \in \mathbb{N} \cup \{0\}. \quad (14)$$

Since T is an α - ψ proximal contraction, we have

$$\alpha(z_{n+1}, x^*) d_G(T(z_n), T(x^*)) \leq \psi(d_G(z_n, x^*)), \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Combining the above inequality with (14), we get

$$\alpha(z_{n+1}, x^*) d_G(z_{n+1}, x^*) \leq \psi(d_G(z_n, x^*)), \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

This implies from (13) that

$$d_G(z_{n+1}, x^*) \leq \psi(d_G(z_n, x^*)), \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

By induction we then derive

$$d_G(z_n, x^*) \leq \psi^n(d_G(z_0, x^*)), \text{ for all } n \in \mathbb{N} \cup \{0\}. \quad (15)$$

Suppose that $z_0 = x^*$, in this case, from (14),

$$d_G(z_1, x^*) = d_G(T(z_0), T(x^*)) = d_G(T(x^*), T(x^*)) = 0,$$

that is $z_1 = x^*$. Continuing this process, by induction, we get that $z_n = x^*$ for all

$n \in \mathbb{N} \cup \{0\}$.

Suppose now that $d_G(z_0, x^*) > 0$. Letting $n \rightarrow \infty$ in (15), we obtain that $z_n \rightarrow x^*$ as $n \rightarrow \infty$. So in all cases, we have $z_n \rightarrow x^*$ as $n \rightarrow \infty$. Similarly, we can prove that

$z_n \rightarrow y^*$ as $n \rightarrow \infty$. By uniqueness of the limit, we obtain that $x^* = y^*$.

□

Example 1. Let $X = [0, \infty)$ and $G(x, y, z) = \frac{1}{4}(|x - y| + |y - z| + |x - z|)$ be a G -metric on X . Then $d_G(x, y) = |x - y|$.

Let $A = \{3, 4, 5, 6, 7\}$ and $B = \{9, 10, 11, 12, 13\}$. Define $T: A \cup B \rightarrow A \cup B$ by

$T(x) = 9$ if $x = 7$ and otherwise $T(x) = x + 6$.

Define $\psi: [0, \infty) \rightarrow [0, \infty)$ by $\psi(t) = t$ and $\alpha(x, y) = 1$.

Clearly, $d_G(A, B) = 2$, $A_0 = \{7\}$, $B_0 = \{9\}$, and $T(A_0) \subseteq B_0$.

Since $d_G(x_1, y_1) = d_G(A, B) = 2$ and $d_G(x_2, y_2) = d_G(A, B) = 2$ then

$d_G(x_1, y_1) = d_G(x_2, y_2)$ so, the pair (A, B) have the P -property.

If $\alpha(x_1, y_1) \geq 1$ and $d_G(u_1, T(x_1)) = d_G(A, B)$ and $d_G(u_2, T(x_2)) = d_G(A, B)$

then $\alpha(u_1, u_2) \geq 1$ for all $x_1, y_1, u_1, u_2 \in A$. So, T is α -proximal admissible.

Furthermore, if we choose $x_0 = x_1 = 7 \in A_0$ then we have:

$d_G(x_1, T(x_0)) = d_G(A, B) = 2$ and $\alpha(x_0, x_1) \geq 1$.

Also, it is clear that T is a continuous α - ψ -proximal contraction.

Therefore, all conditions of Theorem 1 hold and T has a best proximity point. Here $x_0 = 7$ is unique best proximity point.

References

- [1] Abkar, A., & Gabeleh, M. (2011). Best proximity points for cyclic mappings in ordered metric spaces. *Journal of Optimization Theory and Applications*, 150(1), 188-193.
- [2] Al-Thagafi, M. A., & Shahzad, N. (2009). Best proximity pairs and equilibrium pairs for Kakutani multimaps. *Nonlinear Analysis: Theory, Methods & Applications*, 70(3), 1209-1216.
- [3] Al-Thagafi, M. A., & Shahzad, N. (2009). Convergence and existence results for best proximity points. *Nonlinear Analysis: Theory, Methods & Applications*, 70(10), 3665-3671.
- [4] Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta mathematicae*, 3(1), 133-181.
- [5] De la Sen, M., & Ibeas, A. (2013). Fixed points and best proximity points in contractive cyclic self-maps satisfying constraints in closed integral form with some applications. *Applied Mathematics and Computation*, 219(10), 5410-5426.
- [6] Di Bari, C., Suzuki, T., & Vetro, C. (2008). Best proximity points for cyclic Meir-Keeler contractions. *Nonlinear Analysis: Theory, Methods & Applications*, 69(11), 3790-3794.
- [7] Eldred, A. A., & Veeramani, P. (2006). Existence and convergence of best proximity points. *Journal of Mathematical Analysis and Applications*, 323(2), 1001-1006.
- [8] Fan, K. (1969). Extensions of two fixed point theorems of FE Browder. *Mathematische zeitschrift*, 112(3), 234-240.
- [9] Raji, M., Rajpoot, A. K., Al-omeri, W. F., Rathour, L., Mishra, L. N., & Mishra, V. N. (2012). Generalized α - ψ Contractive Type Mappings and Related Fixed Point Theorems with Applications. *Tuijin Jishu/Journal of Propulsion Technology*, 45(10), 2024.
- [10] Kirk, W. A., Srinivasan, P. S., & Veeramani, P. (2003). Fixed points for mappings satisfying cyclical contractive conditions. *Fixed point theory*, 4(1), 79-89.
- [11] Lakshmikantham, V., & Ćirić, L. (2009). Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 70(12), 4341-4349.

- [12] Prolla, João B. "Fixed-point theorems for set-valued mappings and existence of best approximants." *Numerical Functional Analysis and Optimization* 5.4 (1983): 449-455.
- [13] Reich, S. (1978). Approximate selections, best approximations, fixed points, and invariant sets. *Journal of Mathematical Analysis and Applications*, 62(1), 104-113.
- [14] Sadiq Basha, S. (2010). Extensions of Banach's contraction principle. *Numerical Functional Analysis and Optimization*, 31(5), 569-576.
- [15] Basha, S. S. (2011). Best proximity point theorems generalizing the contraction principle. *Nonlinear Analysis: Theory, Methods & Applications*, 74(17), 5844-5850.
- [16] Basha, S. S. (2012). Best proximity point theorems: an exploration of a common solution to approximation and optimization problems. *Applied Mathematics and Computation*, 218(19), 9773-9780.
- [17] Samet, B., & Turinici, M. (2012). Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications.
- [18] Samet, B., Vetro, C., & Vetro, P. (2012). Fixed point theorems for α - ψ -contractive type mappings. *Nonlinear analysis: theory, methods & applications*, 75(4), 2154-2165.
- [19] Mustafa, Z., & Sims, B. (2006). A new approach to generalized metric spaces. *Journal of Nonlinear and convex Analysis*, 7(2), 289.
- [20] Jleli, M., & Samet, B. (2013). Best proximity points for α - ψ -proximal contractive type mappings and applications. *Bulletin des Sciences Mathématiques*, 137(8), 977-995.
- [21] Hussain, N., Latif, A., & Salimi, P. (2014). Best Proximity Point Results in G-Metric Spaces. In *Abstract and Applied Analysis* (Vol. 2014, No. 1, p. 837943). Hindawi Publishing Corporation.