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Improved lower bound of spatial analyticity radius for solutions to nonlinear wave equation

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Abstract. In this paper, the rate of decay for the radius of spatial analyticity for solutions of the nonlinear wave equation

$$\partial_t^2 u - \Delta u + |u|^{p-1}u = 0,$$

on $\mathbb{R}^d \times \mathbb{R}$ is studied. In particular, for a class of analytic initial data with a uniform radius of analyticity σ_0, we obtain an asymptotic lower bound $\sigma(t) \geq a_0|t|^{-\frac{2}{3}}$ when $d = 1$ and $\sigma(t) \geq a_0|t|^{-\frac{3}{2}}$ when $d = 2$ on the uniform radius of analyticity $\sigma(t)$ of solution $u(\cdot,t)$ as $|t| \to +\infty$. This is an improvement of the work [D. O. da Silva, A. J. Castro, Global well-posedness for the nonlinear wave equation in analytic Gevrey spaces, J. Differential Equations 275(2021) 234–249], where the authors obtained a decay rate of order $\sigma(t) \geq a_0(1+|t|)^{-(\frac{p+1}{2})}$ when $d = 1$ and $\sigma(t) \geq a_0(1+|t|)^{-(\frac{p+1-\varepsilon}{1-\varepsilon})}$ when $d = 2$ as $|t| \to +\infty$ for large time t, where $\varepsilon > 0$ is arbitrary. We used an approximate conservation law in a modified Gevrey space, contraction mapping principle, interpolation and Sobolev embedding to obtain the results. [1]

Keywords: Nonlinear wave equation, modified Gevrey space, approximate conservation, radius of analyticity, decay rate for the radius.
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1 Introduction

In this paper, we consider the Cauchy problem for the nonlinear wave (NLW) equation

$$\begin{cases} \partial_t^2 u - \Delta u + |u|^{p-1}u = 0, \quad x \in \mathbb{R}^d,\ t \in \mathbb{R}, \\ u(x,0) = f(x), \\ \partial_t u(x,0) = g(x), \end{cases} \tag{1}$$

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where $u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$, $p > 1$ be any odd integer, and $\Delta = \sum_{j=1}^{d} \partial_{x_j}^2$.

The following energy is conserved under the flow of (1):

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^d} \left(|\partial_t u|^2 + |\nabla u|^2 + \frac{2}{p+1} |u|^{p+1} \right) dx = E(0), \quad \forall t. \tag{2}$$

The NLW equation (1) has a long history and many results are known. For a detailed exposition, we refer the readers that [20] and the references therein. Local well-posedness of (1) in the homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^2)$ has been showed by Lindblad and Sogge [14] with optimal regularity given by

$$s(p) = \begin{cases} \frac{3}{4} - \frac{1}{p-1} & \text{if } 3 \le p \le 5, \\ 1 - \frac{2}{p-1} & \text{if } p \ge 5. \end{cases}$$

We refer the readers also to see the refinements of Nakamura and Ozawa [15]. These results that are obtained for the homogeneous Sobolev spaces can be extended to the inhomogeneous Sobolev spaces H^s by a simple integration in time argument [3].

The main concern of this work is to estimate the radius of spatial analyticity for the solution of the Cauchy problem (1), given initial data in a class of analytic functions. A class of analytic function spaces suitable for our analysis is the modified Gevrey class introduced in [4]. This space is denoted by $H^{\sigma,s} = H^{\sigma,s}(\mathbb{R}^d)$ and consists of functions such that

$$\left\{ \phi \in L_x^2(\mathbb{R}^d) : \|\phi\|_{H^{\sigma,s}} = \left\| \cosh(\sigma|\xi|) \langle \xi \rangle^s) \hat{\phi}(\xi) \right\|_{L_\xi^2(\mathbb{R}^d)} < \infty \right\},$$

where $\langle \xi \rangle^2 = 1 + |\xi|^2$ and $\hat{\phi}(\xi)$ denotes the spatial Fourier transform of $\phi(x)$ which is given by[2]

$$\hat{\phi}(\xi) := \mathscr{F}_x[\phi](\xi) = \int_{\mathbb{R}^d} e^{-i(x\cdot\xi)} \phi(x) dx.$$

The authors in [4] obtained this space from the Gevrey space $G^{\sigma,s}(\mathbb{R}^d)$ by replacing the exponential weight $e^{\sigma|\xi|}$ with the hyperbolic weight $\cosh(\sigma|\xi|)$. The Gevrey space $G^{\sigma,s} = G^{\sigma,s}(\mathbb{R}^d)$ was first introduced in [6] via the norm

$$\|\phi\|_{G^{\sigma,s}} = \left\| e^{\sigma|\xi|} \langle \xi \rangle^s \hat{\phi} \right\|_{L_\xi^2(\mathbb{R}^d)} \quad (\sigma \ge 0, s \in \mathbb{R}).$$

Observe that the Gevrey and modified Gevrey spaces satisfy the following embedding properties. For all $s, s' \in \mathbb{R}$ and $0 \le \sigma < \sigma'$, we have

$$G^{\sigma',s'} \subset G^{\sigma,s}, \quad \text{i.e.,} \quad \|\phi\|_{G^{\sigma,s}} \lesssim \|\phi\|_{G^{\sigma',s'}}. \tag{3}$$

$$H^{\sigma',s'} \subset H^{\sigma,s}, \quad \text{i.e.,} \quad \|\phi\|_{H^{\sigma,s}} \lesssim \|\phi\|_{H^{\sigma',s'}}. \tag{4}$$

For $\sigma = 0$, we have

$$G^{0,s} = H^{0,s} = H^s,$$

[2] $x \cdot \xi$ stands for the usual dot product $\sum_{j=1}^{d} x_j \xi_j$ for all $x_j, \xi_j \in \mathbb{R}$.

where $H^s = H^s(\mathbb{R}^d)$ denotes the standard Sobolev space consisting of functions

$$\|\phi\|_{H^s} = \left\|\langle\xi\rangle^s \hat{\phi}\right\|_{L^2_\xi(\mathbb{R}^d)} < \infty.$$

For $2 \le q < \infty$, we have the Sobolev embedding

$$\|\phi\|_{L^q(\mathbb{R}^d)} \lesssim \|\phi\|_{H^s} \quad \text{if } s \ge d\left(\frac{1}{2} - \frac{1}{q}\right) > 0, \tag{5}$$

$$\|\phi\|_{L^q(\mathbb{R}^d)} \lesssim \|\phi\|_{\dot{H}^s} \quad \text{if } s = d\left(\frac{1}{2} - \frac{1}{q}\right) > 0, \tag{6}$$

where $\dot{H}^s$ denotes the homogeneous Sobolev space of order s. Furthermore, we have [3]

$$\|\phi\|_{L^\infty(\mathbb{R}^d)} \lesssim \|\phi\|_{H^{\frac{d}{2}+}}. \tag{7}$$

By the Paley-Wiener Theorem, the radius of analyticity of a function can be related to decay properties of its Fourier transform. It is therefore natural to take data for (1) in $G^{\sigma,s}$. For $\sigma > 0$, any function in $G^{\sigma,s}$ has a radius of analyticity of at least σ at each point $x \in \mathbb{R}^d$. This fact is contained in the following theorem, whose proof can be found in [13, p. 174] in the case $s = 0$ and $d = 1$; the general case follows from a simple modification.

Lemma 1 (Paley-Wiener Theorem). *For $\sigma \ge 0, s \in \mathbb{R}$, the function $\phi \in G^{\sigma,s}$ if and only if $\phi(\cdot)$ is the restriction to the real line of a function $\Psi(\cdot + iy)$ which is holomorphic in the strip*

$$S_\sigma = \{x + iy \in \mathbb{C} : |y| < \sigma\},$$

and satisfies the bound

$$\sup_{|y|<\sigma} \|\Psi(\cdot + iy)\|_{H^s_x} < \infty.$$

The exponential and hyperbolic weights are equivalent in the sense that

$$\frac{1}{2} e^{\sigma|\xi|} \le \cosh(\sigma|\xi|) \le e^{\sigma|\xi|}, \quad \xi \in \mathbb{R}^d, \tag{8}$$

from which, we find

$$\frac{1}{2}\|\phi\|_{G^{\sigma,s}} \le \|\phi\|_{H^{\sigma,s}} \le \|\phi\|_{G^{\sigma,s}}.$$

In other words, the associated $H^{\sigma,s}$ and $G^{\sigma,s}$–norms are equivalent, i.e.,

$$\|\phi\|_{H^{\sigma,s}} \sim \|\phi\|_{G^{\sigma,s}} = \left\| e^{\sigma|\xi|} \langle\xi\rangle^s \hat{\phi} \right\|_{L^2_\xi(\mathbb{R}^d)}. \tag{9}$$

Therefore, (9) guarantees that the statement of Paley-Wiener Theorem still holds for functions in $H^{\sigma,s}$.

The study of spatial analyticity of solutions to nonlinear Cauchy problems was initiated by Kato and Masuda [12]. Since then, several mathematicians have considered the Cauchy problem for a variety of equations with initial data in $G^{\sigma,s}$–spaces (see, for example e.g., [2, 5, 17, 18, 21–23] and the references

[3] Here, $a+ = a + \varepsilon$ for sufficiently small $\varepsilon > 0$.

therein). For studies with initial data in the modified Gevrey spaces see for example [4, 7–10] and references therein

Coming back to (1), the asymptotic lower bound for the radius of spatial analyticity of the solutions has been established by da Silva and Castro [3], who obtained a decay rate of order $\sigma(t) \geq c(1+|t|)^{-(\frac{p+1}{2})}$ when $d=1$ and $\sigma(t) \geq c(1+|t|)^{-(\frac{p+1-\varepsilon}{1-\varepsilon})}$ when $d=2$ as $|t| \to +\infty$, where $\varepsilon > 0$ is arbitrary. The strategy in [3] is as follows:

1. Prove a local well-posedness by standard fixed-point argument in $G^{\sigma,s} \times G^{\sigma,s-1}$ with a lifespan $T > 0$.
2. Establish an almost conservation law in $G^{\sigma,1} \times G^{\sigma,0}$.
3. By shrinking σ gradually, they used repeatedly the local well-posedness and the almost conservation law on the intervals $[0,T],[T,2T],\cdots$, and obtained a global bound of solution on $[0,\delta]$ for arbitrarily large δ. This idea was introduced by Selberg and Tesfahun in [19], where it was applied to the $1D$ Dirac-Klein-Gordon equations.

The persistence of spatial analyticity of (1) in the periodic setting was dealt by Guo and Titi in [11].

As a consequence of the embedding (4) and the existing well-posedness theory in $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, the Cauchy problem (1) with initial data $f,g \in H^{\sigma_0,1}(\mathbb{R}^d) \times H^{\sigma_0,0}(\mathbb{R}^d)$ for some $\sigma_0 > 0$ has a unique and global in time solution.

The main result in this paper is as follows.

Theorem 1 (Decay rate for the radius of analyticity)**.** *Let $d=1$ or 2 and $p>1$ be any odd integer. If $(f,g) \in H^{\sigma_0,1} \times H^{\sigma_0,0}$ for $\sigma_0 > 0$, then for any $\delta > 0$ the global solution u of (1) satisfies*

$$(u, \partial_t u) \in C\left([0,\delta]; H^{\sigma(t),1} \times H^{\sigma(t),0}\right), \tag{10}$$

where the radius of spatial analyticity $\sigma(t)$ satisfies an asymptotic rate of decay

$$\sigma(t) \geq a_0 |t|^{-\frac{2}{3}} \ \text{when} \ d=1,$$

and

$$\sigma(t) \geq a_0 |t|^{-\frac{3}{2}} \ \text{when} \ d=2,$$

as $|t| \to +\infty$. Here, $a_0 > 0$ is a constant which depends on the initial data norm.

Notation. For any positive numbers a and b, the notation $a \lesssim b$ stands for $a \leq cb$, where c is a positive constant that can be determined by known parameters in a given situation but whose values are not crucial to the problem at hand and may differ from line to line. We also denote $a \sim b$ to mean $b \lesssim a \lesssim b$.

2 Local well-posedness result

The first step in the proof of Theorem 1 is to prove the following local well-posedness result, where the radius of analyticity remains constant.

Theorem 2. *Let $p>1$ be any odd integer and $d=1$ or 2. Given $(f,g)\in H^{\sigma,1}\times H^{\sigma,0}$ for $\sigma>0$, there exists a time $T>0$ and a unique solution*

$$(u,\partial_t u)\in C\left([0,T];H^{\sigma,1}\times H^{\sigma,0}\right),$$

of the Cauchy problem (1) on $\mathbb{R}^d\times[0,T]$. Moreover, the solution depends continuously on the initial data. Furthermore, the existence time is given by

$$T=a_0\left(\|f\|_{H^{\sigma,1}}+\|g\|_{H^{\sigma,0}}\right)^{-(p-1)}, \tag{11}$$

for some constant $a_0>0$. In addition, the local solution u satisfies the bound

$$\sup_{t\in[0,T]}\left(\|u\|_{H^{\sigma,1}}+\|\partial_t u\|_{H^{\sigma,0}}\right)\lesssim\|f\|_{H^{\sigma,1}}+\|g\|_{H^{\sigma,0}}. \tag{12}$$

Proof. Consider the Cauchy problem for the linear wave equation

$$\begin{cases}\partial_t^2 u-\Delta u=F(\cdot,t),\\ u(\cdot,0)=f(\cdot),\\ \partial_t u(\cdot,0)=g(\cdot),\end{cases} \tag{13}$$

whose solution is given by Duhamel's formula

$$u(t)=\partial_t W(t)f+W(t)g+\int_0^t W(t-s)F(\cdot,s)ds, \tag{14}$$

where

$$W(t)=\frac{sin(t|D|)}{|D|}.$$

To obtain an energy inequality from (14), we need the following estimates.

Lemma 2. *([16, Lemma 4.1]) For any $s\in\mathbb{R}$ and any integer $d\geq 1$, we have*

$$\|\partial_t W(t)f\|_{H^s(\mathbb{R}^d)}\lesssim\|f\|_{H^s(\mathbb{R}^d)}, \tag{15}$$

$$\|W(t)g\|_{H^s(\mathbb{R}^d)}\lesssim\|g\|_{H^{s-1}(\mathbb{R}^d)}. \tag{16}$$

Now, applying $\cosh(\sigma|D|)$ to both sides of (14) and taking $H^1(\mathbb{R}^d)$–norm to both sides and then using Lemma 2, we obtain the energy inequality

$$\sup_{t\in[0,T]}\left(\|u\|_{H^{\sigma,1}(\mathbb{R}^d)}+\|\partial_t u\|_{H^{\sigma,0}(\mathbb{R}^d)}\right)\lesssim\left(\|f\|_{H^{\sigma,1}(\mathbb{R}^d)}+\|g\|_{H^{\sigma,0}(\mathbb{R}^d)}\right)+\int_0^T\|F(\cdot,s)\|_{H^{\sigma,0}(\mathbb{R}^d)}ds, \tag{17}$$

for $T>0$ and $d=1,2$.

To use the standard fixed point argument in proving Theorem 2, we need the nonlinear estimate

$$\left\||u|^{p-1}u\right\|_{H^{\sigma,0}(\mathbb{R}^d)}\lesssim\|u\|^p_{H^{\sigma,1}(\mathbb{R}^d)}, \tag{18}$$

Proof. Putting $V = \cosh(\sigma|D|)u$, (18) reduces to

$$\left\| \cosh(\sigma|D|) \left[|\mathrm{sech}(\sigma|D|)V|^{p-1} \mathrm{sech}(\sigma|D|)V \right] \right\|_{L^2_x(\mathbb{R}^d)} \lesssim \|V\|^p_{H^1(\mathbb{R}^d)}. \tag{19}$$

From triangle inequality we have

$$\xi = \sum_{j=1}^{p} \xi_j \Rightarrow |\xi| \leq \sum_{j=1}^{p} |\xi_j|. \tag{20}$$

Morovere, we have from [4] that

$$\cosh(\sigma|\xi|) \prod_{j=1}^{p} \mathrm{sech}(\sigma|\xi_j|) \leq 2^p. \tag{21}$$

By Plancherel's Theorem, (20) and (21), we have

$$\begin{aligned}
\text{LHS of } (19) &= \left\| \mathscr{F}_x \left\{ \cosh(\sigma|D|) \left[|\mathrm{sech}(\sigma|D|)V|^{p-1} \mathrm{sech}(\sigma|D|)V \right] \right\} (\xi) \right\|_{L^2_\xi(\mathbb{R}^d)} \\
&= \left\| \int_{\xi=\sum_{j=1}^p \xi_j} [\cosh(\sigma|\xi|) \prod_{j=1}^{p} \mathrm{sech}(\sigma|\xi_j|)] \prod_{j=1}^{p} \widehat{V}(\xi_j) \prod_{j=1}^{p} d\xi_j \right\|_{L^2_\xi(\mathbb{R}^d)} \\
&\lesssim \left\| \int_{\xi=\sum_{j=1}^p \xi_j} \prod_{j=1}^{p} |\widehat{V}(\xi_j)| \prod_{j=1}^{p} d\xi_j \right\|_{L^2_\xi(\mathbb{R}^d)} \\
&= \|U^p\|_{L^2_x(\mathbb{R}^d)},
\end{aligned}$$

where $U = \mathscr{F}_x^{-1} \left[|\widehat{V}| \right]$.

Here by Sobolev embedding, we obtain

$$\|U^p\|_{L^2_x(\mathbb{R}^d)} = \|U\|^p_{L^{2p}_x(\mathbb{R}^d)} \lesssim \|U\|^p_{H^1(\mathbb{R}^d)} = \|V\|^p_{H^1(\mathbb{R}^d)}, \tag{22}$$

where $p \geq 1$ is any integer. This proves (19), and hence (18). □

Proof of estimate (21). From the definition of cosine hyperbolic function, we have

$$\cosh(\sigma|\xi|) \prod_{j=1}^{p} \mathrm{sech}(\sigma|\xi_j|) = \frac{e^{\sigma|\xi|} + e^{-\sigma|\xi|}}{\prod_{j=1}^{p} (e^{\sigma|\xi_j|} + e^{-\sigma|\xi_j|})} \cdot 2^{p-1}.$$

Since the fraction is less than or equal to one by the key inequality, we obtain

$$\cosh(\sigma|\xi|) \prod_{j=1}^{p} \mathrm{sech}(\sigma|\xi_j|) \leq 1 \cdot 2^{p-1}.$$

Also, since $2^{p-1} \leq 2^p$ for $p \geq 1$, the inequality (21) is proven. That is

$$\cosh(\sigma|\xi|) \prod_{j=1}^{p} \mathrm{sech}(\sigma|\xi_j|) \leq 2^p. \qquad \square$$

Now, the integral form of (1) is given by Duhamel's formula as

$$u(t) = \partial_t W(t) f + W(t) g + \int_0^t W(t-s) |u(s)|^{p-1} u(s) ds. \tag{23}$$

Then define a mapping

$$\Gamma(u)(t) = \partial_t W(t) f + W(t) g + \int_0^t W(t-s) |u(s)|^{p-1} u(s) ds. \tag{24}$$

Consider the closed ball $\mathscr{B}$ in X such that

$$\mathscr{B} = \left\{ u \in X : \|u\|_X \le 2c \left(\|f\|_{H^{\sigma_0,1}} + \|g\|_{H^{\sigma_0,0}} \right) \right\},$$

where

$$X = C\left([0,T]; H^{\sigma,1} \times H^{\sigma,0}\right)$$

equipped with the norm

$$\|u\|_X = \sup_{t\in[0,T]} \left(\|u\|_{H^{\sigma,1}} + \|\partial_t u\|_{H^{\sigma,0}} \right).$$

In view of (17) and (18), we have

$$\begin{aligned} \|\Gamma(u)(t)\|_X &\le c\|f\|_{H^{\sigma_0,1}} + c\|g\|_{H^{\sigma_0,0}} + c\int_0^T \left\| |u(s)|^{p-1} u(s) \right\|_{H^{\sigma,0}} ds \\ &\le c\|f\|_{H^{\sigma_0,1}} + c\|g\|_{H^{\sigma_0,0}} + cT \sup_{t\in[0,T]} \|u\|^p_{H^{\sigma,1}}. \end{aligned} \tag{25}$$

If we choose $0 < T < 1$ sufficiently small such that

$$T = \frac{1}{c2^p \left(\|f\|_{H^{\sigma_0,1}} + \|g\|_{H^{\sigma_0,0}} \right)^{p-1}}, \tag{26}$$

then for any $u \in \mathscr{B}$ and initial data's f, g, (25) yields

$$\|\Gamma(u)(t)\|_X \le 2c \left(\|f\|_{H^{\sigma_0,1}} + \|g\|_{H^{\sigma_0,0}} \right). \tag{27}$$

This implies that Γ maps $\mathscr{B}$ onto itself.

Next, we prove that the map Γ is a contraction map on $\mathscr{B}$. To do this, we need the estimate[4]

$$|a|^{p-1}a - |b|^{p-1}b \lesssim (|a| + |b|)^{p-1} |a-b|. \tag{28}$$

Now, for $u, v \in \mathscr{B}$ with the same choice of T, doing as above and applying (28), we obtain

$$\begin{aligned} \|\Gamma(u)(t) - \Gamma(v)(t)\|_X &\le Tc2^{p-1} \left(\|f\|_{H^{\sigma_0,1}} + \|g\|_{H^{\sigma_0,0}} \right)^{p-1} \|u-v\|_X \\ &\le \frac{1}{2} \|u-v\|_X, \end{aligned}$$

which proves that Γ is a contraction map on $\mathscr{B}$. Therefore by contraction mapping principle, (1) has unique solution in X. Continuous dependence on the initial data can be shown by using the difference estimate. Thus, we proved that the Cauchy problem (1) is locally well-posed in $H^{\sigma,1} \times H^{\sigma,0}$. □

[4]For the justification of this estimate we refer the readers to [1, equation 18 on page 10].

3 Approximate conservation law

The second step in the proof of Theorem 1 is to prove an approximate conservation law for the norm of the solution, that involves a small parameter $\sigma > 0$ and which reduces to the exact energy conservation law in taking the limit as $\sigma \to 0$. To do this, for a solution u of (1), we put

$$v_\sigma = \cosh(\sigma|D|)u,$$

where $D = -i\nabla$ with Fourier symbol ξ. Then we define a modified energy associated with the function v_σ by

$$E_\sigma(t) = \frac{1}{2}\int_{\mathbb{R}^d}\left(|\partial_t v_\sigma|^2 + |\nabla v_\sigma|^2 + \frac{2}{p+1}|v_\sigma|^{p+1}\right)dx.$$

For $\sigma = 0$, we have the conservation

$$E(t) = E(0) \quad \text{for all } \; 0 \le t \le T.$$

However, this fails to hold for $\sigma > 0$. In what follows, we will use equation (1) and Theorem 2 to prove

$$E_\sigma(t) \le E_\sigma(0) + T\left\|\int_{\mathbb{R}^d} \partial_t v_\sigma(x,\cdot)\cdot N(v_\sigma(x,\cdot))dx\right\|_{L_T^\infty}. \tag{29}$$

Proof. Differentiate $E_\sigma(t)$ with respect to time t and then applying integration by parts in the spatial variable and (1), we obtain

$$\begin{aligned}
\frac{d}{dt}E_\sigma(t) &= \int_{\mathbb{R}^d}\partial_t v_\sigma\left[\overline{\partial_t^2 v_\sigma - \Delta v_\sigma} + |v_\sigma|^{p-1}\overline{v_\sigma}\right]dx\\
&= \int_{\mathbb{R}^d}\partial_t v_\sigma\left[\cosh(\sigma|D|)\left[\overline{\partial_t^2 u - \Delta u}\right] + |v_\sigma|^{p-1}\overline{v_\sigma}\right]dx\\
&= \int_{\mathbb{R}^d}\partial_t v_\sigma\left[-\cosh(\sigma|D|)\left[|\operatorname{sech}(\sigma|D|)v_\sigma|^{p-1}\operatorname{sech}(\sigma|D|)\overline{v_\sigma}\right] + |v_\sigma|^{p-1}\overline{v_\sigma}\right]dx\\
&= \int_{\mathbb{R}^d}\partial_t v_\sigma\cdot N(v_\sigma)dx,
\end{aligned}$$

where

$$N(v_\sigma) = -\cosh(\sigma|D|)\left[|\operatorname{sech}(\sigma|D|)v_\sigma|^{p-1}\operatorname{sech}(\sigma|D|)\overline{v_\sigma}\right] + |v_\sigma|^{p-1}\overline{v_\sigma}. \tag{30}$$

Therefore, integrating and using Hölder's inequality in time yields

$$\begin{aligned}
E_\sigma(t) &= E_\sigma(0) + \int_0^t\int_{\mathbb{R}^d}\partial_t v_\sigma(x,s)\cdot N(v_\sigma(x,s))dxds\\
&\le E_\sigma(0) + \int_0^T\left|\int_{\mathbb{R}^d}\partial_t v_\sigma(x,s)\cdot N(v_\sigma(x,s))dx\right|ds
\end{aligned}$$

$$\leq E_\sigma(0) + T \left\| \int_{\mathbb{R}^d} \partial_t v_\sigma(x,\cdot) \cdot N(v_\sigma(x,\cdot))dx \right\|_{L_T^\infty}. \tag{31}$$

□

Lemma 3. *For $\partial_t v_\sigma \in L_x^2$ and $v_\sigma \in H^1$, we have the following estimate*

$$\left| \int_{\mathbb{R}^d} \partial_t v_\sigma \cdot N(v_\sigma)dx \right| \leq C\sigma^{2\theta} \|\partial_t v_\sigma\|_{L_x^2} \|v_\sigma\|_{H^1}^p, \tag{32}$$

for some constant $C > 0$, where $\theta = \frac{3}{4}$ when $d = 1$ and $\theta = \frac{1}{3}$ when $d = 2$.

Proof. Recall that

$$N(v_\sigma) = -\cosh(\sigma|D|)\left[|\operatorname{sech}(\sigma|D|)v_\sigma|^{p-1}\operatorname{sech}(\sigma|D|)\overline{v_\sigma}\right] + |v_\sigma|^{p-1}\overline{v_\sigma}.$$

By the Cauchy-Schwarz inequality, we have

$$\left| \int_{\mathbb{R}^d} \partial_t v_\sigma \cdot N(v_\sigma)dx \right| \leq \|\partial_t v_\sigma\|_{L_x^2} \|N(v_\sigma\|_{L_x^2}.$$

Thus, we are reduced to prove

$$\|N(v_\sigma)\|_{L_x^2} \lesssim \sigma^{2\theta} \|v_\sigma\|_{H^1}^p. \tag{33}$$

To proceed with the proof, we need the following Lemma whose proof is found in [4, Lemma 3].

Lemma 4. *Let $p > 1$ be any odd integer such that $\xi = \sum_{j=1}^p \xi_j$ for $\xi_j \in \mathbb{R}$. Then*

$$\left| 1 - \cosh(|\xi|) \prod_{j=1}^p \operatorname{sech}(|\xi_j|) \right| \leq 2^p \sum_{j \neq k=1}^p |\xi_j||\xi_k|.$$

By symmetry, we may assume that $|\xi_1| \leq |\xi_2| \leq \cdots \leq |\xi_p|$. Then, from Lemma 4 (see [4, Lemma 3]),

$$|K(\sigma\xi)| = \left| 1 - \cosh(\sigma|\xi|) \prod_{j=1}^p \operatorname{sech}(|\sigma\xi_j|) \right| \leq 2^p C\sigma^2 |\xi_{p-1}||\xi_p|. \tag{34}$$

It is easy to see that

$$|K(\sigma\xi)| \leq 1. \tag{35}$$

Now, interpolation between (34) and (35) yields

$$|K(\sigma\xi)| \leq 2^p \left[\sigma^2 |\xi_{p-1}||\xi_p|\right]^\theta,$$

for any $0 \le \theta \le 1$. In particular choosing $\theta = \frac{3}{4}$ when $d = 1$ (respectively, $\theta = \frac{1}{3}$ when $d = 2$) yields

$$|K(\sigma\xi)| \le 2^p \sigma^{\frac{3}{2}} |\xi_{p-1}|^{\frac{3}{4}} |\xi_p|^{\frac{3}{4}} \quad \text{when } d = 1, \tag{36}$$

$$|K(\sigma\xi)| \le 2^p \sigma^{\frac{2}{3}} |\xi_{p-1}|^{\frac{1}{3}} |\xi_p|^{\frac{1}{3}} \quad \text{when } d = 2. \tag{37}$$

By taking the Fourier transform of $N(v_\sigma)$ we obtain

$$\begin{aligned} |\mathscr{F}_x[N(v_\sigma)](\xi)| &= \left| \int_{\xi=\sum_{j=1}^p \xi_j} K(\sigma\xi) \prod_{j=1}^p \overline{\widehat{v_\sigma}(\xi_j)} \prod_{j=1}^p d\xi_j \right| \\ &\lesssim \int_{\xi=\sum_{j=1}^p \xi_j} |K(\sigma\xi)| \prod_{j=1}^p |\widehat{v_\sigma}(\xi_j)| \prod_{j=1}^p d\xi_j. \end{aligned}$$

Now, let

$$w_\sigma = \mathscr{F}_x^{-1}\left(|\widehat{v_\sigma}|\right).$$

If $d = 1$, then using Plancherel's Theorem, (36), Hölder's inequality, and Sobolev embedding, we obtain

$$\begin{aligned} \|N(v_\sigma)\|_{L_x^2} = \|\mathscr{F}_x[N(v_\sigma)](\xi)\|_{L_\xi^2} &\lesssim \sigma^{\frac{3}{2}} \left\| \int_{\xi=\sum_{j=1}^p \xi_j} |\xi_{p-1}|^{\frac{3}{4}} |\xi_p|^{\frac{3}{4}} \prod_{j=1}^p |\widehat{v_\sigma}(\xi_j)| \prod_{j=1}^p d\xi_j \right\|_{L_\xi^2} \\ &\lesssim \sigma^{\frac{3}{2}} \left\| \int_{\xi=\sum_{j=1}^p \xi_j} |\xi_{p-1}|^{\frac{3}{4}} |\xi_p|^{\frac{3}{4}} \prod_{j=1}^p \widehat{w_\sigma}(\xi_j) \prod_{j=1}^p d\xi_j \right\|_{L_\xi^2} \\ &\lesssim \sigma^{\frac{3}{2}} \left\| w_\sigma^{p-2} \cdot |D|^{\frac{3}{4}} w_\sigma \cdot |D|^{\frac{3}{4}} w_\sigma \right\|_{L_x^2} \\ &\lesssim \sigma^{\frac{3}{2}} \left\| w_\sigma^{p-2} \right\|_{L_x^\infty} \left\| \left[|D|^{\frac{3}{4}} w_\sigma \right]^2 \right\|_{L_x^2} \\ &\lesssim \sigma^{\frac{3}{2}} \|w_\sigma\|_{L_x^\infty}^{p-2} \left\| |D|^{\frac{3}{4}} w_\sigma \right\|_{L_x^4}^2 \\ &\lesssim \sigma^{\frac{3}{2}} \|w_\sigma\|_{H^{\frac{1}{2}+}}^{p-2} \left\| |D|^{\frac{3}{4}} w_\sigma \right\|_{H^{\frac{1}{4}}}^2 \\ &\lesssim \sigma^{\frac{3}{2}} \|w_\sigma\|_{H^1}^p \sim \sigma^{\frac{3}{2}} \|v_\sigma\|_{H^1}^p. \end{aligned}$$

Similarly, for $d = 2$, we choose Hölder's exponents $\frac{2}{1-\theta} = 3$ and $\frac{2}{\theta} = 6$, and the by using Plancherel's Theorem, (37), Hölder's inequality, and Sobolev embedding, we obtain

$$\|N(v_\sigma)\|_{L_x^2} = \|\mathscr{F}_x(N(v_\sigma))(\xi)\|_{L_\xi^2} \lesssim \sigma^{\frac{2}{3}} \left\| \int_{\xi=\sum_{j=1}^p \xi_j} |\xi_{p-1}|^{\frac{1}{3}} |\xi_p|^{\frac{1}{3}} \prod_{j=1}^p |\widehat{v_\sigma}(\xi_j)| \prod_{j=1}^p d\xi_j \right\|_{L_\xi^2}$$

$$
\begin{aligned}
&\lesssim \sigma^{\frac{2}{3}} \left\| \int_{\xi=\sum_{j=1}^{p}\xi_j} |\xi_{p-1}|^{\frac{1}{3}} |\xi_p|^{\frac{1}{3}} \prod_{j=1}^{p} \widehat{w_\sigma}(\xi_j) \prod_{j=1}^{p} d\xi_j \right\|_{L^2_\xi} \\
&\lesssim \sigma^{\frac{2}{3}} \left\| w_\sigma^{p-2} \cdot |D|^{\frac{1}{3}} w_\sigma \cdot |D|^{\frac{1}{3}} w_\sigma \right\|_{L^2_x} \\
&\lesssim \sigma^{\frac{2}{3}} \left\| w_\sigma^{p-2} \right\|_{L^6_x} \left\| \left[|D|^{\frac{1}{3}} w_\sigma \right]^2 \right\|_{L^3_x} \\
&\lesssim \sigma^{\frac{2}{3}} \| w_\sigma \|^{p-2}_{L_x^{6(p-1)}} \left\| |D|^{\frac{1}{3}} w_\sigma \right\|^2_{L^6_x} \\
&\lesssim \sigma^{\frac{2}{3}} \| w_\sigma \|^{p-2}_{H^{1-\frac{1}{3(p-1)}}} \left\| |D|^{\frac{1}{3}} w_\sigma \right\|^2_{H^{\frac{2}{3}}} \\
&\lesssim \sigma^{\frac{2}{3}} \| w_\sigma \|^{p}_{H^1} \sim \sigma^{\frac{2}{3}} \| v_\sigma \|^{p}_{H^1}.
\end{aligned}
$$

□

Now, our almost conservation law is stated as follows.

Theorem 3 (Approximate conservation law)**.** *Let $d = 1$ or 2, $\sigma > 0$ and $p > 1$ be any odd integer. Let $(f,g) \in H^{\sigma,1} \times H^{\sigma,0}$ and u be the the local solution of (1) on $\mathbb{R}^d \times [0,T]$ obtained in Theorem 2. Then, we have*

$$
\sup_{t\in[0,T]} E_\sigma(t) \le E_\sigma(0) + cT\sigma^{2\theta}[2E_\sigma(0)]^{\frac{p+1}{2}}, \tag{38}
$$

where θ be as given in Lemma 3.

Proof. We have from (12)

$$
\begin{aligned}
\left(\|v_\sigma\|_{L^\infty_T H^1} + \|\partial_t v_\sigma\|_{L^\infty_T L^2_x} \right)^2 &= \left(\|u\|_{L^\infty_T H^{\sigma,1}} + \|\partial_t u\|_{L^\infty_T H^{\sigma,0}} \right)^2 \\
&\le c \left(\|f\|_{H^{\sigma,1}} + \|g\|_{H^{\sigma,0}} \right)^2 \\
&\le 2c\|v_\sigma(\cdot,0)\|^2_{H^1} + 2C\|\partial_t v_\sigma(\cdot,0)\|^2_{L^2_x} \\
&\le 2cE_\sigma(0),
\end{aligned}
$$

which in turn implies that

$$
\|v_\sigma\|_{L^\infty_T H^1} + \|\partial_t v_\sigma\|_{L^\infty_T L^2_x} \le c\sqrt{2E_\sigma(0)}. \tag{39}
$$

Now, using (29), (32) and (39) we obtain

$$
\begin{aligned}
E_\sigma(t) &\le E_\sigma(0) + cT\|\partial_t v_\sigma\|_{L^\infty_T L^2_x} \|N(v_\sigma\|_{L^\infty_T L^2_x} \\
&\le E_\sigma(0) + cT\sigma^{2\theta} \|\partial_t v_\sigma\|_{L^\infty_T L^2_x} \|v_\sigma\|^p_{L^\infty_T H^1} \\
&\le E_\sigma(0) + cT\sigma^{2\theta} \sqrt{2E_\sigma(0)} \sqrt{2E_\sigma(0)}^p \\
&\sim E_\sigma(0) + cT\sigma^{2\theta} [2E_\sigma(0)]^{\frac{p+1}{2}}.
\end{aligned}
$$

□

4 Proof of Theorem 1

Suppose that $(f,g) \in H^{\sigma_0,1} \times H^{\sigma_0,0}$ for some $\sigma_0 > 0$. This implies

$$v_{\sigma_0}(\cdot,0) = \cosh(\sigma_0|D|)f(\cdot) \in H^1, \quad \partial_t v_{\sigma_0}(\cdot,0) = \cosh(\sigma_0|D|)g(\cdot) \in L_x^2.$$

Then we have by the Sobolev embedding that

$$E_{\sigma_0}(0) \leq \|v_{\sigma_0}(\cdot,0)\|_{H^1}^2 + \|\partial_t v_{\sigma_0}(\cdot,0)\|_{L_x^2}^2 + \|v_{\sigma_0}(\cdot,0)\|_{H^1}^{p+1} < \infty.$$

Now, we can construct a solution on $[0,\delta]$ for arbitrarily large time δ by following the techniques presented in [22]. This is achieved by applying the approximate conservation law Theorem 3, so as to repeat the local result in Theorem 2 on successive short time interval of size T to reach δ by adjusting the strip width parameter $\sigma \in (0,\sigma_0]$ of the solution according to the size of δ. The goal is to prove that for a given parameter $\sigma \in (0,\sigma_0]$ and large δ,

$$\sup_{t\in[0,\delta]} E_\sigma(t) \leq 2E_{\sigma_0}(0) \quad \text{for } \sigma \geq c\delta^{-\frac{1}{2\theta}}, \tag{40}$$

where $c > 0$ depends only on the initial data norm, and θ be as in Theorem 3. This would imply $\mathscr{E}_\sigma(t) < \infty$ for all $t \in [0,\delta]$ and hence for all $t \in [0,\delta]$ the solution u to (1) satisfies

$$(u,\partial_t u)(\cdot,t) \in H^{\sigma,1} \times H^{\sigma,0} \quad \text{for } \sigma \geq c\delta^{-\frac{1}{2\theta}}.$$

It remains to prove (40). Since $\cosh(x)$ is increasing function for $x \geq 0$, for $\sigma \leq \sigma_0$ we have

$$E_\sigma(0) \leq E_{\sigma_0}(0). \tag{41}$$

Now, for a given parameter $\sigma \in (0,\sigma_0]$ and $t_0 \in [0,T]$, we have by Theorem 2, Theorem 3 and (41),

$$\begin{aligned}\sup_{t\in[0,t_0]} E_\sigma(t) &\leq E_\sigma(0) + cT\sigma^{2\theta}E_\sigma^{\frac{p+1}{2}}(0)\\ &\leq E_{\sigma_0}(0) + cT\sigma^{2\theta}E_{\sigma_0}^{\frac{p+1}{2}}(0).\end{aligned}$$

Thus

$$\sup_{t\in[0,t_0]} E_\sigma(t) \leq 2E_{\sigma_0}(0), \tag{42}$$

provided

$$cT\sigma^{2\theta}E_{\sigma_0}^{\frac{p-1}{2}}(0) \leq 1. \tag{43}$$

Then we can apply Theorem 2, with initial time $t = t_0$ and the time step T in (11) to extend the solution from $[0,t_0]$ to $[t_0,t_0+T]$. By Theorem 3 and (42), we obtain

$$\sup_{t\in[t_0,t_0+T]} E_\sigma(t) \leq E_\sigma(t_0) + cT\sigma^{2\theta}\left(2E_{\sigma_0}(0)\right)^{\frac{p+1}{2}}. \tag{44}$$

In this way, we can cover all time intervals $[0,T]$, $[T,2T]$, $[2T,3T]$ etc., and obtain

$$E_\sigma(T) \leq E_\sigma(0) + cT\sigma^{2\theta}\left(2E_{\sigma_0}(0)\right)^{\frac{p+1}{2}},$$

$$\begin{aligned}
E_\sigma(2T) &\le E_\sigma(T) + cT\sigma^{2\theta}\left(2E_{\sigma_0}(0)\right)^{\frac{p+1}{2}} \\
&\le E_\sigma(0) + 2cT\sigma^{2\theta}\left(2E_{\sigma_0}(0)\right)^{\frac{p+1}{2}} \\
E_\sigma(3T) &\le E_\sigma(2T) + cT\sigma^{2\theta}\left(2E_{\sigma_0}(0)\right)^{\frac{p+1}{2}} \\
&\le E_\sigma(T) + 2cT\sigma^{2\theta}\left(2E_{\sigma_0}(0)\right)^{\frac{p+1}{2}} \\
&\le E_\sigma(0) + 3cT\sigma^{2\theta}\left(2E_{\sigma_0}(0)\right)^{\frac{p+1}{2}}, \\
&\cdots, \\
E_\sigma(nT) &\le E_\sigma(0) + ncT\sigma^{2\theta}\left(2E_{\sigma_0}(0)\right)^{\frac{p+1}{2}}.
\end{aligned}$$

This argument continues until

$$cnT\sigma^{2\theta}\left(2E_{\sigma_0}(0)\right)^{\frac{p+1}{2}} \le E_{\sigma_0}(0), \tag{45}$$

and then we have reached the final time

$$\delta = nT,$$

when

$$cnT\sigma^{2\theta}\left(2E_{\sigma_0}(0)\right)^{\frac{p-1}{2}} \le 1.$$

Finally, the condition (43) satisfies for σ such that

$$cnT\sigma^{2\theta}\left(2E_{\sigma_0}(0)\right)^{\frac{p-1}{2}} = 1.$$

Therefore

$$\sigma^{2\theta} = \delta^{-1}\cdot c^{-1}\left(2E_{\sigma_0}(0)\right)^{\frac{1-p}{2}}.$$

It gives (40) if choose $c \le \left(\delta^{-1}\cdot c^{-1}\left(2E_{\sigma_0}(0)\right)^{\frac{1-p}{2}}\right)^{\frac{1}{2\theta}}$. This completes the proof of Theorem 1.

Conflicts of interest

The authors declare that there are no conflicts of interest.

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