

## Improved lower bound of spatial analyticity radius for solutions to nonlinear wave equation

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**Abstract.** In this paper, the rate of decay for the radius of spatial analyticity for solutions of the nonlinear wave equation

$$\partial_t^2 u - \Delta u + |u|^{p-1} u = 0,$$

on  $\mathbb{R}^d \times \mathbb{R}$  is studied. In particular, for a class of analytic initial data with a uniform radius of analyticity  $\sigma_0$ , we obtain an asymptotic lower bound  $\sigma(t) \geq a_0 |t|^{-\frac{2}{3}}$  when  $d = 1$  and  $\sigma(t) \geq a_0 |t|^{-\frac{3}{2}}$  when  $d = 2$  on the uniform radius of analyticity  $\sigma(t)$  of solution  $u(\cdot, t)$  as  $|t| \rightarrow +\infty$ . This is an improvement of the work [D. O. da Silva, A. J. Castro, Global well-posedness for the nonlinear wave equation in analytic Gevrey spaces, J. Differential Equations 275(2021) 234–249], where the authors obtained a decay rate of order  $\sigma(t) \geq a_0(1 + |t|)^{-\left(\frac{p+1}{2}\right)}$  when  $d = 1$  and  $\sigma(t) \geq a_0(1 + |t|)^{-\left(\frac{p+1-\varepsilon}{1-\varepsilon}\right)}$  when  $d = 2$  as  $|t| \rightarrow +\infty$  for large time  $t$ , where  $\varepsilon > 0$  is arbitrary. We used an approximate conservation law in a modified Gevrey space, contraction mapping principle, interpolation and Sobolev embedding to obtain the results.

**Keywords:** Nonlinear wave equation, modified Gevrey space, approximate conservation, radius of analyticity, decay rate for the radius.

**AMS Subject Classification 2010:** 35Q40, 35Q55, 37L70.

## 1 Introduction

In this paper, we consider the Cauchy problem for the nonlinear wave (NLW) equation

$$\begin{cases} \partial_t^2 u - \Delta u + |u|^{p-1} u = 0, & x \in \mathbb{R}^d, t \in \mathbb{R}, \\ u(x, 0) = f(x), \\ \partial_t u(x, 0) = g(x), \end{cases} \quad (1)$$

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where  $u : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{C}$ ,  $p > 1$  be any odd integer, and  $\Delta = \sum_{j=1}^d \partial_{x_j}^2$ .

The following energy is conserved under the flow of (1):

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^d} \left( |\partial_t u|^2 + |\nabla u|^2 + \frac{2}{p+1} |u|^{p+1} \right) dx = E(0), \quad \forall t. \quad (2)$$

The NLW equation (1) has a long history and many results are known. For a detailed exposition, we refer the readers that [20] and the references therein. Local well-posedness of (1) in the homogeneous Sobolev spaces  $\dot{H}^s(\mathbb{R}^2)$  has been showed by Lindblad and Sogge [14] with optimal regularity given by

$$s(p) = \begin{cases} \frac{3}{4} - \frac{1}{p-1} & \text{if } 3 \leq p \leq 5, \\ 1 - \frac{2}{p-1} & \text{if } p \geq 5. \end{cases}$$

We refer the readers also to see the refinements of Nakamura and Ozawa [15]. These results that are obtained for the homogeneous Sobolev spaces can be extended to the inhomogeneous Sobolev spaces  $H^s$  by a simple integration in time argument [3].

The main concern of this work is to estimate the radius of spatial analyticity for the solution of the Cauchy problem (1), given initial data in a class of analytic functions. A class of analytic function spaces suitable for our analysis is the modified Gevrey class introduced in [4]. This space is denoted by  $H^{\sigma,s} = H^{\sigma,s}(\mathbb{R}^d)$  and consists of functions such that

$$\left\{ \phi \in L_x^2(\mathbb{R}^d) : \|\phi\|_{H^{\sigma,s}} = \left\| \cosh(\sigma|\xi|) \langle \xi \rangle^s \hat{\phi}(\xi) \right\|_{L_\xi^2(\mathbb{R}^d)} < \infty \right\},$$

where  $\langle \xi \rangle^2 = 1 + |\xi|^2$  and  $\hat{\phi}(\xi)$  denotes the spatial Fourier transform of  $\phi(x)$  which is given by<sup>1</sup>

$$\hat{\phi}(\xi) := \mathcal{F}_x[\phi](\xi) = \int_{\mathbb{R}^d} e^{-i(x \cdot \xi)} \phi(x) dx.$$

The authors in [4] obtained this space from the Gevrey space  $G^{\sigma,s}(\mathbb{R}^d)$  by replacing the exponential weight  $e^{\sigma|\xi|}$  with the hyperbolic weight  $\cosh(\sigma|\xi|)$ . The Gevrey space  $G^{\sigma,s} = G^{\sigma,s}(\mathbb{R}^d)$  was first introduced in [6] via the norm

$$\|\phi\|_{G^{\sigma,s}} = \left\| e^{\sigma|\xi|} \langle \xi \rangle^s \hat{\phi} \right\|_{L_\xi^2(\mathbb{R}^d)} \quad (\sigma \geq 0, s \in \mathbb{R}).$$

Observe that the Gevrey and modified Gevrey spaces satisfy the following embedding properties. For all  $s, s' \in \mathbb{R}$  and  $0 \leq \sigma < \sigma'$ , we have

$$G^{\sigma',s'} \subset G^{\sigma,s}, \quad \text{i.e.,} \quad \|\phi\|_{G^{\sigma,s}} \lesssim \|\phi\|_{G^{\sigma',s'}}. \quad (3)$$

$$H^{\sigma',s'} \subset H^{\sigma,s}, \quad \text{i.e.,} \quad \|\phi\|_{H^{\sigma,s}} \lesssim \|\phi\|_{H^{\sigma',s'}}. \quad (4)$$

For  $\sigma = 0$ , we have

$$G^{0,s} = H^{0,s} = H^s,$$

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<sup>1</sup>  $x \cdot \xi$  stands for the usual dot product  $\sum_{j=1}^d x_j \xi_j$  for all  $x_j, \xi_j \in \mathbb{R}$ .

where  $H^s = H^s(\mathbb{R}^d)$  denotes the standard Sobolev space consisting of functions

$$\|\phi\|_{H^s} = \|\langle \xi \rangle^s \hat{\phi}\|_{L^2_\xi(\mathbb{R}^d)} < \infty.$$

For  $2 \leq q < \infty$ , we have the Sobolev embedding

$$\|\phi\|_{L^q(\mathbb{R}^d)} \lesssim \|\phi\|_{H^s} \quad \text{if } s \geq d \left( \frac{1}{2} - \frac{1}{q} \right) > 0, \quad (5)$$

$$\|\phi\|_{L^q(\mathbb{R}^d)} \lesssim \|\phi\|_{\dot{H}^s} \quad \text{if } s = d \left( \frac{1}{2} - \frac{1}{q} \right) > 0, \quad (6)$$

where  $\dot{H}^s$  denotes the homogeneous Sobolev space of order  $s$ . Furthermore, we have <sup>2</sup>

$$\|\phi\|_{L^\infty(\mathbb{R}^d)} \lesssim \|\phi\|_{H^{\frac{d}{2}+}}. \quad (7)$$

By the Paley-Wiener Theorem, the radius of analyticity of a function can be related to decay properties of its Fourier transform. It is therefore natural to take data for (1) in  $G^{\sigma,s}$ . For  $\sigma > 0$ , any function in  $G^{\sigma,s}$  has a radius of analyticity of at least  $\sigma$  at each point  $x \in \mathbb{R}^d$ . This fact is contained in the following theorem, whose proof can be found in [13, p. 174] in the case  $s = 0$  and  $d = 1$ ; the general case follows from a simple modification.

**Lemma 1** (Paley-Wiener Theorem). *For  $\sigma \geq 0, s \in \mathbb{R}$ , the function  $\phi \in G^{\sigma,s}$  if and only if  $\phi(\cdot)$  is the restriction to the real line of a function  $\Psi(\cdot + iy)$  which is holomorphic in the strip*

$$S_\sigma = \{x + iy \in \mathbb{C} : |y| < \sigma\},$$

and satisfies the bound

$$\sup_{|y| < \sigma} \|\Psi(\cdot + iy)\|_{H^s_x} < \infty.$$

The exponential and hyperbolic weights are equivalent in the sense that

$$\frac{1}{2} e^{\sigma|\xi|} \leq \cosh(\sigma|\xi|) \leq e^{\sigma|\xi|}, \quad \xi \in \mathbb{R}^d, \quad (8)$$

from which, we find

$$\frac{1}{2} \|\phi\|_{G^{\sigma,s}} \leq \|\phi\|_{H^{\sigma,s}} \leq \|\phi\|_{G^{\sigma,s}}.$$

In other words, the associated  $H^{\sigma,s}$  and  $G^{\sigma,s}$ -norms are equivalent, i.e.,

$$\|\phi\|_{H^{\sigma,s}} \sim \|\phi\|_{G^{\sigma,s}} = \left\| e^{\sigma|\xi|} \langle \xi \rangle^s \hat{\phi} \right\|_{L^2_\xi(\mathbb{R}^d)}. \quad (9)$$

Therefore, (9) guarantees that the statement of Paley-Wiener Theorem still holds for functions in  $H^{\sigma,s}$ .

The study of spatial analyticity of solutions to nonlinear Cauchy problems was initiated by Kato and Masuda [12]. Since then, several mathematicians have considered the Cauchy problem for a variety of equations with initial data in  $G^{\sigma,s}$ -spaces (see, for example e.g., [2, 5, 17, 18, 21–23] and the references

<sup>2</sup>Here,  $a+ = a + \varepsilon$  for sufficiently small  $\varepsilon > 0$ .

therein). For studies with initial data in the modified Gevrey spaces see for example [4, 7–10] and references therein

Coming back to (1), the asymptotic lower bound for the radius of spatial analyticity of the solutions has been established by da Silva and Castro [3], who obtained a decay rate of order  $\sigma(t) \geq c(1+|t|)^{-\left(\frac{p+1}{2}\right)}$  when  $d = 1$  and  $\sigma(t) \geq c(1+|t|)^{-\left(\frac{p+1-\varepsilon}{1-\varepsilon}\right)}$  when  $d = 2$  as  $|t| \rightarrow +\infty$ , where  $\varepsilon > 0$  is arbitrary. The strategy in [3] is as follows:

1. Prove a local well-posedness by standard fixed-point argument in  $G^{\sigma,s} \times G^{\sigma,s-1}$  with a lifespan  $T > 0$ .
2. Establish an almost conservation law in  $G^{\sigma,1} \times G^{\sigma,0}$ .
3. By shrinking  $\sigma$  gradually, they used repeatedly the local well-posedness and the almost conservation law on the intervals  $[0, T], [T, 2T], \dots$ , and obtained a global bound of solution on  $[0, \delta]$  for arbitrarily large  $\delta$ . This idea was introduced by Selberg and Tesfahun in [19], where it was applied to the 1D Dirac-Klein-Gordon equations.

The persistence of spatial analyticity of (1) in the periodic setting was dealt by Guo and Titi in [11].

As a consequence of the embedding (4) and the existing well-posedness theory in  $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ , the Cauchy problem (1) with initial data  $f, g \in H^{\sigma_0,1}(\mathbb{R}^d) \times H^{\sigma_0,0}(\mathbb{R}^d)$  for some  $\sigma_0 > 0$  has a unique and global in time solution.

The main result in this paper is as follows.

**Theorem 1** (Decay rate for the radius of analyticity). *Let  $d = 1$  or  $2$  and  $p > 1$  be any odd integer. If  $(f, g) \in H^{\sigma_0,1} \times H^{\sigma_0,0}$  for  $\sigma_0 > 0$ , then for any  $\delta > 0$  the global solution  $u$  of (1) satisfies*

$$(u, \partial_t u) \in C\left([0, \delta]; H^{\sigma(t),1} \times H^{\sigma(t),0}\right), \quad (10)$$

where the radius of spatial analyticity  $\sigma(t)$  satisfies an asymptotic rate of decay

$$\sigma(t) \geq a_0 |t|^{-\frac{2}{3}} \text{ when } d = 1,$$

and

$$\sigma(t) \geq a_0 |t|^{-\frac{3}{2}} \text{ when } d = 2,$$

as  $|t| \rightarrow +\infty$ . Here,  $a_0 > 0$  is a constant which depends on the initial data norm.

**Notation.** For any positive numbers  $a$  and  $b$ , the notation  $a \lesssim b$  stands for  $a \leq cb$ , where  $c$  is a positive constant that can be determined by known parameters in a given situation but whose values are not crucial to the problem at hand and may differ from line to line. We also denote  $a \sim b$  to mean  $b \lesssim a \lesssim b$ .

## 2 Local well-posedness result

The first step in the proof of Theorem 1 is to prove the following local well-posedness result, where the radius of analyticity remains constant.

**Theorem 2.** Let  $p > 1$  be any odd integer and  $d = 1$  or  $2$ . Given  $(f, g) \in H^{\sigma,1} \times H^{\sigma,0}$  for  $\sigma > 0$ , there exists a time  $T > 0$  and a unique solution

$$(u, \partial_t u) \in C([0, T]; H^{\sigma,1} \times H^{\sigma,0}),$$

of the Cauchy problem (1) on  $\mathbb{R}^d \times [0, T]$ . Moreover, the solution depends continuously on the initial data. Furthermore, the existence time is given by

$$T = a_0 (\|f\|_{H^{\sigma,1}} + \|g\|_{H^{\sigma,0}})^{-(p-1)}, \quad (11)$$

for some constant  $a_0 > 0$ . In addition, the local solution  $u$  satisfies the bound

$$\sup_{t \in [0, T]} (\|u\|_{H^{\sigma,1}} + \|\partial_t u\|_{H^{\sigma,0}}) \lesssim \|f\|_{H^{\sigma,1}} + \|g\|_{H^{\sigma,0}}. \quad (12)$$

*Proof.* Consider the Cauchy problem for the linear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = F(\cdot, t), \\ u(\cdot, 0) = f(\cdot), \\ \partial_t u(\cdot, 0) = g(\cdot), \end{cases} \quad (13)$$

whose solution is given by Duhamel's formula

$$u(t) = \partial_t W(t)f + W(t)g + \int_0^t W(t-s)F(\cdot, s)ds, \quad (14)$$

where

$$W(t) = \frac{\sin(t|D|)}{|D|}.$$

To obtain an energy inequality from (14), we need the following estimates.

**Lemma 2.** ([16, Lemma 4.1]) For any  $s \in \mathbb{R}$  and any integer  $d \geq 1$ , we have

$$\|\partial_t W(t)f\|_{H^s(\mathbb{R}^d)} \lesssim \|f\|_{H^s(\mathbb{R}^d)}, \quad (15)$$

$$\|W(t)g\|_{H^s(\mathbb{R}^d)} \lesssim \|g\|_{H^{s-1}(\mathbb{R}^d)}. \quad (16)$$

Now, applying  $\cosh(\sigma|D|)$  to both sides of (14) and taking  $H^1(\mathbb{R}^d)$ -norm to both sides and then using Lemma 2, we obtain the energy inequality

$$\sup_{t \in [0, T]} (\|u\|_{H^{\sigma,1}(\mathbb{R}^d)} + \|\partial_t u\|_{H^{\sigma,0}(\mathbb{R}^d)}) \lesssim (\|f\|_{H^{\sigma,1}(\mathbb{R}^d)} + \|g\|_{H^{\sigma,0}(\mathbb{R}^d)}) + \int_0^T \|F(\cdot, s)\|_{H^{\sigma,0}(\mathbb{R}^d)} ds, \quad (17)$$

for  $T > 0$  and  $d = 1, 2$ .

To use the standard fixed point argument in proving Theorem 2, we need the nonlinear estimate

$$\| |u|^{p-1} u \|_{H^{\sigma,0}(\mathbb{R}^d)} \lesssim \|u\|_{H^{\sigma,1}(\mathbb{R}^d)}^p, \quad (18)$$

*Proof.* Putting  $V = \cosh(\sigma|D|)u$ , (18) reduces to

$$\left\| \cosh(\sigma|D|) \left[ |\operatorname{sech}(\sigma|D|)V|^{p-1} \operatorname{sech}(\sigma|D|)V \right] \right\|_{L_x^2(\mathbb{R}^d)} \lesssim \|V\|_{H^1(\mathbb{R}^d)}^p. \quad (19)$$

From triangle inequality we have

$$\xi = \sum_{j=1}^p \xi_j \Rightarrow |\xi| \leq \sum_{j=1}^p |\xi_j|. \quad (20)$$

Moreover, we have from [4] that

$$\cosh(\sigma|\xi|) \prod_{j=1}^p \operatorname{sech}(\sigma|\xi_j|) \leq 2^p. \quad (21)$$

By Plancherel's Theorem, (20) and (21), we have

$$\begin{aligned} \text{LHS of (19)} &= \left\| \mathcal{F}_x \left\{ \cosh(\sigma|D|) \left[ |\operatorname{sech}(\sigma|D|)V|^{p-1} \operatorname{sech}(\sigma|D|)V \right] \right\}(\xi) \right\|_{L_\xi^2(\mathbb{R}^d)} \\ &= \left\| \int_{\xi = \sum_{j=1}^p \xi_j} [\cosh(\sigma|\xi|) \prod_{j=1}^p \operatorname{sech}(\sigma|\xi_j|)] \prod_{j=1}^p \widehat{V}(\xi_j) \prod_{j=1}^p d\xi_j \right\|_{L_\xi^2(\mathbb{R}^d)} \\ &\lesssim \left\| \int_{\xi = \sum_{j=1}^p \xi_j} \prod_{j=1}^p |\widehat{V}(\xi_j)| \prod_{j=1}^p d\xi_j \right\|_{L_\xi^2(\mathbb{R}^d)} \\ &= \|U^p\|_{L_x^2(\mathbb{R}^d)}, \end{aligned}$$

where  $U = \mathcal{F}_x^{-1} [|\widehat{V}|]$ .

Here by Sobolev embedding, we obtain

$$\|U^p\|_{L_x^2(\mathbb{R}^d)} = \|U\|_{L_x^{2p}(\mathbb{R}^d)}^p \lesssim \|U\|_{H^1(\mathbb{R}^d)}^p = \|V\|_{H^1(\mathbb{R}^d)}^p, \quad (22)$$

where  $p \geq 1$  is any integer. This proves (19), and hence (18).  $\square$

**Proof of estimate (21).** From the definition of cosine hyperbolic function, we have

$$\cosh(\sigma|\xi|) \prod_{j=1}^p \operatorname{sech}(\sigma|\xi_j|) = \frac{e^{\sigma|\xi|} + e^{-\sigma|\xi|}}{\prod_{j=1}^p (e^{\sigma|\xi_j|} + e^{-\sigma|\xi_j|})} \cdot 2^{p-1}.$$

Since the fraction is less than or equal to one by the key inequality, we obtain

$$\cosh(\sigma|\xi|) \prod_{j=1}^p \operatorname{sech}(\sigma|\xi_j|) \leq 1 \cdot 2^{p-1}.$$

Also, since  $2^{p-1} \leq 2^p$  for  $p \geq 1$ , the inequality (21) is proven. That is

$$\cosh(\sigma|\xi|) \prod_{j=1}^p \operatorname{sech}(\sigma|\xi_j|) \leq 2^p. \quad \square$$

Now, the integral form of (1) is given by Duhamel's formula as

$$u(t) = \partial_t W(t)f + W(t)g + \int_0^t W(t-s)|u(s)|^{p-1}u(s)ds. \quad (23)$$

Then define a mapping

$$\Gamma(u)(t) = \partial_t W(t)f + W(t)g + \int_0^t W(t-s)|u(s)|^{p-1}u(s)ds. \quad (24)$$

Consider the closed ball  $\mathcal{B}$  in  $X$  such that

$$\mathcal{B} = \{u \in X : \|u\|_X \leq 2c(\|f\|_{H^{\sigma_0,1}} + \|g\|_{H^{\sigma_0,0}})\},$$

where

$$X = C([0, T]; H^{\sigma,1} \times H^{\sigma,0})$$

equipped with the norm

$$\|u\|_X = \sup_{t \in [0, T]} (\|u\|_{H^{\sigma,1}} + \|\partial_t u\|_{H^{\sigma,0}}).$$

In view of (17) and (18), we have

$$\begin{aligned} \|\Gamma(u)(t)\|_X &\leq c\|f\|_{H^{\sigma_0,1}} + c\|g\|_{H^{\sigma_0,0}} + c \int_0^T \| |u(s)|^{p-1}u(s) \|_{H^{\sigma,0}} ds \\ &\leq c\|f\|_{H^{\sigma_0,1}} + c\|g\|_{H^{\sigma_0,0}} + cT \sup_{t \in [0, T]} \|u\|_{H^{\sigma,1}}^p. \end{aligned} \quad (25)$$

If we choose  $0 < T < 1$  sufficiently small such that

$$T = \frac{1}{c2^p(\|f\|_{H^{\sigma_0,1}} + \|g\|_{H^{\sigma_0,0}})^{p-1}}, \quad (26)$$

then for any  $u \in \mathcal{B}$  and initial data's  $f, g$ , (25) yields

$$\|\Gamma(u)(t)\|_X \leq 2c(\|f\|_{H^{\sigma_0,1}} + \|g\|_{H^{\sigma_0,0}}). \quad (27)$$

This implies that  $\Gamma$  maps  $\mathcal{B}$  onto itself.

Next, we prove that the map  $\Gamma$  is a contraction map on  $\mathcal{B}$ . To do this, we need the estimate<sup>3</sup>

$$|a|^{p-1}a - |b|^{p-1}b \lesssim (|a| + |b|)^{p-1}|a - b|. \quad (28)$$

Now, for  $u, v \in \mathcal{B}$  with the same choice of  $T$ , doing as above and applying (28), we obtain

$$\begin{aligned} \|\Gamma(u)(t) - \Gamma(v)(t)\|_X &\leq Tc2^{p-1}(\|f\|_{H^{\sigma_0,1}} + \|g\|_{H^{\sigma_0,0}})^{p-1}\|u - v\|_X \\ &\leq \frac{1}{2}\|u - v\|_X, \end{aligned}$$

which proves that  $\Gamma$  is a contraction map on  $\mathcal{B}$ . Therefore by contraction mapping principle, (1) has unique solution in  $X$ . Continuous dependence on the initial data can be shown by using the difference estimate. Thus, we proved that the Cauchy problem (1) is locally well-posed in  $H^{\sigma,1} \times H^{\sigma,0}$ .  $\square$

<sup>3</sup>For the justification of this estimate we refer the readers to [1, equation 18 on page 10].

### 3 Approximate conservation law

The second step in the proof of Theorem 1 is to prove an approximate conservation law for the norm of the solution, that involves a small parameter  $\sigma > 0$  and which reduces to the exact energy conservation law in taking the limit as  $\sigma \rightarrow 0$ . To do this, for a solution  $u$  of (1), we put

$$v_\sigma = \cosh(\sigma|D|)u,$$

where  $D = -i\nabla$  with Fourier symbol  $\xi$ . Then we define a modified energy associated with the function  $v_\sigma$  by

$$E_\sigma(t) = \frac{1}{2} \int_{\mathbb{R}^d} \left( |\partial_t v_\sigma|^2 + |\nabla v_\sigma|^2 + \frac{2}{p+1} |v_\sigma|^{p+1} \right) dx.$$

For  $\sigma = 0$ , we have the conservation

$$E(t) = E(0) \quad \text{for all } 0 \leq t \leq T.$$

However, this fails to hold for  $\sigma > 0$ . In what follows, we will use equation (1) and Theorem 2 to prove

$$E_\sigma(t) \leq E_\sigma(0) + T \left\| \int_{\mathbb{R}^d} \partial_t v_\sigma(x, \cdot) \cdot N(v_\sigma(x, \cdot)) dx \right\|_{L_T^\infty}. \quad (29)$$

**Proof.** Differentiate  $E_\sigma(t)$  with respect to time  $t$  and then applying integration by parts in the spatial variable and (1), we obtain

$$\begin{aligned} \frac{d}{dt} E_\sigma(t) &= \int_{\mathbb{R}^d} \partial_t v_\sigma \left[ \overline{\partial_t^2 v_\sigma - \Delta v_\sigma} + |v_\sigma|^{p-1} \overline{v_\sigma} \right] dx \\ &= \int_{\mathbb{R}^d} \partial_t v_\sigma \left[ \cosh(\sigma|D|) \left[ \overline{\partial_t^2 u - \Delta u} \right] + |v_\sigma|^{p-1} \overline{v_\sigma} \right] dx \\ &= \int_{\mathbb{R}^d} \partial_t v_\sigma \left[ -\cosh(\sigma|D|) \left[ |\operatorname{sech}(\sigma|D|) v_\sigma|^{p-1} \operatorname{sech}(\sigma|D|) \overline{v_\sigma} \right] + |v_\sigma|^{p-1} \overline{v_\sigma} \right] dx \\ &= \int_{\mathbb{R}^d} \partial_t v_\sigma \cdot N(v_\sigma) dx, \end{aligned}$$

where

$$N(v_\sigma) = -\cosh(\sigma|D|) \left[ |\operatorname{sech}(\sigma|D|) v_\sigma|^{p-1} \operatorname{sech}(\sigma|D|) \overline{v_\sigma} \right] + |v_\sigma|^{p-1} \overline{v_\sigma}. \quad (30)$$

Therefore, integrating and using Hölder's inequality in time yields

$$\begin{aligned} E_\sigma(t) &= E_\sigma(0) + \int_0^t \int_{\mathbb{R}^d} \partial_t v_\sigma(x, s) \cdot N(v_\sigma(x, s)) dx ds \\ &\leq E_\sigma(0) + \int_0^T \left| \int_{\mathbb{R}^d} \partial_t v_\sigma(x, s) \cdot N(v_\sigma(x, s)) dx \right| ds \end{aligned}$$

$$\leq E_\sigma(0) + T \left\| \int_{\mathbb{R}^d} \partial_t v_\sigma(x, \cdot) \cdot N(v_\sigma(x, \cdot)) dx \right\|_{L_T^\infty}. \quad (31)$$

□

**Lemma 3.** For  $\partial_t v_\sigma \in L_x^2$  and  $v_\sigma \in H^1$ , we have the following estimate

$$\left| \int_{\mathbb{R}^d} \partial_t v_\sigma \cdot N(v_\sigma) dx \right| \leq C \sigma^{2\theta} \|\partial_t v_\sigma\|_{L_x^2} \|v_\sigma\|_{H^1}^p, \quad (32)$$

for some constant  $C > 0$ , where  $\theta = \frac{3}{4}$  when  $d = 1$  and  $\theta = \frac{1}{3}$  when  $d = 2$ .

*Proof.* Recall that

$$N(v_\sigma) = -\cosh(\sigma|D|) \left[ |\operatorname{sech}(\sigma|D|) v_\sigma|^{p-1} \operatorname{sech}(\sigma|D|) \overline{v_\sigma} \right] + |v_\sigma|^{p-1} \overline{v_\sigma}.$$

By the Cauchy-Schwarz inequality, we have

$$\left| \int_{\mathbb{R}^d} \partial_t v_\sigma \cdot N(v_\sigma) dx \right| \leq \|\partial_t v_\sigma\|_{L_x^2} \|N(v_\sigma)\|_{L_x^2}.$$

Thus, we are reduced to prove

$$\|N(v_\sigma)\|_{L_x^2} \lesssim \sigma^{2\theta} \|v_\sigma\|_{H^1}^p. \quad (33)$$

To proceed with the proof, we need the following Lemma whose proof is found in [4, Lemma 3].

**Lemma 4.** Let  $p > 1$  be any odd integer such that  $\xi = \sum_{j=1}^p \xi_j$  for  $\xi_j \in \mathbb{R}$ . Then

$$\left| 1 - \cosh(|\xi|) \prod_{j=1}^p \operatorname{sech}(|\xi_j|) \right| \leq 2^p \sum_{j \neq k=1}^p |\xi_j| |\xi_k|.$$

By symmetry, we may assume that  $|\xi_1| \leq |\xi_2| \leq \dots \leq |\xi_p|$ . Then, from Lemma 4 (see [4, Lemma 3]),

$$|K(\sigma\xi)| = \left| 1 - \cosh(\sigma|\xi|) \prod_{j=1}^p \operatorname{sech}(\sigma|\xi_j|) \right| \leq 2^p C \sigma^2 |\xi_{p-1}| |\xi_p|. \quad (34)$$

It is easy to see that

$$|K(\sigma\xi)| \leq 1. \quad (35)$$

Now, interpolation between (34) and (35) yields

$$|K(\sigma\xi)| \leq 2^p \left[ \sigma^2 |\xi_{p-1}| |\xi_p| \right]^\theta,$$

for any  $0 \leq \theta \leq 1$ . In particular choosing  $\theta = \frac{3}{4}$  when  $d = 1$  (respectively,  $\theta = \frac{1}{3}$  when  $d = 2$ ) yields

$$|K(\sigma\xi)| \leq 2^p \sigma^{\frac{3}{2}} |\xi_{p-1}|^{\frac{3}{4}} |\xi_p|^{\frac{3}{4}} \quad \text{when } d = 1, \quad (36)$$

$$|K(\sigma\xi)| \leq 2^p \sigma^{\frac{2}{3}} |\xi_{p-1}|^{\frac{1}{3}} |\xi_p|^{\frac{1}{3}} \quad \text{when } d = 2. \quad (37)$$

By taking the Fourier transform of  $N(v_\sigma)$  we obtain

$$\begin{aligned} |\mathcal{F}_x[N(v_\sigma)](\xi)| &= \left| \int_{\xi = \sum_{j=1}^p \xi_j} K(\sigma\xi) \prod_{j=1}^p \widehat{v_\sigma}(\xi_j) \prod_{j=1}^p d\xi_j \right| \\ &\lesssim \int_{\xi = \sum_{j=1}^p \xi_j} |K(\sigma\xi)| \prod_{j=1}^p |\widehat{v_\sigma}(\xi_j)| \prod_{j=1}^p d\xi_j. \end{aligned}$$

Now, let

$$w_\sigma = \mathcal{F}_x^{-1}(|\widehat{v_\sigma}|).$$

If  $d = 1$ , then using Plancherel's Theorem, (36), Hölder's inequality, and Sobolev embedding, we obtain

$$\begin{aligned} \|N(v_\sigma)\|_{L_x^2} &= \|\mathcal{F}_x[N(v_\sigma)](\xi)\|_{L_\xi^2} \lesssim \sigma^{\frac{3}{2}} \left\| \int_{\xi = \sum_{j=1}^p \xi_j} |\xi_{p-1}|^{\frac{3}{4}} |\xi_p|^{\frac{3}{4}} \prod_{j=1}^p |\widehat{v_\sigma}(\xi_j)| \prod_{j=1}^p d\xi_j \right\|_{L_\xi^2} \\ &\lesssim \sigma^{\frac{3}{2}} \left\| \int_{\xi = \sum_{j=1}^p \xi_j} |\xi_{p-1}|^{\frac{3}{4}} |\xi_p|^{\frac{3}{4}} \prod_{j=1}^p \widehat{w_\sigma}(\xi_j) \prod_{j=1}^p d\xi_j \right\|_{L_\xi^2} \\ &\lesssim \sigma^{\frac{3}{2}} \left\| w_\sigma^{p-2} \cdot |D|^{\frac{3}{4}} w_\sigma \cdot |D|^{\frac{3}{4}} w_\sigma \right\|_{L_x^2} \\ &\lesssim \sigma^{\frac{3}{2}} \left\| w_\sigma^{p-2} \right\|_{L_x^\infty} \left\| |D|^{\frac{3}{4}} w_\sigma \right\|_{L_x^2}^2 \\ &\lesssim \sigma^{\frac{3}{2}} \|w_\sigma\|_{L_x^\infty}^{p-2} \left\| |D|^{\frac{3}{4}} w_\sigma \right\|_{L_x^4}^2 \\ &\lesssim \sigma^{\frac{3}{2}} \|w_\sigma\|_{H^{\frac{1}{2}+}}^{p-2} \left\| |D|^{\frac{3}{4}} w_\sigma \right\|_{H^{\frac{1}{4}}}^2 \\ &\lesssim \sigma^{\frac{3}{2}} \|w_\sigma\|_{H^1}^p \sim \sigma^{\frac{3}{2}} \|v_\sigma\|_{H^1}^p. \end{aligned}$$

Similarly, for  $d = 2$ , we choose Hölder's exponents  $\frac{2}{1-\theta} = 3$  and  $\frac{2}{\theta} = 6$ , and the by using Plancherel's Theorem, (37), Hölder's inequality, and Sobolev embedding, we obtain

$$\|N(v_\sigma)\|_{L_x^2} = \|\mathcal{F}_x[N(v_\sigma)](\xi)\|_{L_\xi^2} \lesssim \sigma^{\frac{2}{3}} \left\| \int_{\xi = \sum_{j=1}^p \xi_j} |\xi_{p-1}|^{\frac{1}{3}} |\xi_p|^{\frac{1}{3}} \prod_{j=1}^p |\widehat{v_\sigma}(\xi_j)| \prod_{j=1}^p d\xi_j \right\|_{L_\xi^2}$$

$$\begin{aligned}
&\lesssim \sigma^{\frac{2}{3}} \left\| \int_{\xi=\sum_{j=1}^p \xi_j} |\xi_{p-1}|^{\frac{1}{3}} |\xi_p|^{\frac{1}{3}} \prod_{j=1}^p \widehat{w_\sigma}(\xi_j) \prod_{j=1}^p d\xi_j \right\|_{L_\xi^2} \\
&\lesssim \sigma^{\frac{2}{3}} \left\| w_\sigma^{p-2} \cdot |D|^{\frac{1}{3}} w_\sigma \cdot |D|^{\frac{1}{3}} w_\sigma \right\|_{L_x^2} \\
&\lesssim \sigma^{\frac{2}{3}} \left\| w_\sigma^{p-2} \right\|_{L_x^6} \left\| \left[ |D|^{\frac{1}{3}} w_\sigma \right]^2 \right\|_{L_x^3} \\
&\lesssim \sigma^{\frac{2}{3}} \|w_\sigma\|_{L_x^{6(p-1)}}^{p-2} \left\| |D|^{\frac{1}{3}} w_\sigma \right\|_{L_x^6}^2 \\
&\lesssim \sigma^{\frac{2}{3}} \|w_\sigma\|_{H^{1-\frac{1}{3(p-1)}}}^{p-2} \left\| |D|^{\frac{1}{3}} w_\sigma \right\|_{H^{\frac{2}{3}}}^2 \\
&\lesssim \sigma^{\frac{2}{3}} \|w_\sigma\|_{H^1}^p \sim \sigma^{\frac{2}{3}} \|v_\sigma\|_{H^1}^p.
\end{aligned}$$

□

Now, our almost conservation law is stated as follows.

**Theorem 3** (Approximate conservation law). *Let  $d = 1$  or  $2$ ,  $\sigma > 0$  and  $p > 1$  be any odd integer. Let  $(f, g) \in H^{\sigma,1} \times H^{\sigma,0}$  and  $u$  be the local solution of (1) on  $\mathbb{R}^d \times [0, T]$  obtained in Theorem 2. Then, we have*

$$\sup_{t \in [0, T]} E_\sigma(t) \leq E_\sigma(0) + cT\sigma^{2\theta} [2E_\sigma(0)]^{\frac{p+1}{2}}, \quad (38)$$

where  $\theta$  be as given in Lemma 3.

*Proof.* We have from (12)

$$\begin{aligned}
\left( \|v_\sigma\|_{L_T^\infty H^1} + \|\partial_t v_\sigma\|_{L_T^\infty L_x^2} \right)^2 &= \left( \|u\|_{L_T^\infty H^{\sigma,1}} + \|\partial_t u\|_{L_T^\infty H^{\sigma,0}} \right)^2 \\
&\leq c (\|f\|_{H^{\sigma,1}} + \|g\|_{H^{\sigma,0}})^2 \\
&\leq 2c \|v_\sigma(\cdot, 0)\|_{H^1}^2 + 2C \|\partial_t v_\sigma(\cdot, 0)\|_{L_x^2}^2 \\
&\leq 2c E_\sigma(0),
\end{aligned}$$

which in turn implies that

$$\|v_\sigma\|_{L_T^\infty H^1} + \|\partial_t v_\sigma\|_{L_T^\infty L_x^2} \leq c\sqrt{2E_\sigma(0)}. \quad (39)$$

Now, using (29), (32) and (39) we obtain

$$\begin{aligned}
E_\sigma(t) &\leq E_\sigma(0) + cT \|\partial_t v_\sigma\|_{L_T^\infty L_x^2} \|N(v_\sigma)\|_{L_T^\infty L_x^2} \\
&\leq E_\sigma(0) + cT\sigma^{2\theta} \|\partial_t v_\sigma\|_{L_T^\infty L_x^2} \|v_\sigma\|_{L_T^\infty H^1}^p \\
&\leq E_\sigma(0) + cT\sigma^{2\theta} \sqrt{2E_\sigma(0)} \sqrt{2E_\sigma(0)}^p \\
&\sim E_\sigma(0) + cT\sigma^{2\theta} [2E_\sigma(0)]^{\frac{p+1}{2}}.
\end{aligned}$$

□

#### 4 Proof of Theorem 1

Suppose that  $(f, g) \in H^{\sigma_0, 1} \times H^{\sigma_0, 0}$  for some  $\sigma_0 > 0$ . This implies

$$v_{\sigma_0}(\cdot, 0) = \cosh(\sigma_0|D|)f(\cdot) \in H^1, \quad \partial_t v_{\sigma_0}(\cdot, 0) = \cosh(\sigma_0|D|)g(\cdot) \in L_x^2.$$

Then we have by the Sobolev embedding that

$$E_{\sigma_0}(0) \leq \|v_{\sigma_0}(\cdot, 0)\|_{H^1}^2 + \|\partial_t v_{\sigma_0}(\cdot, 0)\|_{L_x^2}^2 + \|v_{\sigma_0}(\cdot, 0)\|_{H^1}^{p+1} < \infty.$$

Now, we can construct a solution on  $[0, \delta]$  for arbitrarily large time  $\delta$  by following the techniques presented in [22]. This is achieved by applying the approximate conservation law Theorem 3, so as to repeat the local result in Theorem 2 on successive short time interval of size  $T$  to reach  $\delta$  by adjusting the strip width parameter  $\sigma \in (0, \sigma_0]$  of the solution according to the size of  $\delta$ . The goal is to prove that for a given parameter  $\sigma \in (0, \sigma_0]$  and large  $\delta$ ,

$$\sup_{t \in [0, \delta]} E_{\sigma}(t) \leq 2E_{\sigma_0}(0) \quad \text{for } \sigma \geq c\delta^{-\frac{1}{2\theta}}, \quad (40)$$

where  $c > 0$  depends only on the initial data norm, and  $\theta$  be as in Theorem 3. This would imply  $\mathcal{E}_{\sigma}(t) < \infty$  for all  $t \in [0, \delta]$  and hence for all  $t \in [0, \delta]$  the solution  $u$  to (1) satisfies

$$(u, \partial_t u)(\cdot, t) \in H^{\sigma, 1} \times H^{\sigma, 0} \quad \text{for } \sigma \geq c\delta^{-\frac{1}{2\theta}}.$$

It remains to prove (40). Since  $\cosh(x)$  is increasing function for  $x \geq 0$ , for  $\sigma \leq \sigma_0$  we have

$$E_{\sigma}(0) \leq E_{\sigma_0}(0). \quad (41)$$

Now, for a given parameter  $\sigma \in (0, \sigma_0]$  and  $t_0 \in [0, T]$ , we have by Theorem 2, Theorem 3 and (41),

$$\begin{aligned} \sup_{t \in [0, t_0]} E_{\sigma}(t) &\leq E_{\sigma}(0) + cT\sigma^{2\theta} E_{\sigma}^{\frac{p+1}{2}}(0) \\ &\leq E_{\sigma_0}(0) + cT\sigma^{2\theta} E_{\sigma_0}^{\frac{p+1}{2}}(0). \end{aligned}$$

Thus

$$\sup_{t \in [0, t_0]} E_{\sigma}(t) \leq 2E_{\sigma_0}(0), \quad (42)$$

provided

$$cT\sigma^{2\theta} E_{\sigma_0}^{\frac{p+1}{2}}(0) \leq 1. \quad (43)$$

Then we can apply Theorem 2, with initial time  $t = t_0$  and the time step  $T$  in (11) to extend the solution from  $[0, t_0]$  to  $[t_0, t_0 + T]$ . By Theorem 3 and (42), we obtain

$$\sup_{t \in [t_0, t_0 + T]} E_{\sigma}(t) \leq E_{\sigma}(t_0) + cT\sigma^{2\theta} (2E_{\sigma_0}(0))^{\frac{p+1}{2}}. \quad (44)$$

In this way, we can cover all time intervals  $[0, T]$ ,  $[T, 2T]$ ,  $[2T, 3T]$  etc., and obtain

$$E_{\sigma}(T) \leq E_{\sigma}(0) + cT\sigma^{2\theta} (2E_{\sigma_0}(0))^{\frac{p+1}{2}},$$

$$\begin{aligned}
E_\sigma(2T) &\leq E_\sigma(T) + cT\sigma^{2\theta} (2E_{\sigma_0}(0))^{\frac{p+1}{2}} \\
&\leq E_\sigma(0) + 2cT\sigma^{2\theta} (2E_{\sigma_0}(0))^{\frac{p+1}{2}} \\
E_\sigma(3T) &\leq E_\sigma(2T) + cT\sigma^{2\theta} (2E_{\sigma_0}(0))^{\frac{p+1}{2}} \\
&\leq E_\sigma(T) + 2cT\sigma^{2\theta} (2E_{\sigma_0}(0))^{\frac{p+1}{2}} \\
&\leq E_\sigma(0) + 3cT\sigma^{2\theta} (2E_{\sigma_0}(0))^{\frac{p+1}{2}}, \\
&\dots, \\
E_\sigma(nT) &\leq E_\sigma(0) + ncT\sigma^{2\theta} (2E_{\sigma_0}(0))^{\frac{p+1}{2}}.
\end{aligned}$$

This argument continues until

$$cnT\sigma^{2\theta} (2E_{\sigma_0}(0))^{\frac{p+1}{2}} \leq E_{\sigma_0}(0), \quad (45)$$

and then we have reached the final time

$$\delta = nT,$$

when

$$cnT\sigma^{2\theta} (2E_{\sigma_0}(0))^{\frac{p+1}{2}} \leq 1.$$

Finally, the condition (43) satisfies for  $\sigma$  such that

$$cnT\sigma^{2\theta} (2E_{\sigma_0}(0))^{\frac{p+1}{2}} = 1.$$

Therefore

$$\sigma^{2\theta} = \delta^{-1} \cdot c^{-1} (2E_{\sigma_0}(0))^{\frac{1-p}{2}}.$$

It gives (40) if choose  $c \leq \left( \delta^{-1} \cdot c^{-1} (2E_{\sigma_0}(0))^{\frac{1-p}{2}} \right)^{\frac{1}{2\theta}}$ . This completes the proof of Theorem 1.

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