Journal of Mathematical Modeling
Vol. 14, No. 1, 2026, pp. 315-325. Research Article

JMM

Spline-interpolation solution of Cauchy problem for a harmonic function in a simply connected 3D domain

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Abstract. Here we construct an approximate spline-interpolation solution of the Cauchy problem for the Laplace equation. Our construction describes two different methods based on solution of integral equations. The first method involves singular integral equation, and the second one is based on solution of a Fredholm equation. We present the linear and the polynomial examples clarifying the construction approaches.

Keywords: Cauchy problem, integral equation, holomorphic function, spline, approximate solution
AMS Subject Classification 2010: 30-08

1 Introduction

Recall the formulation of the problem. Let D be a simply connected domain in the XYH space, $S = \{(x(s,h), y(s,h))\}$ be a connected component of the surface of D in this space with parameters $s \in [s_0(h), s_1(h)]$, $h \in [\min\{Pr_H(D)\}, \max\{Pr_H(D)\}]$, here $Pr_H(D)$ is the projection of D onto the coordinate axis H. There are given two real-valued functions $\phi(s)$ and $\psi(s)$, $s \in S$. The harmonic in D function $u(x,y,h)$ with the properties

$$u(x(s,h), y(s,h)) = \phi(s,h), \quad \frac{\partial u(x,y)}{\partial \overrightarrow{\nu}}|_{x=x(s,h), y=y(s,h)} = \psi(s,h), \tag{1}$$

where $s \in [s_0(h), s_1(h)]$, $h \in [\min\{Pr_H(D)\}, \max\{Pr_H(D)\}]$, and $\overrightarrow{\nu}$ is the exterior normal to S, is to be found in the whole domain D.

The problem is clearly an ill-posed one since we do not have any uniquely predefined rule how to reconstruct the solution.

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Received: 28 January 2025/ Revised: 29 August 2025/ Accepted: 21 October 2025
DOI: 10.22124/jmm.2025.29691.2646

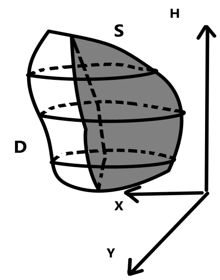

Figure 1: Modified problem

There exist extensive studies on the solution of the problem [9, 12]. The authors mainly apply regularization method or cost functions and a boundary control to solve this problem [3]. It is possible to apply purely imaginary quaternions to similar problems [1, 2], though the result, in general, will also be a quaternion-valued function. Also in this case commutative multiplication is lost and the expressions become cumbersome.

Here we construct the spline-interpolation solution of Cauchy problem for 3D Laplace equation. This paper generalizes articles [6, 7] applying the technique of [5]. More properly, we combine the 3D problem solution technique of [5] with the plane Cauchy problem solutions of [6, 7] in order to construct the solution of the spatial Cauchy problem by reducing it to the set of plane problems.

Our first goal is to describe the appearing variety of the similar plane problems. In order to do this we consider plane sections of the set S and reconstruct the polynomial on h solution in each slice of the initial solid D between two adjacent plane sections. We present two different methods of solution. The first is based on solution of the singular equation under condition of smooth boundary surface. We also apply Theorem 2 of [7]. The second approach is based on approximate solution of the integral Fredholm equation that is sometimes called Symm equation [11].

First we reduce the 3D problem to the set of the plane problems. Then we construct two different linear spline-interpolation solutions. Finally, we present the solution of degree 3 on the coordinate h.

2 Construction of the approximate solution

Let us reduce the initial spatial problem to the set of plane problems.

Assume that the solid D is bounded by the surface ∂D. Assume also that $S \subset \partial D$ is a connected subset of this surface.

Let us introduce the planes p_l parallel to the plane XOY. Assume that their equations are $h = h_l$, $l = 0, \ldots, L$. Consider the curves Γ_l that are the sections $S \bigcap p_l$ of the surface S by the planes p_l, see Fig. 1.

The normal to the surface S vector $\overrightarrow{\nu}$ naturally splits into two components, namely, the vector $\overrightarrow{\nu}_h$ parallel to the XY-plane, and the vector $\overrightarrow{\nu}_v$ parallel to the axis H.

Note that since we know $u|_S$ it is possible to reconstruct the directional derivative $\frac{\partial u}{\partial \overrightarrow{\tau}}(s)$, here $\overrightarrow{\tau}$ is the vector tangent to S and normal to $\Gamma_l(s)$ in the XY plane for $s \in \Gamma_l$, $l = 0, \ldots, L$.

Since the vectors $\overrightarrow{\nu}(s)$ and $\overrightarrow{\tau}(s)$ form a base of the plane normal to $\Gamma_l(s)$ the vector $\overrightarrow{\nu}_h(s)$ is a linear combination of these vectors. So for any $l = 0, \dots, L$, there exist functions $n_{1l}(s)$, $n_{2l}(s)$ such that $\overrightarrow{\nu}_h(s) = n_{1l}(s)\overrightarrow{\nu}(s) + n_{2l}(s)\overrightarrow{\tau}(s)$. This allows us to reduce the spatial problem to the set of plane problems by passing from the boundary restriction on $\frac{\partial u}{\partial \overrightarrow{\nu}}$ to that on $\frac{\partial u}{\partial \overrightarrow{\nu}_h}$.

Indeed, the second boundary condition then provides us with the boundary condition at the plane p_l parallel to XOY, $l = 0, \dots, L$. Consider two adjacent sections of S by the planes $h = h_0$ and $h = h_1$. Denote these curves by Γ_0 and Γ_1. So we have

$$u|_{\Gamma_0} = U_0, \ \frac{\partial u}{\partial \overrightarrow{\nu}_h}|_{\Gamma_0} = V_0, \tag{2}$$

and

$$u|_{\Gamma_1} = U_1, \ \frac{\partial u}{\partial \overrightarrow{\nu}_h}|_{\Gamma_1} = V_1. \tag{3}$$

We now reduce our problem (1) to the set of problems between adjacent sections p_{l-1} and p_l of the surface subset S, $l = 1, \dots, L$. The formulation of the modified problem is as follows: given the values of the functions $U_0(s)$, $V_0(s)$, $s \in \Gamma_0$, and that of the functions $U_1(s)$, $V_1(s)$, $s \in \Gamma_1$, it should be possible to reconstruct the harmonic function in this slice bounded by ∂D. It is well-known that the plane Cauchy problem is equivalent to construction of the analytic function in the domain bounded by the curve $\Gamma_j \bigcup \Gamma'_j$ given its values at the points of Γ_j, $j = 0, 1$. Here we define the curve Γ'_j as $(\partial D \setminus S) \bigcap p_j$, $j = 0, 1$.

Our next goal is to reduce this spatial problem to the set of plane Cauchy problems. In order to solve the last we naturally turn to the complex analysis. We first introduce the complex variable $z = x + iy$. Next we search for the solution u polynomial on h: $u(z, \bar{z}, h) = \sum_{k=0}^{n} u_k(z, \bar{z}) h^k$. The boundary conditions on the set of curves Γ_l, $l = 0, \dots, L$, allow us to find the coefficients $u_k(z, \bar{z})$, $k = 1, \dots, n$. In the case of linear spline we cannot introduce in our constructions the boundary condition on $\frac{\partial u}{\partial \overrightarrow{\tau}}$. In order to do this, we should assume k to be greater than 1. The approximate solution function $u(x, y, h)$ is then a spline in h over the set of intervals $[h_{l-1}, h_l]$, $l = 1, \dots, L$.

3 Linear spline

Let us search for the spline-interpolation solution $u(x, y, h)$ linear on h in each slice: $u(z, \bar{z}, h) = u_0(z, \bar{z}) + u_1(z, \bar{z})h$, $h \in [p_0, p_1]$. Then the functions u_0, u_1 are both harmonic functions on $z, \bar{z}$ at each end of the slice and we find the missing values of the functions U_0, V_0 and U_1, V_1 serving as boundary conditions (2, 3) at the points of the surface $\partial D \setminus S$ at these ends as in [6].

We reconstruct the surface step by step for each adjacent splice between p_{l-1} and p_l, $l = 1, \dots, L$. As in [5] the approximation of the boundary conditions at the surface ∂D is $O(\Delta(h))$, here $\Delta(h)$ is the step of the partition $h_0, h_1, \dots, h_L$.

It remains to find the approximation of the upper and lower auxiliary curves, i.e. the values of the harmonic functions $u_0(z, \bar{z})$ and $u_1(z, \bar{z})$ in each slice.

Since we construct the spline in one slice between two adjacent plane sections of D it suffices to consider only the first slice bounded by the planes p_0, p_1.

We already presented two methods of the curve reconstruction [6]. Both of them involve the mapping of the unit disc $\mathbb{D}^1$ onto the domain D_l partly bounded by Γ_l, $l = 0, \dots, L$.

3.1 Singular integral equation method

The first method involves Sokhotsky's formula. We interpret this formula as the system of equations on the unknown points on the curve completing Γ_l, $l = 0,1$ to the closed curve. We assume that the upper part of the unit circle maps onto Γ_l, $l = 0,1$. After this we reconstruct the complete boundary of the unknown solid as the ruled surface with directrices given by $\Gamma_0 \bigcup \Gamma_0'$ and $\Gamma_1 \bigcup \Gamma_1'$.

Recall the construction scheme for each curve. Due to [4, 10] the necessary and sufficient condition for $f(z(s))$, $s \in s \in [s_0(h_l), s_1(h_l)]$, and for all $l \in \{1,\dots,L\}$ to be the boundary values of the holomorphic in D_l function is the relation

$$f(z(s)) = \frac{1}{\pi \mathrm{i}} \int_0^T \frac{f(z(t))z'(t)}{z(t)-z(s)} dt, \;\; s \in [s_0(h_l), s_1(h_l)]. \tag{4}$$

Let us separate the values of $f(z(t))$ on the known part of the curve from that on the unknown part of the curve:

$$\begin{aligned}\phi'(s) + \mathrm{i}\psi(s) =& \frac{1}{\pi \mathrm{i}} \int_{\Gamma} \frac{(\phi'(t) + \mathrm{i}\psi(t))z'(t)}{z(t)-z(s)} dt \\ &+ \frac{1}{\pi \mathrm{i}} \int_{\Gamma'} \frac{(\tilde{\phi}(t) + \mathrm{i}\tilde{\chi}(t))z'(t)}{z(t)-z(s)} dt, s \in [s_0(h_l), s_1(h_l)],\end{aligned} \tag{5}$$

$$\begin{aligned}\tilde{\phi}(s) + \mathrm{i}\tilde{\chi}(s) =& \frac{1}{\pi \mathrm{i}} \int_{\Gamma} \frac{(\phi'(t) + \mathrm{i}\psi(t))z'(t)}{z(t)-z(s)} dt \\ &+ \frac{1}{\pi \mathrm{i}} \int_{\Gamma'} \frac{(\tilde{\phi}(t) + \mathrm{i}\tilde{\chi}(t))z'(t)}{z(t)-z(s)} dt, s \in [s_1(h_l), s_0(h_l)].\end{aligned} \tag{6}$$

We consider relations (5) and (6) as two singular integral equations [4, 10]. The free term $\frac{1}{\pi \mathrm{i}} \int\limits_{[s_0(h_l), s_1(h_l)]} \frac{(\phi(t)+\mathrm{i}\chi(t))z'(t)}{z(t)-z(s)} dt$ of equation (6) contains the known functions $\phi(s)$ and $\chi(s)$ and is thus also known. So we can solve (6) with respect to $\tilde{\phi}(s) + \mathrm{i}\tilde{\chi}(s)$, $s \in [s_1(h_l), s_0(h_l)]$. After finding $\tilde{\phi}(s) + \mathrm{i}\tilde{\chi}(s)$ we check if the function $\phi'(s) + \mathrm{i}\psi(s)$, $s \in [s_0(h_l), s_1(h_l)]$, satisfies equation (5). Now if $\phi'(s) + \mathrm{i}\psi(s)$ meets (5) then the function $f(z(t))$ is the boundary value of a holomorphic in D_l function by (4). Therefore we formulate the following

Theorem 1. *Let D_l for any $l \in \{1,\dots,L\}$ be a one-connected domain with the smooth boundary $\Gamma \cup \Gamma'$, where $\Gamma = \{z(s) = x(s) + \mathrm{i}y(s), s \in [s_0(h_l), s_1(h_l)]\}$, s being the natural parameter of Γ, $\Gamma' = \{z(s) = x(s) + \mathrm{i}y(s), s \in [s_1(h_l), s_0(h_l)]\}$. Let the data defined by the formulation of the Cauchy problem be given at the points of Γ. Let $\phi'(s)$, $s \in [s_0(h_l), s_1(h_l)]$, be of the Hölder class and $\psi(s)$, $s \in [s_0(h_l), s_1(h_l)]$, be continuous. Then this Cauchy problem is solvable if and only if the known function $\phi'(s) + \mathrm{i}\int_0^s \psi(t)dt$,*

$s \in [s_0(h_l), s_1(h_l)]$, *satisfies the relation*

$$\phi'(s) + \mathrm{i}\psi(s) = \frac{1}{\pi \mathrm{i}} \int\limits_{\Gamma} \frac{(\phi'(t) + \mathrm{i}\psi(t)d\tau)z'(t)}{z(t) - z(s)} dt + \frac{1}{\pi \mathrm{i}} \int\limits_{\Gamma'} \frac{(\tilde{\phi}'(t) + \mathrm{i}\tilde{\chi}(t))z'(t)}{z(t) - z(s)} dt, \; s \in [s_0(h_l), s_1(h_l)],$$

where $\tilde{\phi}(s) + \mathrm{i}\tilde{\chi}(s)$, $s \in [s_1(h_l), s_0(h_l)]$, *is the solution of the singular integral equation*

$$\tilde{\phi}(s) + \mathrm{i}\tilde{\chi}(s) = \frac{1}{\pi \mathrm{i}} \int\limits_{\Gamma'} \frac{(\tilde{\phi}(t) + \mathrm{i}\tilde{\chi}(t))z'(t)}{z(t) - z(s)} dt + \frac{1}{\pi \mathrm{i}} \int\limits_{\Gamma} \frac{(\phi'(t) + \mathrm{i}\psi(t))z'(t)}{z(t) - z(s)} dt, \; s \in [s_1(h_l), s_0(h_l)].$$

The solution of the solvable Cauchy problem has the form

$$w(x,y) = \mathrm{Re}[\int \left\{ \frac{1}{2\pi \mathrm{i}} \int\limits_{\Gamma} \frac{(\phi'(t) + \mathrm{i}\psi(t))z'(t)}{z(t) - x - \mathrm{i}y} dt + \frac{1}{2\pi \mathrm{i}} \int\limits_{\Gamma'} \frac{(\tilde{\phi}(t) + \mathrm{i}\tilde{\chi}(t))z'(t)}{z(t) - x - \mathrm{i}y} dt \right\} dz].$$

In order to find the domains D_j, $j = 0, \ldots, L$, we construct the curves Γ'_j. In order to do this in its turn we consider integral equation (6) as a system over the unknown points on this curve. Note that under the assumption that the curve Γ_j is the image of the semicircle and the curve Γ'_j is the image of the other semicircle, the problem is solvable [6, 7].

This statement is a natural generalization of [6, Theorem 1]. The only difference is that we simultaneously solve the plane problem for all the domains D_j, $j = 0, \ldots, L$. This difference itself poses the stability requirement on the solution of the algebraic linear equation system relative to singular equation (6). Note that the relative system matrix is nondegenerate [7] and continuously depends on the curve Γ' of the system.

Also by [5] the convergence rate of the approximate solution to the exact one equals $O(\max\{\Delta_h(\phi), \Delta_h(\psi)\}, \frac{A}{N})$. Here $\Delta_h\phi$ is the continuity modulus of the function ϕ, $\Delta_h\psi$ is the continuity modulus of the function ψ, A is a constant, and N is the number of nodes at the boundary curve Γ.

Example 1. We fix the cylinder $\mathbb{D}^1 \times [0,1]$ as D and $\Gamma \times [0,1]$ as S. Here Γ is an upper half-circle $\{\cos(t), \sin(t)\}$, $t \in [0, \pi]$. Consider the mapping $(\cos(t), \sin(t)) \mapsto (\cos(t), \sin(t) + 0.2\sin(4t))$ at the lower section, $l = 0$, and $(\cos(t), \sin(t) + 0.15\sin(3t) + 0.2\sin(4t))$ at the upper section, $l = 1$, $t \in [0, \pi]$. We reconstruct from equation (6) the curve Γ'_l, $l = 0, 1$. In this case the matrix of the system relative to Theorem 1 is the same for both ends of the cylinder. Only difference lies in the right-hand sides of the linear systems. So the construction of the approximate linear spline-interpolation solution is correct. Fig. 2 shows the result of calculations.

This construction naturally extends to the general case of a non-cylindrical solid D. We should consider the combination of the mapping f_1 that maps the boundary of the cylindrical slice $\mathbb{S}^1 \times [h_{l-1}, h_l]$ into the relative slice of D and the boundary data on this slice.

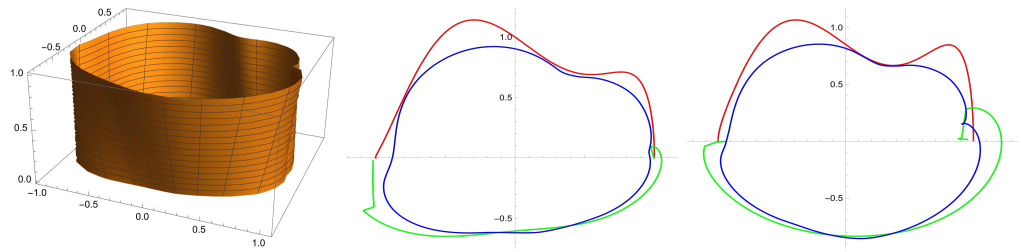

Figure 2: The first part is the approximation of solution as the closed surface. The second part is the approximation of the solution function at upper section. The last part is the approximation of the solution function at the lower section.

3.2 Method of an auxiliary conformal mapping

The second method applies the auxiliary conformal mapping.

First we consider closures Γ_l' of the curves Γ_l, $l = 0,\dots,L$, so that the poles of the functions given on these curves do not belong to the domains D_l bounded by the closures $\Gamma_l' \bigcup \Gamma_l$.

Next we reconstruct the conformal mappings of the unit disk $\mathbb{D}^1$ onto the domains D_l, $l = 0,\dots,L$.

The construction of this map is given, e.g. in [8] and involves reparametrization of the initial boundary curve. We construct this reparametrization through approximate solution of the Fredholm integral equation.

Theorem 2 ([8])**.** *For any domain D_l, $l = 0,\dots,L$, with boundary — a simple smooth closed curve Γ, defined by a Fourier polynomial, one can construct with any accuracy a function mapping D_l onto the unit circle by solving an integral equation reduced to a finite system of linear equations.*

Finally, we reconstruct the function u as the linear on h spline approximating the auxiliary conformal mappings.

Example 2. We fix the cylinder $\mathbb{D}^1 \times [0,1]$ as D and $\Gamma \times [0,1]$ as S. Here Γ is an upper half-circle $\{\cos(t), \sin(t)\}, t \in [0,\pi]$. Assume that the boundary conditions are given at the half-circles $(\cos(t), \sin(t))$, as $(\cos(t), \sin(t) - 0.1\sin(2t) + 0.3\cos(3t))$ and $(\cos(t), \sin(t) - 0.25\sin(2t) + 0.3\cos(3t))$ at the lower and upper sections, $t \in [0,\pi]$.

In order to complete the contours, we simply extend the boundary data to the parameter values $t \in [\pi, 2\pi]$. So we have to find the approximate mappings of the unit disc onto the domains bounded by the curves $(\cos(t), \sin(t) - 0.1\sin(2t) + 0.3\cos(3t))$ and $(\cos(t), \sin(t) - 0.25\sin(2t) + 0.3\cos(3t))$ at the lower and upper sections. Fig. 3 presents the solution in this case.

This construction as in Example 1 naturally extends to the general case of a non-cylindrical solid D. But here we consider the combination of the analytic mappings f_{l-1} and f_l that map the ends of the cylindrical slice $\mathbb{S}^1 \times [h_{l-1}, h_l]$ into the relative ends of the slice of the initial domain D and the boundary data on this slice.

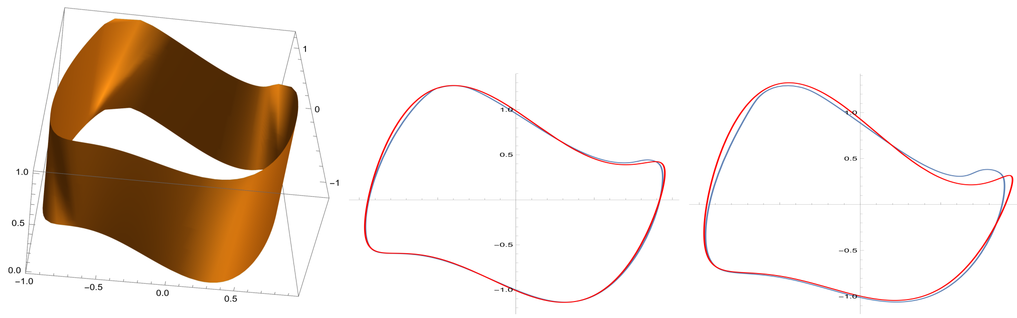

Figure 3: The first part is the approximation of the solution as the closed surface. The second part is the approximation of the solution function at the upper section, red contour is the initial curve, green curve is the reconstructed part, blue one is the approximation. The last part is the approximation of the solution function at the lower section, red contour is the initial curve, green curve is the reconstructed part, blue one is the approximation.

4 Polynomial spline

We search for the solution $u(z,\bar{z},h)$ polynomial on h: $u(z,\bar{z},h) = \sum_{k=0}^{n} u_k(z,\bar{z})h^k$ in each slice between the planes p_{l-1} and p_l, $l = 1,\dots,L$.

The curves $\Gamma_j \bigcup \Gamma'_j$ bound the domain D_j, $j = 0,\dots,L$. In order to glue the derivatives $u_h(z,\bar{z})$, $z \in D_j$, $j = 1,\dots,L-1$ in the adjacent slices and take into consideration the condition on $\frac{\partial u}{\partial \vec{\tau}}$ we necessarily consider splines of degree greater than 1 [5].

The construction is similar to one given in the previous section. The only difference is the recurrent formula allowing us to reconstruct u_0, u_1 through the terms with greater. degrees of h:

$$k(k-1)u_k(z,\bar{z}) = 4\partial_z\partial_{\bar{z}}u_{k-2}(z,\bar{z}). \tag{7}$$

So we split the coefficients into two groups, namely, the even and the odd sets. In order to glue the solutions in the lower end of the slice we consider the even coefficients. Odd coefficient set allows us both to meet the boundary condition at the top curve and to minimize the integral norm of the differences between the derivatives with respect to h at the common lower end of the slice.

For example, for $n = 3$ we have

$$u_0(z,\bar{z}) = f_0(z) + \overline{f_0(z)} + \frac{1}{2}(F_2(z)\bar{z} + \overline{F_2(z)}z),$$

$$u_1(z,\bar{z}) = f_1(z) + \overline{f_1(z)} + \frac{3}{2}(F_3(z)\bar{z} + \overline{F_3(z)}z).$$

Here $F_j(z) = \int f_j(z)dz$, $F_j(0) = 0$, $j = 2,3$.

Note that we reconstruct the solution starting from $f_2(z)$ and $f_3(z)$. In order to do this we consider the second derivative of the boundary function with respect to h. Assume that $h_0 = 0$, $h_1 = 1$. Then we achieve the system

$$\begin{cases} u_{h,h}(z,\bar{z},h_0)|_{\Gamma_0} = 2\mathrm{Re} f_2(z), \\ u_{h,h}(z,\bar{z},h_1)|_{\Gamma_1} = 6\mathrm{Re} f_3(z) + 2\mathrm{Re} f_2(z). \end{cases} \tag{8}$$

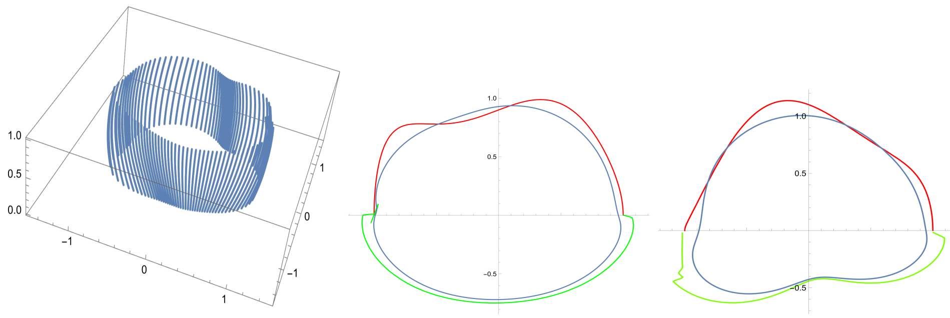

Figure 4: The polynomial approximation of the approximate solution as the closed surface. The second part is the approximation of the derivative $u_{h,h}$ at the upper section, red contour is the initial curve, green curve is the reconstructed part, blue one is the approximation. The last part is the approximation of the derivative $u_{h,h}$ at the lower section, red contour is the initial curve, green curve is the reconstructed part, blue one is the approximation.

From system (8) we find first $\mathrm{Re} f_3(z)$, $\mathrm{Re} f_2(z)$ and then $f_3(z)$, $f_2(z)$ on the curves Γ_l, $l = 0,1$.

We put these functions into the system

$$\begin{cases} u(z,\bar{z},h_0)|_{\Gamma_0} = \mathrm{Re} f_0(z) + \frac{1}{2}\mathrm{Re}(F_2(z)\bar{z}), \\ u_h(z,\bar{z},h_1)|_{\Gamma_1} = \mathrm{Re} f_1(z) + \frac{3}{2}\mathrm{Re}(F_3(z)\bar{z}). \end{cases} \tag{9}$$

Finally we reconstruct the analytic functions $f_0(z)$, $f_1(z)$ from (9). Here we apply Theorem 1 four times, twice for each end of the slice.

So, we have the spline-iterpolation solution of degree 3 on the variable h.

By [5] the convergence rate of the approximate solution to the exact one equals $O(\max\{\Delta_h^2(\phi), \Delta_h^2(\psi)\}, \frac{A}{N})$. Here $\Delta_h^2\phi$ is the continuity modulus of order 2 of the function ϕ, $\Delta_h^2\psi$ is the continuity modulus of order 2 of the function ψ, A is a constant, and N is the minimal number of nodes at the boundary curve Γ_l, $l = 0,\dots,L$.

Example 3. We again fix the cylinder $\mathbb{D}^1 \times [0,1]$ as D and $\Gamma \times [0,1]$ as S. Here Γ is an upper half-circle $\{\cos(t), \sin(t)\}$, $t \in [0,\pi]$. Assume that the boundary conditions are given at the half-circles $(\cos(t), \sin(t))$, $t \in [0,\pi]$, as in Example 1. Let us add to the boundary conditions of Example 1 the assertions on the values of $u_{h,h}$ given by $\frac{1}{4}(\cos(t), \sin(t) + 0.1\sin(3t) - 0.1\sin(4t))$ and $\frac{1}{4}(\cos(t), \sin(t) - 0.1\sin(3t) + 0.1\sin(4t))$ at the lower and the upper curve, respectively, $t \in [0,\pi]$.

We reconstruct four auxiliary curves for each boundary condition.

Note that it is easier to reconstruct the boundary surface S as the set of curves since then we simply reconstruct the set of cubic parabolas by their values at the ends. Fig. 4 presents the solution in this case. Here we combine the solutions of Example 1 with that for the second derivative $u_{h,h}$.

5 Examples

Here we compare the approximate solutions given in the first sections of the article to the exact. Consider two sections of the solid at $h = 0$ and $h = 1$ (Fig. 5) given by the curves $(\cos t, \sin t)$, $t \in [0,\pi]$, and $(\cos(t) - 0.00530516\sin(4t), 0.606104\sin(t) + 0.063662\sin(3t) - 0.212206\cos(2t) + 0.00530516\cos(4t))$,

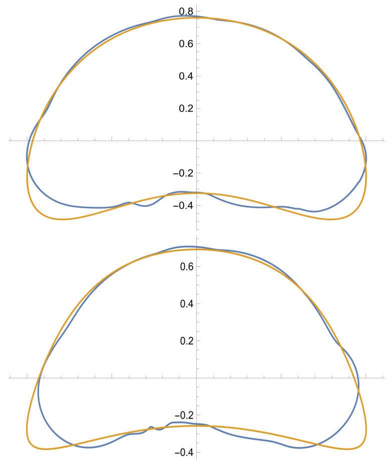

Figure 5: Approximations of the upper and lower sections of the solid by conformal mappings of the unit disk

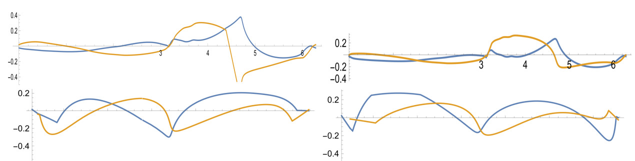

Figure 6: Example 4. The differences of the real and imaginary parts of the approximate and exact solutions at the upper (left column) and lower (right column) sections of the solid. Upper row – solutions with an auxiliary conformal mapping. Lower row – solutions of the singular integral equation

$t \in [\pi, 2\pi]$; and $(\cos t, \sin t)$, $t \in [0, \pi]$, and $(0.0143997\sin(-4s)+\cos(-s), -0.025\sin(-3s)-0.5\sin(-s)-0.0143997\cos(-4s)-0.231305\cos(-2s))$, $t \in [\pi, 2\pi]$, respectively. Assume that the data is given at the semicircle $(\cos t, \sin t)$, $t \in [0, \pi]$. Naturally the differences of the exact solutions with the approximate happen at the points with the greatest curvature. It is the illustration of the problems connected with conformal approximation of the thin domains or domains with corner points.

Example 4. Consider the function $(1+\frac{h}{10})z^{\frac{1}{3}}$. The function given by $1.1\cos(\frac{t}{3})+1.1\imath\sin(\frac{t}{3})$ at the upper section and by $\cos(\frac{t}{3})+\imath\sin(\frac{t}{3})$ at the lower one. Here we have 0 as the branching point of the reconstructed function (Fig. 6).

It is important to note that the reconstructed solution at the second part of the curve belongs to the same branch of the Riemann surface as the given data.

Example 5. Consider the function $(1-\frac{h}{10})z^2$. The function given by $\cos(2t)+i\sin(2t)$ at the upper section and by $0.9\cos(2t)+0.9i\sin(2t)$ at the lower one. Here the reconstructed function possesses no poles in the domains (Fig. 7).

Here the differences between the exact and the approximate solutions real and imaginary parts possess additional peaks due to the existence of such extrema for the initial function.

Example 6. Consider the function $(1+\frac{h}{10})(z+\frac{1}{z})$. The function given by $2\cos(t)$ at the upper section and by $2.1\cos(t)$ at the lower one. Here the reconstructed function has the pole at 0 (Fig. 8).

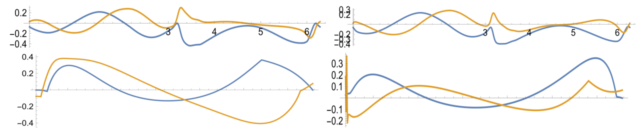

Figure 7: Example 5. The differences of the real and imaginary parts of the approximate and exact solutions at the upper (left column) and lower (right column) sections of the solid. Upper row – solutions with an auxiliary conformal mapping. Lower row – solutions of the singular integral equation

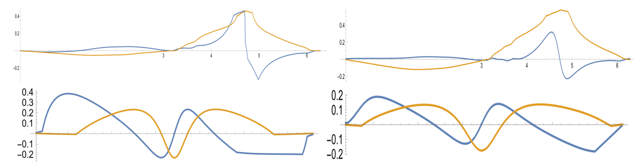

Figure 8: Example 6. The differences of the real and imaginary parts of the approximate and exact solutions at the upper (left column) and lower (right column) sections of the solid. Upper row – solutions with an auxiliary conformal mapping. Lower row – solutions of the singular integral equation

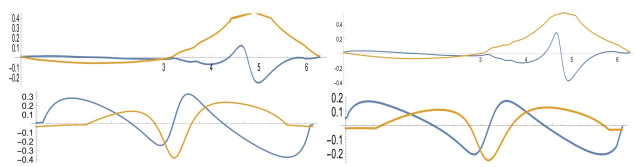

Figure 9: Example 7. The differences of the real and imaginary parts of the approximate and exact solutions at the upper (left column) and lower (right column) sections of the solid. Upper row – solutions with an auxiliary conformal mapping. Lower row – solutions of the singular integral equation

Example 7. Consider the function $(1+\frac{h}{10})\frac{0.1}{z^2}+\frac{1}{z})$. The function given by $i(-0.1\sin(2t)-\sin(t))+\cos(t)+0.1\cos(2t)$ at the upper section and by $i(-0.2\sin(2t)-\sin(t))+\cos(t)+0.2\cos(2t)$ at the lower one. The pole at 0 of the reconstructed function is of order 2 (Fig. 9).

Conflicts of interest

The authors declare that there are no conflicts of interest.

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