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# The k-order variations on the k-Fibonacci universal code for k ≥ 4

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ABSTRACT

In the current paper, first we study the $k-$order variations on the $k$-Fibonacci universal code for $k \geq 4$ (denoted by $GH_{-2}(k, n)$) and get Table 3. Also in Tables 1 and 2, we obtain the $k$-Fibonacci representation and the $k$-Fibonacci code for $k \geq 4$ and $1 \leq n \leq 50$, respectively. Finally, we examine a new blocking algorithm using $k-$Fibonacci code and Gopala-Hemachandra code.

## 1. Introduction

In mathematics, there are a lot of integer sequences, which are used in almost every field modern sciences. Many authors studied on these sequences (see [3, 6, 7, 9, 10]). The Fibonacci and generalized Fibonacci sequence ($k$-Fibonacci sequence) are famous sequences in mathematics.

Now for $k \geq 2$, the $k$-Fibonacci sequence, $\{F_n^k\}^{\infty}_0$ is definted by:

$$F_n^k = F_{n-1}^k + F_{n-2}^k + \cdots + F_{n-k}^k, \quad n \geq k,$$

with the initial conditions $F_0^k = 0$, $F_1^k = 0, \ldots$, $F_{k-2}^k = 0$, and $F_{k-1}^k = 1$ (see [8]). For example, $\{F_n^3\}^{\infty}_0 = \{0, 0, 1, 1, 2, 4, 7, \ldots\}$ and $\{F_n^4\}^{\infty}_0 = \{0, 0, 0, 1, 1, 2, 4, 8, \ldots\}$.

In 1986, Apostolic and Fraenkle introduced the Fibonacci code (one of the type of the universal code used to encoding positive integers is the Fibonacci code) which used in the source coding as well as in cryptography [1]. In [14], Zeckendrof proved that every positive integer has a unique representation as the sum of the nonconsecutive element of Fibonacci numbers. Let $A$ be a positive integer. Then, $A$ can be written as a binary string $r_1 \ldots r_s$ with the length $s$ such that $A = r_1F_1^2$

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$+r_2F_2^2 + . . . +r_sF_s^2$. In this method, for $A$, we first get the largest Fibonacci number $F_S^2$ at most $A$. Then, continue recursively with $A - F_S^2$. For instance,

$$15 = 0\times1+1\times2+0\times3+0\times5+0\times8+1\times13,$$

so, 010001 is the Fibonacci representation for 15.

For every integer $A$ have uniquely represented by the string $r_1 . . . r_s$ such that $A = r_1F_1^k + r_2F_2^k +... + r_sF_s^k$ applying the iterative encoding procedure mentioned above. Therefore, for $k \geq 3$, every positive integer $A$ can be representation by a binary string [11]. In [12], studied on the 3−Fibonacci representation. Based on this fact, for $k = 3$, we have

$$9 = 0 \times 1 + 1 \times 2 + 0 \times 4 + 1 \times 7.$$

Therefore, 0101 is the 3−Fibonacci representation for 9.

The $k$-Fibonacci code is gotten to add $(k - 1)1$ bits to the $k$-Fibonacci representation. For example, we have $13 = 1 \times 1 + 0 \times 2 + 1 \times 4 + 1 \times 8$ for $k = 4$. So, 1011 is the 4−Fibonacci representation and 1011111 is 4−Fibonacci code for 13. For simplicity, we set $1011^4 = 1011111$. In ***Tables 1 and 2***, we obtain the $k$-Fibonacci representation and the $k$-Fibonacci code for $k \geq 4$ and $1 \leq n \leq 50$, respectively.

Section 2, devotes to studying the Gopala-Hemachandra ($GH$) sequence and tables for the ($GH$) code for $k \geq 4$ are displayed. In Section 3, we use the $k$−Fibonacci code and the Gopala-Hemachandra code to give a new blocking algorithm.

## 2. Gopala-Hemachandra (GH) sequence and codes

The aim of this section is to get the $k$−order variant $k$−Fibonacci sequence and the $k$−order variations on the $k$-Fibonacci universal code for $k \geq 4$.

The Gopala-Hemachandra ($GH$) sequence defined as follows:

$$a, b, a + b, a + 2b, 2a + 3b, ... ,$$

for any pair $a$, $b$, and the Fibonacci numbers are obtained for the case $a = 1$ and $b = 2$ (see [13]).

In [2], authors introduced the second order variant Fibonacci sequence and got tits Gopala-Hemachandra $GH$ code. Also, they found that Gopala-Hemachandra ($GH$) code exists for $a = -2, ... , -20$, and $1 \leq n \leq 100$. The second order variant Fibonacci sequence (2-Fibonacci sequence), denoted by $VF_a(2, n)$, is defined as the ($GH$) sequence $\{a, b, a + b, a + 2b, 2a + 3b,...\}$ where $b = 1 - a$. $VF_a(2, n)$ is as

$$VF_a(2, n) = VF_a(2, n - 1) + VF_a(2, n - 2), \quad n \geq 3,$$

where $VF_a(2, 1) = a$ ($a \in$ ), $VF_a(2, 2) = 1 - a$. For example, we have $\{-3, 4, 1, 5,...\}$ for $a = -3$.

In [4], Daykin proved that only the Fibonacci sequence gives a unique Zeckendorf's representation for all positive integers. But the variant Fibonacci sequences allow for multiple Zeckendorf's representation for the same positive integers. In [9], was showed that there is no Zeckendorf's representation for $n = 5, 12$. A. Nalli and C. Ozylimaz studied the third order variant 3−Fibonacci sequence and get $GH$ code for $a = -2, -20$, and $1 \leq n \leq 100$ (see [12]).

***Table 1.*** The $k$-Fibonacci representation for $k \geq 4$

| $n$ | 4-Fibonacci code | 5-Fibonacci code | 6-Fibonacci code | 7-Fibonacci code | ... | $k$-Fibonacci code |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | ... | 1 |
| 2 | 01 | 01 | 01 | 01 | ... | 01 |
| 3 | 11 | 11 | 11 | 11 | ... | 11 |
| 4 | 001 | 001 | 001 | 001 | ... | 001 |
| 5 | 101 | 101 | 101 | 101 | ... | 101 |
| 6 | 011 | 011 | 011 | 011 | ... | 011 |
| 7 | 111 | 111 | 111 | 111 | ... | 111 |
| 8 | 0001 | 0001 | 0001 | 0001 | ... | 0001 |
| 9 | 1001 | 1001 | 1001 | 1001 | ... | 1001 |
| 10 | 0101 | 0101 | 0101 | 0101 | ... | 0101 |
| 11 | 1101 | 1101 | 1101 | 1101 | ... | 1101 |
| 12 | 0011 | 0011 | 0011 | 0011 | ... | 0011 |
| 13 | 1011 | 1011 | 1011 | 1011 | ... | 1011 |
| 14 | 0111 | 0111 | 0111 | 0111 | ... | 0111 |
| 15 | 00001 | 1111 | 1111 | 1111 | ... | 1111 |
| 16 | 10001 | 00001 | 00001 | 00001 | ... | 00001 |
| 17 | 01001 | 10001 | 10001 | 10001 | ... | 10001 |
| 18 | 11001 | 01001 | 01001 | 01001 | ... | 01001 |
| 19 | 00101 | 11001 | 11001 | 11001 | ... | 11001 |
| 20 | 10101 | 00011 | 001011 | 001011 | ... | 001011 |
| 21 | 01101 | 10011 | 10101 | 10101 | ... | 10101 |
| 22 | 11101 | 01101 | 01101 | 01101 | ... | 01101 |
| 23 | 00011 | 11101 | 11101 | 11101 | ... | 11101 |
| 24 | 10011 | 00011 | 00011 | 00011 | ... | 00011 |
| 25 | 01011 | 10011 | 10011 | 10011 | ... | 10011 |
| 26 | 11011 | 01011 | 01011 | 01011 | ... | 01011 |
| 27 | 00111 | 11011 | 11011 | 11011 | ... | 11011 |
| 28 | 10111 | 00111 | 00111 | 00111 | ... | 00111 |
| 29 | 000001 | 10111 | 10111 | 10111 | ... | 10111 |
| 30 | 100001 | 01111 | 01111 | 01111 | ... | 01111 |
| 31 | 010001 | 000001 | 11111 | 11111 | ... | 11111 |
| 32 | 110001 | 100001 | 000001 | 000001 | ... | 000001 |
| 33 | 001001 | 010001 | 100001 | 100001 | ... | 100001 |
| 34 | 101001 | 110001 | 010001 | 010001 | ... | 010001 |
| 35 | 011001 | 001001 | 110001 | 110001 | ... | 110001 |
| 36 | 111001 | 101001 | 001001 | 001001 | ... | 001001 |
| 37 | 000101 | 011001 | 101001 | 101001 | ... | 101001 |
| 38 | 100101 | 111001 | 011001 | 011001 | ... | 011001 |
| 39 | 010101 | 000101 | 111001 | 111001 | ... | 111001 |
| 40 | 110101 | 100101 | 000101 | 000101 | ... | 000101 |
| 41 | 001101 | 010101 | 100101 | 100101 | ... | 100101 |
| 42 | 101101 | 110101 | 010101 | 010101 | ... | 010101 |
| 43 | 011101 | 001101 | 110101 | 110101 | ... | 110101 |
| 44 | 000011 | 101101 | 001101 | 001101 | ... | 001101 |
| 45 | 100011 | 011101 | 101101 | 101101 | ... | 101101 |
| 46 | 010011 | 111101 | 011101 | 011101 | ... | 011101 |
| 47 | 110011 | 000011 | 111101 | 111101 | ... | 111101 |
| 48 | 001011 | 100011 | 000011 | 000011 | ... | 000011 |
| 49 | 101011 | 010011 | 100011 | 100011 | ... | 100011 |
| 50 | 011011 | 110011 | 010011 | 010011 | ... | 010011 |

***Table 2.*** The k-Fibonacci code for $k \geq 4$

| n | 4-Fibonacci code | 5-Fibonacci code | 6-Fibonacci code | 7-Fibonacci code | : : : | k-Fibonacci code |
|---|---|---|---|---|---|---|
| 1 | 1111 | 11111 | 111111 | 1111111 | : : : | $1^k$ |
| 2 | 01111 | 011111 | 0111111 | 01111111 | : : : | $01^k$ |
| 3 | 11111 | 111111 | 1111111 | 11111111 | : : : | $11^k$ |
| 4 | 001111 | 0011111 | 00111111 | 001111111 | : : : | $001^k$ |
| 5 | 101111 | 1011111 | 10111111 | 101111111 | : : : | $101^k$ |
| 6 | 011111 | 0111111 | 01111111 | 011111111 | : : : | $011^k$ |
| 7 | 111111 | 1111111 | 11111111 | 111111111 | : : : | $111^k$ |
| 8 | 0001111 | 00011111 | 000111111 | 0001111111 | : : : | $0001^k$ |
| 9 | 1001111 | 10011111 | 100111111 | 1001111111 | : : : | $1001^k$ |
| 10 | 0101111 | 01011111 | 010111111 | 0101111111 | : : : | $0101^k$ |
| 11 | 1101111 | 11011111 | 110111111 | 1101111111 | : : : | $1101^k$ |
| 12 | 0011111 | 00111111 | 001111111 | 0011111111 | : : : | $0011^k$ |
| 13 | 1011111 | 10111111 | 101111111 | 1011111111 | : : : | $1011^k$ |
| 14 | 0111111 | 01111111 | 011111111 | 0111111111 | : : : | $0111^k$ |
| 15 | 00001111 | 11111111 | 111111111 | 1111111111 | : : : | $1111^k$ |
| 16 | 10001111 | 000011111 | 0000111111 | 00001111111 | : : : | $00001^k$ |
| 17 | 01001111 | 100011111 | 1000111111 | 10001111111 | : : : | $10001^k$ |
| 18 | 11001111 | 010011111 | 0100111111 | 01001111111 | : : : | $01001^k$ |
| 19 | 00101111 | 110011111 | 1100111111 | 11001111111 | : : : | $11001^k$ |
| 20 | 10101111 | 000111111 | 0010111111 | 00101111111 | : : : | $001011^k$ |
| 21 | 011011111 | 100111111 | 1010111111 | 10101111111 | : : : | $10101^k$ |
| 22 | 111011111 | 0110111111 | 0110111111 | 01101111111 | : : : | $01101^k$ |
| 23 | 00011111 | 111011111 | 1110111111 | 11101111111 | : : : | $11101^k$ |
| 24 | 10011111 | 000111111 | 0001111111 | 00011111111 | : : : | $00011^k$ |
| 25 | 01011111 | 100111111 | 1001111111 | 10011111111 | : : : | $10011^k$ |
| 26 | 11011111 | 010111111 | 0101111111 | 01011111111 | : : : | $01011^k$ |
| 27 | 00111111 | 110111111 | 1101111111 | 11011111111 | : : : | $11011^k$ |
| 28 | 10111111 | 0011111111 | 0011111111 | 00111111111 | : : : | $00111^k$ |
| 29 | 000001111 | 101111111 | 1011111111 | 10111111111 | : : : | $10111^k$ |
| 30 | 100001111 | 011111111 | 0111111111 | 01111111111 | : : : | $01111^k$ |
| 31 | 010001111 | 0000011111 | 1111111111 | 11111111111 | : : : | $11111^k$ |
| 32 | 110001111 | 1000011111 | 00000111111 | 000001111111 | : : : | $000001^k$ |
| 33 | 001001111 | 0100011111 | 10000111111 | 100001111111 | : : : | $100001^k$ |
| 34 | 101001111 | 1100011111 | 01000111111 | 010001111111 | : : : | $010001^k$ |
| 35 | 011001111 | 0010011111 | 11000111111 | 110001111111 | : : : | $110001^k$ |
| 36 | 111001111 | 1010011111 | 00100111111 | 001001111111 | : : : | $001001^k$ |
| 37 | 000101111 | 0110011111 | 10100111111 | 101001111111 | : : : | $101001^k$ |
| 38 | 100101111 | 1110011111 | 01100111111 | 011001111111 | : : : | $011001^k$ |
| 39 | 010101111 | 0001011111 | 11100111111 | 111001111111 | : : : | $111001^k$ |
| 40 | 110101111 | 1001011111 | 00010111111 | 000101111111 | : : : | $000101^k$ |
| 41 | 001101111 | 0101011111 | 10010111111 | 100101111111 | : : : | $100101^k$ |
| 42 | 101101111 | 1101011111 | 01010111111 | 010101111111 | : : : | $010101^k$ |
| 43 | 011101111 | 0011011111 | 11010111111 | 110101111111 | : : : | $110101^k$ |
| 44 | 000011111 | 1011011111 | 00110111111 | 001101111111 | : : : | $001101^k$ |
| 45 | 100011111 | 0111011111 | 10110111111 | 101101111111 | : : : | $101101^k$ |
| 46 | 010011111 | 1111011111 | 01110111111 | 011101111111 | : : : | $011101^k$ |
| 47 | 110011111 | 0000111111 | 11110111111 | 111101111111 | : : : | $111101^k$ |
| 48 | 001011111 | 1000111111 | 00001111111 | 000011111111 | : : : | $000011^k$ |
| 49 | 101011111 | 0100111111 | 10001111111 | 100011111111 | : : : | $100011^k$ |
| 50 | 011011111 | 1100111111 | 01001111111 | 010011111111 | : : : | $010011^k$ |

Here, we define the $k$-order variant $k-$Fibonacci sequence.

**Definition 1.** *The k-order variant k−Fibonacci sequence, denoted by* $VF_a(k, n)$, *is defined as the GH sequence* $\{a, b, a + b, 2a + 2b, 4a + 4b, ...\}$ *where* $b = 1 - a$. $VF_a(k, n)$ *is as follows:*

$$VF_a(k, n) = VF_a(k, n-1) + VF_a(k, n-2) + \cdots + VF_a(k, n-k), \qquad n > k,$$

*where* $VF_a(k, 1) = a\ (a \in \quad), VF_a(k, 2) = 1 - a,\ VF_a(k, 3) = 1, \ldots,\ VF_a(k, k) = 2^{k-3}$.

**Example 1.** *(i) From Definition* 1, *for* $k = 4$ *and* $a = -2$, *we have*

$$VF_a(4, n) = VF_a(4, n-1) + VF_a(4, n-2) + VF_a(4, n-3) + VF_a(4, n-4), \qquad n \geq 4,$$

*where* $VF_a(k, 1) = a\ (a \in \quad),\ VF_a(k, 2) = 1 - a,\ VF_a(k, 3) = 1, \ldots,\ VF_a(k, k) = 2$. *So, we get* $\{-2, 3, 1, 2, 4, 10, 17, ...\}$ *for* $a = -2$.

*(i) Let* $a = -3$, $k = 4$, *we get* $VF_{-3}(4, n) = \{-3, 4, 1, 2, 4, ...\}$.

*(ii) If* $a = -7$ *and* $k = 5$, *then* $VF_{-7}(5, n) = \{-7, 8, 1, 2, 4, 8, ...\}$.

Now, we get a new universal source code by these $k-$order variant $k-$Fibonacci sequences. We obtain the $k-$order variations on the $k-$Fibonacci universal code, denoted by $GH_a(k, n)$, and are listed in ***Table 3*** for $k \geq 4$, $1 \leq n \leq 50$. The results are as follows:

($i$) There exists GH code for $1 \leq n \leq 50$, $k \geq 4$ and $a = -3$.

($ii$) There is not GH code for $a = -3$ and $k \geq 4$.

($iii$) There is not GH code for $a = -7$ and $k \geq 5$.

Consequently, from ***Table 3***, $GH_{-2}(k, n)$ ($k \geq 4$) exist for $1 \leq n \leq 50$. So, these sequences can be useful for cryptography.

## 3. Blocking method as use of Gopala-Hemachandra (GH) codes

Here, we consider the use of Gopala-Hemachandra ($GH$) codes and introduce a blocking method using of these codes.

Now, we explain the blocking algorithm.

(1) Construct a matrix $P$. We get the message in a square matrix $P$ of size $2m$. Note that between the two words put zero and the size of the message matrix is even. If the size of the message matrix is not even, then the remaining elements of the matrix is set zero.

(2) Block the matrix $P$. We divide the matrix $P$ into the submatrices $E^i (1 \leq i \leq m^2)$, of the size $2 \times 2$ from left to right. The number of the submatrices $E^i$ is denoted by $e$.

(3) We choose $k$, $n$ and $a$ such that $k$ is equal the number of alphabet in the message

$$n = \left\lfloor \frac{e}{2} \right\rfloor, \quad a = \begin{cases} -e, & \text{if } 2 \leq e \leq 10 \\ 4 & \text{if } e \geq 10. \end{cases}$$

***Table 3.*** The k order GH codes for $1 \leq n \leq 50$ and a = -2

| n | $GH_a(4; n)$ | $GH_a(5; n)$ | $GH_a(6; n)$ | $GH_a(7; n)$ | | : : : | $GH_a(k; n)$ |
|---|---|---|---|---|---|---|---|
| 1 | 001111 | 0011111 | 00111111 | 001111111 | | : : : | $001^k$ |
| 2 | 0001111 | 00011111 | 000111111 | 0001111111 | | : : : | $0001^k$ |
| 3 | 01111 | 011111 | 0111111 | 01111111 | | : : : | $01^k$ |
| 4 | 00001111 | 000011111 | 00000111111 | 00001111111 | | : : : | $00001^k$ |
| 5 | 00101111 | 001011111 | 0010111111 | 00101111111 | | : : : | $00101^k$ |
| 6 | 00011111 | 000111111 | 00010111111 | 000101111111 | | : : : | $000101^k$ |
| 7 | 01001111 | 010011111 | 0011111111 | 00111111111 | | : : : | $00111^k$ |
| 8 | 100001111 | 0000011111 | 00000111111 | 000001111111 | | : : : | $000001^k$ |
| 9 | 01011111 | 0010011111 | 00100111111 | 001001111111 | | : : : | $001001^k$ |
| 10 | 000001111 | 0001011111 | 00010111111 | 000101111111 | | : : : | $000101^k$ |
| 11 | 001001111 | 0100011111 | 01000111111 | 010001111111 | | : : : | $010001^k$ |
| 12 | 000101111 | 0000111111 | 00001111111 | 000011111111 | | : : : | $000011^k$ |
| 13 | 010001111 | 0010111111 | 00101111111 | 001011111111 | | : : : | $001011^k$ |
| 14 | 000011111 | 0001111111 | 100000111111 | 1000001111111 | | : : : | $1000001^k$ |
| 15 | 1000001111 | 0100111111 | 101000111111 | 1010001111111 | | : : : | $1010001^k$ |
| 16 | 1010001111 | 10000011111 | 000000111111 | 0000001111111 | | : : : | $0000001^k$ |
| 17 | 0000001111 | 10100011111 | 001000111111 | 0010001111111 | | : : : | $0010001^k$ |
| 18 | 0010001111 | 00000011111 | 000100111111 | 000100111111 | | : : : | $0001001^k$ |
| 19 | 0001001111 | 00100011111 | 010000111111 | 010000111111 | | : : : | $0100001^k$ |
| 20 | 0100001111 | 00010011111 | 000010111111 | 0000101111111 | | : : : | $0000101^k$ |
| 21 | 0000101111 | 01000011111 | 001010111111 | 0010101111111 | | : : : | $0010101^k$ |
| 22 | 0010101111 | 00001011111 | 000110111111 | 0001101111111 | | : : : | $0001101^k$ |
| 23 | 0001101111 | 00101011111 | 010010111111 | 0100101111111 | | : : : | $0100101^k$ |
| 24 | 0100101111 | 00011011111 | 000001111111 | 0000011111111 | | : : : | $0000011^k$ |
| 25 | 0110101111 | 01001011111 | 001001111111 | 0010011111111 | | : : : | $0010011^k$ |
| 26 | 0101101111 | 00000111111 | 000101111111 | 0001011111111 | | : : : | $0001011^k$ |
| 27 | 0000011111 | 00100111111 | 010001111111 | 0100011111111 | | : : : | $0100011^k$ |
| 28 | 0010011111 | 00010111111 | 000011111111 | 0000111111111 | | : : : | $0000111^k$ |
| 29 | 0001011111 | 010000111111 | 001011111111 | 0010111111111 | | : : : | $0010111^k$ |
| 30 | 0100011111 | 100000011111 | 000111111111 | 10000001111111 | | : : : | $10000001^k$ |
| 31 | 10000001111 | 101000011111 | 010011111111 | 10100001111111 | | : : : | $10100001^k$ |
| 32 | 10100001111 | 100100011111 | 1000000111111 | 00000001111111 | | : : : | $00000001^k$ |
| 33 | 00000001111 | 000000011111 | 1010000111111 | 00100001111111 | | : : : | $00100001^k$ |
| 34 | 00100001111 | 001000011111 | 0000000111111 | 00010001111111 | | : : : | $00010001^k$ |
| 35 | 00010001111 | 000100011111 | 0010000111111 | 01000001111111 | | : : : | $01000001^k$ |
| 36 | 01000001111 | 010000011111 | 0001000111111 | 00001001111111 | | : : : | $00001001^k$ |
| 37 | 00001001111 | 000010011111 | 0100000111111 | 00101001111111 | | : : : | $00101001^k$ |
| 38 | 00101001111 | 001010011111 | 0000100111111 | 00011001111111 | | : : : | $00011001^k$ |
| 39 | 00011001111 | 000110011111 | 0010100111111 | 01001001111111 | | : : : | $01001001^k$ |
| 40 | 01001001111 | 010010011111 | 0001100111111 | 00000101111111 | | : : : | $00000101^k$ |
| 41 | 01101001111 | 000001011111 | 0100100111111 | 00100101111111 | | : : : | $00100101^k$ |
| 42 | 01011001111 | 001001011111 | 0000010111111 | 00010101111111 | | : : : | $00010101^k$ |
| 43 | 00000101111 | 000101011111 | 0010010111111 | 01000101111111 | | : : : | $01000101^k$ |
| 44 | 0010010111 | 010001011111 | 0001010111111 | 00001101111111 | | : : : | $00001101^k$ |
| 45 | 00010101111 | 000011011111 | 0100010111111 | 00101101111111 | | : : : | $00101101^k$ |
| 46 | 01000101111 | 001011011111 | 0000110111111 | 00011101111111 | | : : : | $00011101^k$ |
| 47 | 00001101111 | 000111011111 | 0010110111111 | 01001101111111 | | : : : | $01001101^k$ |
| 48 | 00101101111 | 010011011111 | 0001110111111 | 00000011111111 | | : : : | $00000011^k$ |
| 49 | 00011101111 | 011011011111 | 0100110111111 | 00100011111111 | | : : : | $00100011^k$ |
| 50 | 00000011111 | 010111011111 | 0110110111111 | 00010011111111 | | : : : | $00010011^k$ |

***Table 4.*** English Alphabet

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| A | B | C | D | E | F | G | H | I |
| $F_{n+1}^k$ | $F_{n+2}^k$ | $F_{n+3}^k$ | $F_{n+4}^k$ | $F_{n+5}^k$ | $F_{n+6}^k$ | $F_{n+7}^k$ | $F_{n+8}^k$ | $F_{n+9}^k$ |
| J | K | L | M | N | O | P | Q | R |
| $F_{n+10}^k$ | $F_{n+11}^k$ | $F_{n+12}^k$ | $F_{n+13}^k$ | $F_{n+14}^k$ | $F_{n+15}^k$ | $F_{n+16}^k$ | $F_{n+17}^k$ | $F_{n+18}^k$ |
| S | T | U | V | W | X | Y | Z | 0 |
| $F_{n+19}^k$ | $F_{n+20}^k$ | $F_{n+21}^k$ | $F_{n+22}^k$ | $F_{n+23}^k$ | $F_{n+24}^k$ | $F_{n+25}^k$ | $F_{n+26}^k$ | $F_{n+27}^k$ |

(4) By considering $n$ and $k$, we get ***Table 4*** according to module 28.

From ***Table 2***, we get the $k$-Fibonacci codes of them and denoted by

$$E^i = \begin{bmatrix} e_{11}^i & e_{12}^i \\ e_{21}^i & e_{22}^i \end{bmatrix}, \quad 1 \le i \le m^2$$

(5) We get $T_i = e^i_{11} \oplus e^i_{22}$ ($mod$ 2), where $1 \le i \le m^2$ ($\oplus$ is the string summation of elements to the modulo 2 and defined by the elements of two strings are summed one by one from left to right to the mode 2. If the length of one of them is less than other one, we will add zero elements to shorter string for having strings with equal lengths. For example, $1011101 \oplus 01110 = 1100101$($mod$ 2).

(6) We calculate $G_i = GH_a(k, n) \oplus e^i_{12}$($mod$ 2).

(7) We get the coding matrix $C = \left[T_i, G_i, e_{21}^i, e_{22}^i\right]$ , $1 \le i \le m^2$ .

(8) End of algorithm.

Decoding blocking algorithm:

(1) We get $x_1^i = T_i \oplus e^i_{22}$($mod$ 2), and $x_1^i \to e^i_{11}$.

(2) We calculate $x_2^i = G_i \oplus GH_a(k, n)$($mod$ 2), and replacing $x_2^i$ by $e^i_{12}$.

(3) Construct $E^i$.

(4) We get $P$.

(5) End of algorithm.

**Example 2.** *Consider the message "SUMMER IS HoT". According the blocking algorithm, we have the matrix P as follows:*

$$P = \begin{bmatrix} S & U & M & M \\ E & R & \circ & I \\ S & \circ & H & O \\ T & \circ & \circ & \circ \end{bmatrix}.$$

We divide $P$ . Therefore, we have

$$E^1=\begin{bmatrix} S & U \\ E & R \end{bmatrix},\ E^2=\begin{bmatrix} M & N \\ \circ & I \end{bmatrix},\ E^3=\begin{bmatrix} S & \circ \\ T & \circ \end{bmatrix},E^4=\begin{bmatrix} H & O \\ \circ & \circ \end{bmatrix},$$

and $k=11, n=2, a=-2$. Also by ***Table 2 and 4***, we obtain

$$E^1 \to \begin{bmatrix} F_{21}^{11} & F_{23}^{11} \\ F_7^{11} & F_{20}^{11} \end{bmatrix} \to \begin{bmatrix} 10101^{11} & 11101^{11} \\ 111^{11} & 001011^{11} \end{bmatrix},$$

$$E^2 \to \begin{bmatrix} F_{15}^{11} & F_{15}^{11} \\ F_{29}^{11} & F_{211}^{11} \end{bmatrix} \to \begin{bmatrix} 1111^{11} & 11111^{11} \\ 10111^{11} & 1101^{11} \end{bmatrix},$$

$$E^3 \to \begin{bmatrix} F_{21}^{11} & F_{29}^{11} \\ F_{22}^{11} & F_{25}^{11} \end{bmatrix} \to \begin{bmatrix} 10101^{11} & 10111^{11} \\ 01101^{11} & 10111^{11} \end{bmatrix},$$

$$E^4 \to \begin{bmatrix} F_{21}^{11} & F_{17}^{11} \\ F_{29}^{11} & F_{29}^{11} \end{bmatrix} \to \begin{bmatrix} 0101^{11} & 10001^{11} \\ 10111^{11} & 10111^{11} \end{bmatrix}.$$

Using the relation $T_1 = e_{11}^i \oplus e_{22}^i \pmod 2$, we have

$$T_1 = 10101^{11} \oplus 001011^{11} = 10^{13}1 (mod\ 2),$$

$$T_2 = 1111^{11} \oplus 1101^{11} = 0010^{11} (mod\ 2),$$

$$T_3 = 10101^{11} \oplus 10111^{11} = 00010^{11} (mod\ 2),$$

$$T_4 = 0101^{11} \oplus 10111^{11} = 1110^{10}1 (mod\ 2).$$

On the other hands, we get

$$G_1 = 11110^{10}1,\ G_2 = 1110^{10},\ G_3 = 1010^{11}1,\ G_4 = 10010^{10}1.$$

Therefore, we have

$$C=\begin{bmatrix} 10^{13}1 & 11110^{10} & 111^{11} & 001011^{11} \\ 0010^{11} & 1110^{11} & 10111^{11} & 1101^{11} \\ 00010^{11} & 10101^{11} & 01101^{11} & 10111^{11} \\ 10101^{11} & 10010^{11} & 10111^{11} & 10111^{11} \end{bmatrix}.$$

Now, we obtain decoding blocking algorithm. First, by the relation $x_1^i = T_i \oplus e^i_{22}$ (mod 2), we have

$$x_1^1 = 10101^{11},\ x_1^2 = 1111^{11},\ x_1^3 = 10101^{11},\ x_1^4 = 0101^{11}.$$

Similarly, by the relation $x_2^i = G_i \oplus GH_a(k, n) (mod\ 2)$, we get

$$x_2^1 = 11101^{11},\ x_2^2 = 1111^{11},\ x_2^3 = 10111^{11},\ x_2^4 = 10001^{11}$$

Therefore, according to above information and ***Table 2***, we obtain $E^i$, $1 \le i \le 4$.

$$E^1 \to \begin{bmatrix} 10101^{11} & 11101^{11} \\ 111^{11} & 001011^{11} \end{bmatrix} \to \begin{bmatrix} F_{21}^{11} & F_{23}^{11} \\ F_7^{11} & F_{20}^{11} \end{bmatrix},$$

$$E^2 \to \begin{bmatrix} 1111^{11} & 11111^{11} \\ 10111^{11} & 1101^{11} \end{bmatrix} \to \begin{bmatrix} F_{15}^{11} & F_{15}^{11} \\ F_{29}^{11} & F_{211}^{11} \end{bmatrix}$$

$$E^3 \to \begin{bmatrix} 10101^{11} & 10111^{11} \\ 01101^{11} & 10111^{11} \end{bmatrix} \to \begin{bmatrix} F_{21}^{11} & F_{29}^{11} \\ F_{22}^{11} & F_{25}^{11} \end{bmatrix}$$

$$E^4 \to \begin{bmatrix} 0101^{11} & 10001^{11} \\ 10111^{11} & 10111^{11} \end{bmatrix} \to \begin{bmatrix} F_{21}^{11} & F_{17}^{11} \\ F_{29}^{11} & F_{29}^{11} \end{bmatrix}.$$

By ***Table 1***, we have

$$E^1 = \begin{bmatrix} S & U \\ E & R \end{bmatrix}, \quad E^2 = \begin{bmatrix} M & N \\ \circ & I \end{bmatrix}, E^3 = \begin{bmatrix} S & \circ \\ T & \circ \end{bmatrix}, E^4 = \begin{bmatrix} H & O \\ \circ & \circ \end{bmatrix},$$

Consequently

$$P = \begin{bmatrix} S & U & M & M \\ E & R & \circ & I \\ S & \circ & H & O \\ T & \circ & \circ & \circ \end{bmatrix}.$$

So, we get the massage *"SUMMER IS HoT"*.

## 4. Conclusion

In [2, 12], for $k$ = 2, 3, $GH_a(k, n)$ were studied for $n$ = 1, 2 ,..., 100 and $-20 \leq a \leq -2$, respectively. In the current paper, we obtained the $k-$order variations on the $k-$Fibonacci ($k \geq 4$) universal code for $1 \leq n \leq 50$ and $a = -2$ and got that the GH-code for exists for it. We can use the results in the cryptographic.

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