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Investigate new soliton type solutions to a nonlinear partial Schrödinger differential equations with the new auxiliary equation approach and its modification

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ABSTRACT

The central and fundamental subject of present research is to investigate the effective method of the new auxiliary equation approach for generalized Schrödinger's equation. Finding solutions to this equation has always been of great importance due to its applications in quantum mechanics and light propagation in nonlinear optical fibers. The results show that the proposed approaches are quite effective and efficient in obtaining exact solutions for nonlinear partial differential equations. Due to the complexity of these equations' calculations, Wolfram Mathematica software has been used to validate the results of the proposed techniques.

1. Introduction

We must accept the tremendous impact of some ideas proposed and confirmed in various subjects, including mathematics. For example, we see that directing the topics of mathematical physics and chemical reactions including the heat conduction, electro-chemistry, superfluidity [1], neural networks [2], plasma diagnostics, pulses of sound reflections [3], physical electronics, nuclear physical, optics and astrophysics [4], to nonlinear partial differential equations (NPDEs) can solve the ambiguous points of some problems.

PDE is one of those approach which deal in science, engineering and other fields. Finding exact soliton solutions of PDEs is very attractive and innovative for these equations with innovative analytical techniques. It is natural that based on this point of view, various analytical techniques,

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such as traveling wave solutions of the fractional regularized long-wave in [5], Khater method for solving nonlinear PDEs in [6], new auxiliary equation technique (AET) for nonlinear Sasa-Satsuma equation to demonstrate a new optical forms of solitary traveling wave solutions in [7], the AET to demonstrate a large family of optical solutions for the nonlinear ordinary Kundu-Eckhaus DEs in [8], the AET to gain the exact solitary wave solutions for nonlinear underground water model in [9], homotopy perturbation transform method in [10,11], F-expansion method [12,13], generalized exponential rational function method [14], kudryashov method [15] have been developed by researchers for solving PDEs and ODEs. Kumar et al. investigated some novel forms of explicit complex hyperbolic and complex trigonometric function solutions based on the extended sinh-Gordon equation expansion method in the following system of coupled nonlinear Schrödinger-KdV equations, with a real coupling constant ν

$$\begin{cases} i\,\frac{\partial(B(x,t))}{\partial t}+\frac{\partial^2(B(x,t))}{\partial x^2}-\nu\,B(x,t)\,S(x,t)+|B(x,t)|^2\,B(x,t)=0, \\ i\,\frac{\partial(S(x,t))}{\partial t}+\frac{\partial^3(S(x,t))}{\partial x^3}+\frac{\partial(S(x,t))}{\partial x}\,S(x,t)-\frac{\nu}{2}\frac{\partial(\,(B(x,t))\hat{}2\,)}{\partial x}=0, \end{cases}$$

in various forms such as dark, bright, combined dark-bright, singular, combined singular optical, periodic wave, dipole soliton and other related solutions, where $B(x,t)$ is a complex function that stands for the short wave profile and $S(x,t)$ is a real function which represents long wave profile [16]. Sadaf et al. examined the dynamical wave structures provided by the exact wave solutions of the Camassa-Holm nonlinear Schrödinger equation $(3+1)-D$ Boussinesq equation by implementation of $\frac{G'}{G}$-expansion method to evaluate the spectrum of traveling wave solutions of the considered equations including solitons and and solitary waves [17].

In this manuscript, Consider the generalized Schrödinger's equation as:

$$i\left(\frac{\partial(B(x,t))}{\partial t}+\frac{\partial^3(B(x,t))}{\partial x^3}+\nu\,\frac{\partial(B(x,t))}{\partial x}\,|B(x,t)|^2\right)+|B(x,t)|^2\,(a\,B(x,t))+i\,k\,\frac{\partial(|B(x,t)|^2)}{\partial x}\,B(x,t)=0, \quad (1)$$

where ν, k are parameters related to nonlinear dispersion effects and a is a parameter for the nonlinear coefficients.

With this approach, Sect.2, presents the Implementation of techniques. In Sect.3, the calculation of solitons for Schrödinger differential equations is presented using the new auxiliary equation approach and its modification. The simulations and discussion are presented in Section ‘‘Analysis of graphical representation’’. Eventually, Sect.5 serves as the conclusion for the paper.

2. Summarize of Techniques

As the first step, let us consider the following wave transformations are utilized,

$$B(x,t)=B(\xi)\times exp\,(i\phi),\ \ \xi=\mu\,x+\omega\,t,\ \ \phi=\tau\,x+\lambda\,t, \quad (2)$$

including unknown parameters μ, ω, τ and λ. Using hypothesis (2) in ***Eq. (1)***, ignoring the imaginary part, and applying a few other manipulations, we have:

$$12\,\mu^3 k^2\, B''(\xi) + (4\,\omega\, k^2 - 3\,a^2\,\mu)\,B(\xi) + \mu\, k^2\,(\nu + 2\,k\,)\,B^3(\xi) = 0, \quad (3)$$

We now, shall focus on basic ideas of both Technique. The auxiliary equation method has been used solving the fractional burgers system regularized long-wave [18].

2.1. The new AET (model I)

The fundamental steps of the new AET for solving NPDEs,

$$\Omega\,(B(x,t), B_x(x,t), B_t(x,t), B_{xx}(x,t), \dots) = 0 \quad (4)$$

is provided below, with taking the following new variables in a few steps.

Step 1.

$$B(x,t) = B(\xi),\ where \quad \xi = \mu\, x + \omega\, t, \quad (5)$$

where μ and ω are unknown parameters that need to be determined. ***Eq. (4)*** is transferred to the following NODE form,

$$\Omega\,(B(\xi), \mu\, B_x(\xi), -\omega\, B_t(\xi), \mu^2\, B_{xx}(\xi), \omega^2 B_{tt}(\xi), \dots) = 0, \quad (6)$$

Step 2. Let the solution of ***Eq (3)*** take the following symbolic structure form,

$$B(\xi) = e_0 + \sum_{j=1}^{\upsilon} e_j\left(\Upsilon^{\beta(\xi)}\right)^j + \sum_{l=1}^{\upsilon} f_l\left(\Upsilon^{-\beta(\xi)}\right)^l, \quad (7)$$

where $\upsilon$is calculated by using the principle of homogeneous rules and $e_0, e_j, f_l\ (1 \le j, l \le \upsilon)$ are constants to be determined later, and $\beta(\xi)$satisfies a first-order NODE,

$$\beta'(\xi) = \frac{1}{ln(\beta)}\left(g\,\Upsilon^{-\beta(\xi)} + h + c\,\Upsilon^{\beta(\xi)}\right), \quad (8)$$

The parameters g, h and c are real the constants and ***Eq. (8)*** has the following list of possible solutions. Assuming

Case 1: $c \neq 0$ and $-h^2 + 4cg < 0$,

$$\Upsilon^{\beta(\xi)} = -\frac{h}{2c} - \frac{\xi + \xi_0}{2c}\sqrt{-4cg + h^2}\, tanh\left(\frac{1}{2}\sqrt{-4cg + h^2}\right), \quad (9)$$

Case 2: $c \neq 0$ and $-h\text{^}2 + 4cg > 0$,

$$\Upsilon^{\beta(\xi)} = -\frac{h}{2c} + \frac{\xi + \xi_0}{2c}\sqrt{4cg - h^2}\ tan\left(\frac{1}{2}\sqrt{4cg - h^2}\right), \quad (10)$$

Case 3: $c \neq 0, h = 0, c > 0$ and $g < 0$,

$$\Upsilon^{\beta(\xi)} = -\sqrt{-\frac{g}{c}}\ tanh\left(\sqrt{-cg}\ (\xi + \xi_0)\right), \quad (11)$$

Case 4: $c \neq 0$ and $4cg > 0$,

$$\Upsilon^{\beta(\xi)} = \sqrt{\frac{g}{c}}\ tan\left(\sqrt{cg}\ (\xi + \xi_0)\right), \quad (12)$$

Case 5: $g = c$ and $h = 0$,

$$\Upsilon^{\beta(\xi)} = tan(g\ (\xi + \xi_0)), \quad (13)$$

Case 6: $c = 0$,

$$\Upsilon^{\beta(\xi)} = \frac{e^{h\,(\xi + \xi_0)}}{h} - \frac{g}{h}, \quad (14)$$

Case 7: $g, c = 0$,

$$\Upsilon^{\beta(\xi)} = e^{h\,(\xi + \xi_0)}, \quad (15)$$

Case 8: $g = 0$ and $c = h$,

$$\Upsilon^{\beta(\xi)} = \frac{e^{h\,(\xi + \xi_0)}}{-1 + e^{h\,(\xi + \xi_0)}}, \quad (16)$$

Case 9: $g = h$ and $c = 0$,

$$\Upsilon^{\beta(\xi)} = -1 + \frac{e^{h\,(\xi + \xi_0)}}{h}, \quad (17)$$

Step 3. Inserting ***Eq. (7)*** along with its associated derivatives into ***Eq. (6)*** and collecting all powers of $\Upsilon^{\beta(\xi)}$ a system of polynomial equations is obtained.

Step 4. Finally, solve the obtained system of algebraic equations from the previous step, and insert the solutions into ***Eq. (7)*** to determine the values of the unknown parameters. This causes one to obtain the analytical solutions of Schrödinger equation.

2.2. An improvement of the AET (model II)

The fundamental steps of the new AET for solving NPDE,

$$\Omega\,(B(x,t), B_x(x,t), B_t(x,t), B_{xx}(x,t), \ldots)\ =\ 0 \quad \textbf{(18)}$$

is provided below.

Step 1. Taking the following new variables,

$$B(x,t) = B(\xi),\ where\ \ \xi = \mu\, x + \omega\, t, \quad \textbf{(19)}$$

where μ and ω are unknown parameters that need to be determined. ***Eq. (18)*** is transferred to the following NODE form,

$$\Omega\,(\,B(\xi), \mu\, B_x(\xi), -\omega\, B_t(\xi), \mu^2\, B_{xx}(\xi), \omega^2 B_{tt}(\xi), \ldots)\ =\ 0, \quad \textbf{(20)}$$

Step 2. Let the solution of ***Eq. (20)*** take the following symbolic structure form,

$$B(\xi) = e_0 + \sum_{j=1}^{\upsilon} e_j \left(\Upsilon^{\beta(\xi)}\right)^j + \sum_{l=1}^{\upsilon} f_l \left(\Upsilon^{-\beta(\xi)}\right)^l + \sum_{d=1}^{\upsilon} d_k \left(\frac{\Upsilon^{\beta(\xi)'}}{\Upsilon^{\beta(\xi)}}\right)^l, \quad \textbf{(21)}$$

where υ is calculated by using the principle of homogeneous rules and $e_0, e_j, f_l,\ d_k (1 \le\ j, l\ , k \le \upsilon)$ are constants to be determined later, and $\beta(\xi)$ satisfies a first-order NODE,

$$\beta'(\xi) = \frac{1}{ln(\beta)}\left(g\,\Upsilon^{-\beta(\xi)} + h + c\,\Upsilon^{\beta(\xi)}\right), \quad \textbf{(22)}$$

The parameters g, h and c are real the constants and ***Eq. (22)*** lists the possible solutions Eqs. (9)-(17).

Step 3. Substituting ***Eq. (21)*** along with its associated derivatives into ***Eq. (20)*** and collecting all powers of $\Upsilon^{\beta(\xi)}$ a system of polynomial equations is obtained.

Step 4. Finally, solve the obtained system of algebraic equations from the previous step, and insert the solutions into ***Eq. (21)*** to determine the values of the unknown parameters. This causes one to obtain the analytical solutions of Schrödinger equation.

3. Procedure of Solution

In this subsection, the application of AET in two models I and II, to ***Eq. (1)*** is presented. Then, balancing $B^3(\xi)$ and $B''(\xi)$ in ***Eq. (3)*** yields $\upsilon = 1$.

3.1. Implementation of model I

After pursuing the steps for the model I, outlined in the section 3, we achieve the following results. Inserting $\upsilon = 1$ into ***Eq. (7)***, it reduces to

$$B(\xi) = e_0 + e_1\,\Upsilon^{\beta(\xi)} + f_1 \Upsilon^{-\beta(\xi)}, \quad \textbf{(23)}$$

Using ***Eq. (23)*** in ***Eq. (24)*** and after completing the steps of the technique I, the following families of solutions of ***Eq. (1)*** are obtained,

$$e_0 = -\frac{\sqrt{6}h\mu}{\sqrt{-2k-\nu}}\,, e_1 = -\frac{\sqrt{6}c\mu}{\sqrt{-2k-\nu}}\,, f_1 = 0\,, \omega = \frac{3}{4}\mu\left(\frac{a^2}{k^2}+2(-4cg+h^2)\mu^2\right). \tag{24}$$

Hence, we arrive at the analytical solutions to ***Eq. (1)*** as the below mentioned sets:

$$B_1(x,t) = \frac{\sqrt{6}}{\sqrt{-2k-\nu}}\, exp(i\phi)\sqrt{-4cg + h^2}\,\mu\, tanh\left(\frac{1}{2}\sqrt{-4cg + h^2}\,(\xi + \xi_0)\right),$$
$$B_2(x,t) = -\frac{1}{\sqrt{-2k-\nu}}\, exp(i\phi)\sqrt{24cg-6h^2}\,\mu\, tan\left(\frac{1}{2}\sqrt{4cg - h^2}\,(\xi + \xi_0)\right),$$
$$B_3(x,t) = \frac{\sqrt{6}}{\sqrt{-2k-\nu}}\, exp(i\phi)\,\mu\left(h-2c\sqrt{-\frac{g}{c}}\, tanh\left(\sqrt{-cg}(\xi + \xi_0)\right)\right),$$
$$B_4(x,t) = -\frac{\sqrt{6}}{\sqrt{-2k-\nu}}\, exp(i\phi)\,\mu\left(h+2c\sqrt{cg}\, tan\left(\sqrt{cg}(\xi + \xi_0)\right)\right),$$
$$B_5(x,t) = -\frac{\sqrt{6}}{\sqrt{-2k-\nu}}\, exp(i\phi)\,\mu\left(h+2c\, tan\left(g(\xi + \xi_0)\right)\right),$$
$$B_6(x,t) = -\frac{\sqrt{6}}{h\sqrt{-2k-\nu}}\, exp(i\phi)\,\mu\left(2c\left(exp\left(h(\xi + \xi_0)\right)-g\right)+h^2\right)\mu,$$
$$B_7(x,t) = -\frac{\sqrt{6}}{\sqrt{-2k-\nu}}\, exp(i\phi)\,\mu\left(2c\, exp\left(h(\xi + \xi_0)\right)+h\right)\mu,$$
$$B_8(x,t) = -\frac{\sqrt{6}\, exp(i\phi)\,\mu\left(2c\, exp\left(h(\xi+\xi_0)\right)+h-exp\left(h(\xi+\xi_0)\right)h\right)\mu}{\left(-1+exp\left(h(\xi+\xi_0)\right)\right)\sqrt{-2k-\nu}},$$
$$B_9(x,t) = -\frac{\sqrt{6}}{h\sqrt{-2k-\nu}}\, exp(i\phi)\,\mu\left(2c\left(exp\left(h(\xi + \xi_0)\right)-h\right)+h^2\right)\mu,$$

3.2. Implementation of model II

After pursuing the steps for Technique II, outlined in the Section 2.2, we achieve the following results. Inserting $\upsilon = 1$ into ***Eq. (21)***, it reduces to,

$$B(\xi) = e_0 + e_1\,\Upsilon^{\beta(\xi)} + f_1\Upsilon^{-\beta(\xi)} + d_1\,\frac{{\Upsilon^{\beta(\xi)}}'}{\Upsilon^{\beta(\xi)}}, \tag{25}$$

Using ***Eq. (25)*** in ***Eq. (3)*** and after completing the steps of the technique I, the following Families of solutions of ***Eq. (1)*** are obtained,

$$e_0 = h\left(-d_1 + \frac{\sqrt{6}h\mu^2}{\sqrt{-h^2\mu^2(2k+\nu)}}\right),\ \ e_1 = -c\, d_1 + \frac{\sqrt{6}h\mu^2}{\sqrt{-h^2\mu^2(2k+\nu)}},$$
$$f_1 = -g\, d_1,\ \ \omega = \frac{3}{4}\,\mu\left(\frac{a^2}{k^2}+2(-4cg+h^2)\mu^2\right). \tag{26}$$

Hence, we arrive at the analytical solutions to ***Eq. (1)*** as the below mentioned sets:

$$B_1(x,t) = -\frac{\sqrt{6}exp(i\phi)h\sqrt{-4cg+h^2}\,\mu^2\, tanh\left(\frac{1}{2}\sqrt{-4cg+h^2}\,(\xi+\xi_0)\right)}{\sqrt{-h^2\mu^2(2k+\nu)}},$$
$$B_2(x,t) = \frac{exp(i\phi)h\sqrt{24cg-6h^2}\,\mu^2\, tan\left(\frac{1}{2}\sqrt{4cg-h^2}\,(\xi+\xi_0)\right)}{\sqrt{-h^2\mu^2(2k+\nu)}},$$
$$B_3(x,t) = \frac{\sqrt{6}exp(i\phi)\,h\mu^2\left(h-2c\sqrt{-\frac{g}{c}}\, tanh\left(\sqrt{-cg}(\xi+\xi_0)\right)\right)}{\sqrt{-h^2\mu^2(2k+\nu)}},$$

$$B_4(x,t) = \frac{\sqrt{6}exp(i\phi)\, h\mu^2 \left(h+2c\sqrt{cg}\, tan\left(\sqrt{cg}(\xi+\xi_0)\right)\right)}{\sqrt{-h^2\mu^2(2k+\nu)}},$$

$$B_5(x,t) = \frac{\sqrt{6}exp(i\phi)\, h\mu^2 \left(h+2c\, tan\left(g(\xi+\xi_0)\right)\right)}{\sqrt{-h^2\mu^2(2k+\nu)}},$$

$$B_6(x,t) = \frac{\sqrt{6exp(i\phi)\ \left(2c\left(exp\left(h(\xi+\xi_0)\right)-g\right)+h^2\right)\mu^2}}{\sqrt{-h^2\mu^2(2k+\nu)}},$$

$$B_7(x,t) = \frac{\sqrt{6}\, exp(i\phi)\, h\left(2c\, exp\left(h(\xi+\xi_0)\right)+h\right)\mu}{\sqrt{-h^2\mu^2(2k+\nu)}},$$

$$B_8(x,t) = -\frac{\sqrt{6}\, exp(i\phi)\, h\left(2c\, exp\left(h(\xi+\xi_0)\right)+h-exp\left(h(\xi+\xi_0)\right)h\right)\mu^2}{\left(-1+exp\left(h(\xi+\xi_0)\right)\right)\sqrt{-h^2\mu^2(2k+\nu)}},$$

$$B_9(x,t) = \frac{\sqrt{6}exp(i\phi)\ \left(2c\left(exp\left(h(\xi+\xi_0)\right)-h\right)+h^2\right)\mu^2}{\sqrt{-h^2\mu^2(2k+\nu)}},$$

4. Analysis of Graphical Representation

This section discusses the construction of soliton solutions for a model of the Schr“dinger equation using AET. Soliton solutions are novel and will have significant value in quantum mechanics and optical fiber propagation. Furthermore, these solitons have been plotted in the form of 3D and contour plots over a specific time interval by choosing appropriate values of the desired parameters. These diagrams will help in understanding and describing the behavior of the Schrödinger equation. Contour plots can be useful when understanding three-dimensional behavior is difficult, providing information about wave behavior or the overlap of different waves .

In ***Figure 1 (1a-1f)***, the dynamic behavior of $B_1(x,t)$ is plotted for $c=1,\ g=-2.5,\ \mu=1, \omega=\frac{5}{3},\ \xi_0=-1,\ h=2,\ \lambda=1, \tau=1,\ \nu=1$ and $k=8$. Which represents the hybrid periodic bright soliton solution in both the real and imaginary states, as well as the Caspon soliton solution in the absolute value state.

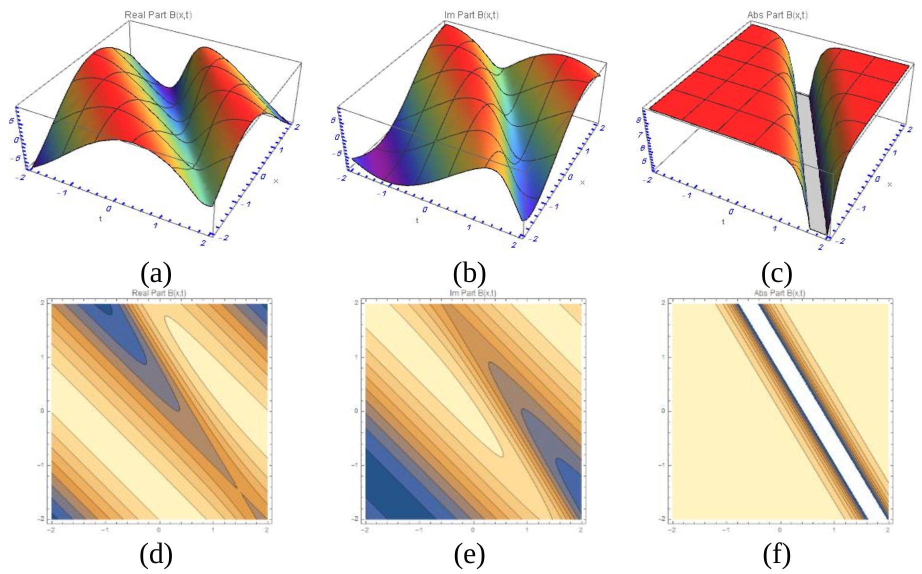

Figure 1. 3D and contour graphs soliton solutions of Eq. $B_1(x,t)$.

In ***Figure 2 (2a-2f)***, the dynamic behavior of $B_2(x,t)$ is plotted for $c=3,\ g=1,\ \mu=1,\ \omega=0.3,\ \xi_0=0,\ h=\frac{2}{3},\ \lambda=1.1,\ \tau=0.1,\ \nu=2.4$ and $k=-1$. Which represents the envelope bright solitons solution in both the real and imaginary states, as well as the Gap soliton solution in the absolute value state.

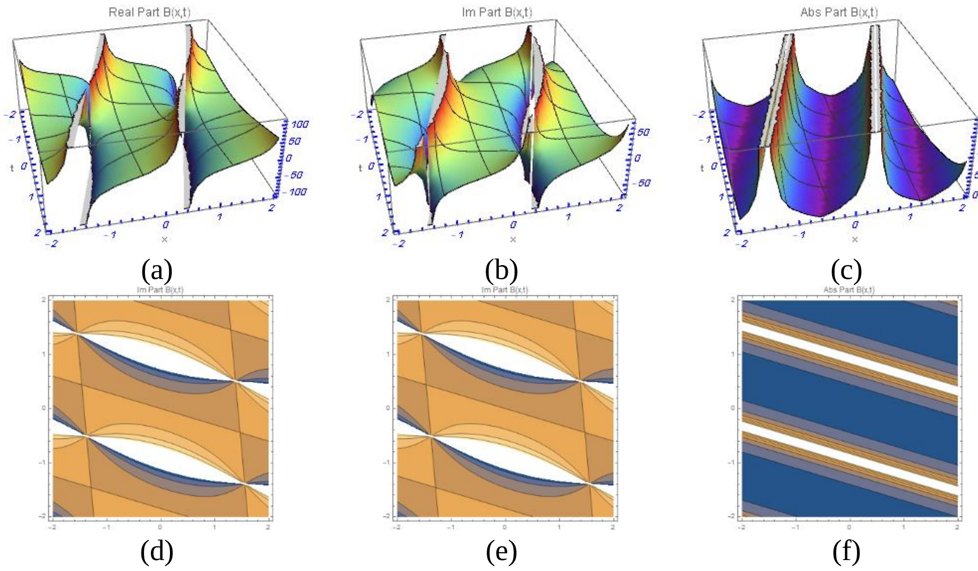

Figure 2. 3D and contour graphs soliton solutions of Eq. $B_2(x,t)$.

In ***Figure 3 (3a-3f)***, the dynamic behavior of $B_3(x,t)$ is plotted for $c=-4,\ g=2,\ \mu=1,\ \omega=3,\ \xi_0=1,\ h=4,\ \lambda=\frac{5}{7},\ \tau=5.4,\ \nu=0.1$ and $k=3$. Which represents the Gap bright soliton solution in both the real and imaginary states, as well as the Kink soliton solution in the absolute value state.

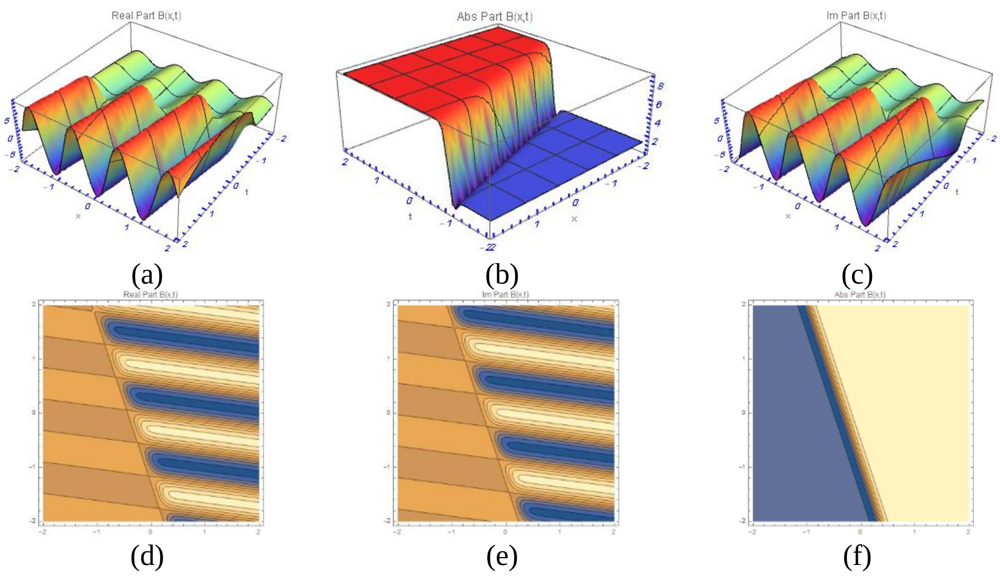

Figure 3. 3D and contour graphs soliton solutions of Eq. $B_3(x,t)$.

In ***Figure 4 (4a-4f)***, the dynamic behavior of $B_4(x,t)$ is plotted for $c=0.2, g=4,\ \mu=0,\ \omega=2,\ \xi_0=1.1,\ h=2,\ \lambda=0.2,\ \tau=0.3,\ \nu=-1$ and $k=-2$. Which represents the hybrid periodic bright soliton solution in both the real and imaginary states, as well as the Gap soliton solution in the absolute value state.

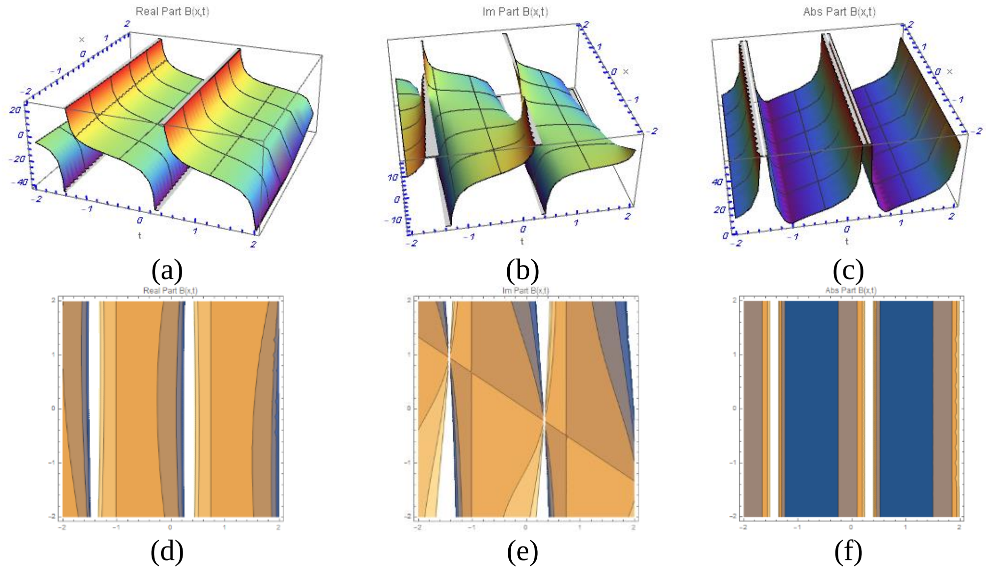

Figure 4. 3D and contour graphs soliton solutions of Eq. $B_4(x,t)$.

In ***Figure 5 (5a-5f)***, the dynamic behavior of $B_5(x,t)$ is plotted for $c = 1.5, g = 0.1, \mu = -1,\ \omega = 2,\ \xi_0 = -1,\ h = 1,\ \lambda = 3,\ \tau = 0.2,\ \nu = 5$ and $k = -1$. Which represents the hybrid periodic bright soliton solution in both the real and imaginary states, as well as the Breather soliton solution in the absolute value state.

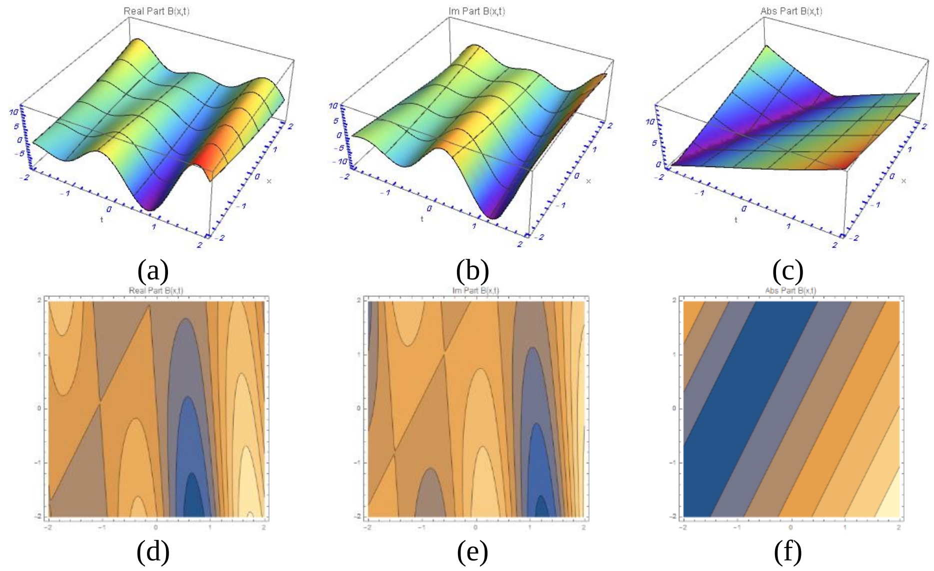

Figure 5. 3D and contour graphs soliton solutions of Eq. $B_5(x,t)$.

In ***Figure 6 (6a-6f)***, the dynamic behavior of $B_6(x,t)$ is plotted for $c = 1.5, g = 0.1, \mu = -1, \omega = 4,\ \xi_0 = -1,\ h = 5, \lambda = 1, \tau = 0.2, \nu = 5$ and $k = -1$. Which represents the Kink soliton solution.

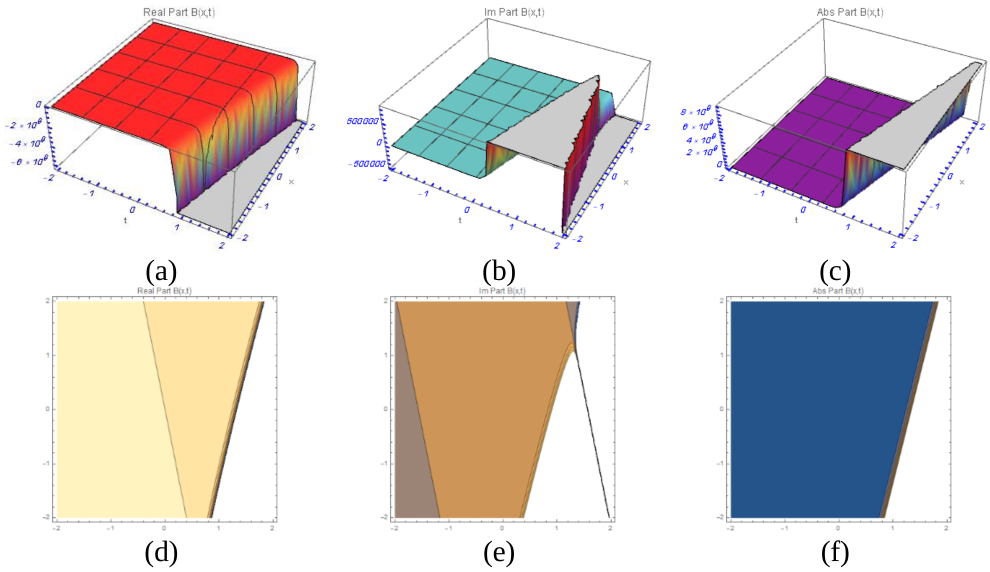

Figure 6. 3D and contour graphs soliton solutions of Eq. $B_6(x,t)$.

In ***Figure 7(7a-7f)***, the dynamic behavior of $B_7(x,t)$ is plotted for $c = 8, g = \frac{1}{3}, \mu = -2, \omega = 4,\ \xi_0 = 1,\ h = \frac{7}{3}, \lambda = 1.6, \tau = 2.1, \nu = 2$ and $k = -5$. Which represents the Peakons soliton solution in both the real and imaginary states, as well as the Kink soliton solution in the absolute value state.

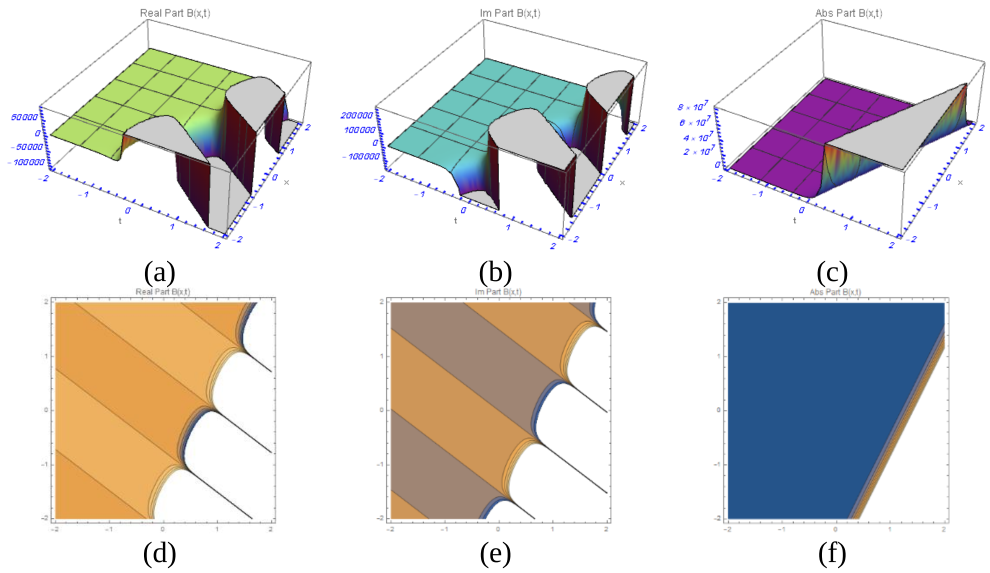

Figure 7. 3D and contour graphs soliton solutions of Eq. $B_7(x,t)$.

In ***Figure 8 (8a-8f)***, the dynamic behavior of $B_8(x,t)$ is plotted for $c = 0.8, g = 1, \mu = 1, \omega = 2,\ \xi_0 = -2,\ h = 1,\ \lambda = -4, \tau = 2, \nu = 3$ and $k = -7$. Which represents the hybrid periodic bright soliton solution in both the real and imaginary states, as well as the Peakons soliton solution in the absolute value state.

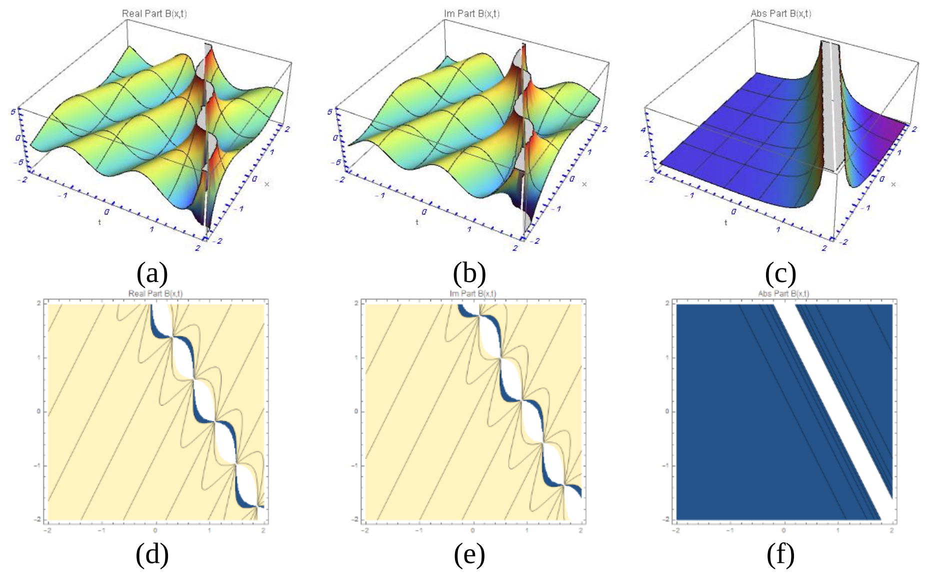

Figure 8. 3D and contour graphs soliton solutions of Eq. $B_8(x,t)$.

In ***Figure 9 (9a-9f)***, the dynamic behavior of $B_9(x,t)$ is plotted for $c = 4, g = 2, \mu = 1.4, \omega = 3, \xi_0 = 0, h = 3, \lambda = 2, \tau = 1, \nu = 1$ and $k = -4$. Which represents the peakons soliton solution in both the real and imaginary states, as well as the Caspon soliton solution in the absolute value state.

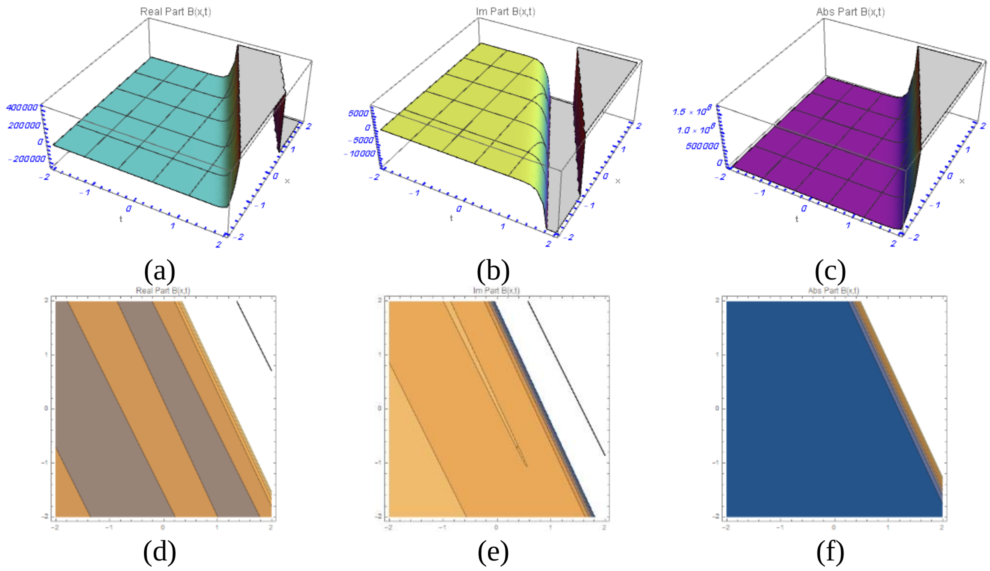

Figure 9. 3D and contour graphs soliton solutions of Eq. $B_9(x,t)$.

5. Conclusion

In this paper, traveling wave solutions such as hyperbolic, trigonometric, and exponential functions for the nonlinear Schrödinger equation, which has a great emphasis in quantum mechanics and optical fiber propagation, are obtained analytically using the method of AET. These findings provide a better understanding of the nonlinear model by graphically representing the soliton solutions. We believe that these solutions can have significant applications in the field of quantum engineering.

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