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Numerical solution of Fredholm integral equations by the least squares method using QR decomposition

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Abstract. This paper presents a numerical method for solving the second-kind Fredholm integral equations using the least squares approach via QR decomposition. The proposed technique employs Gaussian quadrature for numerical integration and efficiently applies QR decomposition for minimization. This strategy not only simplifies the implementation but also significantly enhances computational performance. A convergence analysis is provided, and several numerical examples demonstrate the accuracy and effectiveness of the method compared to traditional technique.

Keywords: Least squares method, QR decomposition, Gauss quadrature rule, Fredholm integral equations.
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1 Introduction

Fredholm integral equations of the second kind arise frequently in various fields of engineering and physics, where many problems can be reformulated as such equations. For example, in plasma physics [17], medical imaging and signal processing [14], and atmospheric radiative transfer [29], all of which often require efficient numerical solutions. Extensive research has focused on developing and analyzing numerical methods for solving these equations (e.g., [6–8, 10]). Several efficient algorithms have been proposed for the rapid numerical solution of Fredholm integral equations of the second kind in [9, 10, 16, 17, 23, 28, 31].

We focus on the following Fredholm integral equation:

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt. \tag{1}$$

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We consider an approximate solution to the Fredholm integral equation of the second kind, expressed as a linear combination of Chebyshev polynomials. To determine the coefficients of this linear combination, we formulate and solve a minimization problem by minimizing the norm of the residual function associated with the approximate solution. Gaussian quadrature is employed to approximate the integral in the norm of residual function, leading to a non-square linear system. The least squares solution to this system is then obtained using QR decomposition.

The least squares solution to this system is then obtained using QR decomposition, while other decomposition methods are available [18], QR decomposition is chosen here due to its direct relevance to the minimization process. The use of Gaussian quadrature with Chebyshev nodes provides an accurate approximation of the integral. The use of Gaussian quadrature with Chebyshev nodes provides an accurate approximation of the integral. Moreover, Chebyshev polynomials have been widely used in numerical approximation due to their favorable properties [20, 21, 24] and the least squares method has been extensively employed in [1–5, 11–13, 22, 25, 26]. This paper focuses on the computational implementation of the least squares method. The efficiency of the least squares method over other well-known numerical techniques has been reported in [15, 33]. Our primary contribution is to demonstrate that a solver based on QR decomposition offers significant performance advantages compared to the traditional zero-gradient approach for the required minimization. To further contextualize this efficiency, a numerical comparison with the Galerkin method is also presented. This analysis confirms that our proposed QR-based method is substantially more efficient, reducing CPU time significantly while delivering the same level of absolute error. This paper presents several crucial contributions, which can be summarized as follows:

- The integral equation is reformulated as a minimization problem for the residual norm. By applying Gaussian quadrature for numerical integration, the residual norm is transformed into a non-square linear system, thereby reducing the minimization problem to one of numerical linear algebra problems.

- The least squares solution of this non-square system is efficiently obtained using the QR decomposition method, which offers both ease of implementation and high computational speed.

- Rigorous proofs are provided for the convergence of the residual norm as well as the convergence of the approximate solution. Moreover, the minimum order of convergence is presented.

- Several numerical examples are included to illustrate the efficiency of the proposed method over zero-gradient method and Galerkin method.

The rest of paper is organized as follows. Section 2 introduces the proposed method for transforming the Fredholm integral equation of the second kind into a least squares problem and outlines the use of QR decomposition to obtain the minimum-norm solution. Section 3 provides a rigorous convergence analysis of the method and establishes the convergence rate of the approximate solution. Section 4 presents numerical examples that illustrate the superior efficiency and accuracy of the QR decomposition method in comparison with the zero-gradient approach and Galerkin method. Finally, Section 5 offers concluding remarks and summarizes the main findings of the study.

2 Description of the method

For convenience, we briefly express the Fredholm integral equation (1) in the form $u = f + \lambda Ku$ where K denotes the Fredholm integral operator on function u. We consider the approximate solution, in the form of a polynomial expressed in terms of Chebyshev polynomials as follows:

$$u_n(x) = \sum_{j=0}^{n} c_j T_j(x).$$

The residual function is defined as a function in the following form:

$$R(c_0,\ldots,c_n)(x) := u_n(x) - f(x) - \lambda \int_a^b k(x,t)u_n(t)dt.$$

The unknown coefficients c_i's are determined such that the weighted Chebyshev norm of the residual function is minimized (without loss of generality, consider $[a,b] = [-1,1]$):

$$\min_{c_0,\ldots,c_n} \|R(c_0,\ldots,c_n)\|_{L_w^2}^2 = \min_{c_0,\ldots,c_n} \int_{-1}^{1} w(x) \left(R(c_0,\ldots,c_n)(x)\right)^2 dx. \tag{2}$$

Traditionally, to minimize the norm of the residual function, the partial derivatives of the norm of the residual $R(c_0,\ldots,c_n)$ are set to zero and the solution of the following system yields the minimum:

$$\frac{\partial \|R(c_0,\ldots,c_n)\|_{L_w^2}}{\partial c_k} = 0, \quad k = 0,\ldots,n.$$

For convenience, we call this traditional method ($\bigtriangledown \|R(c_0,\ldots,c_n)\|_{L_w^2} = 0$), as the zero-gradient method throughout this paper.

However, this paper shows that this method is very time-consuming even for moderately large N. To address this issue, we propose a solution that, in an ideal scenario, considers the coefficients c_i's such that the norm of the residual function equals zero:

$$0 = \|R(c_0,\ldots,c_n)\|_{L_w^2}^2 = \int_{-1}^{1} w(x) \left(\sum_{j=0}^{n} c_j T_j(x) - f(x) - \lambda K \left(\sum_{j=0}^{n} c_j T_j \right)(x) \right)^2 dx \tag{3}$$

$$\approx \sum_{i=0}^{M} w_i \left(\sum_{j=0}^{n} c_j T_j(x_i) - f(x_i) - \lambda K \left(\sum_{j=0}^{n} c_j T_j \right)(x_i) \right)^2. \tag{4}$$

If we consider polynomial T_i as Chebyshev polynomial, therefore, the last equality effectively approximates the integral using Gaussian quadrature with a sufficiently large number of nodes, denoted by $M+1$. Since the weights w_i's are positive, each term in the summation is nonnegative. As a result, the following linear system should hold:

$$\begin{aligned} 0 &= \sum_{j=0}^{n} c_j T_j(x_i) - f(x_i) - \lambda K \left(\sum_{j=0}^{n} c_j T_j \right)(x_i) \\ &= \sum_{j=0}^{n} c_j \left(T_j - \lambda K T_j\right)(x_i) - f(x_i) \end{aligned}$$

for $i = 0,1,2,\ldots,M$. This non-square system can be written as

$$A_{(M+1)\times(n+1)}C_{(n+1)\times 1} = F_{(M+1)\times 1}, \tag{5}$$

where

$$C = \left(c_0, c_1, \ldots, c_n\right)^T_{1\times(n+1)},$$

$$F = \left(f(x_0), f(x_1), \ldots, f(x_M)\right)^T_{1\times(M+1)}$$

and

$$A_{ij} := \left(T_j - \lambda KT_j\right)(x_i), \quad i = 0,1,\ldots,M, \quad j = 0,1\ldots,n.$$

Note that the computation of the integral operator KT_j at x_i may involve complexities. In such cases, Gaussian quadrature is used to efficiently approximate $KT_j(x_i)$. The number M will be chosen as a sufficiently large number to facilitate an accurate approximation of the integral (3). Note that $M+1 > n+1$ and the above matrix is not square matrix. The least squares solution of the linear system (5) is a vector C which minimizes the function $C \longmapsto \| AC - F \|_{L^2_w}$ on $C \in \mathbb{R}^{n+1}$. The QR decomposition of matrix A yields the least squares solution of $AC = F$ by solving the following linear system:

$$RC = Q^T F, \tag{6}$$

where R is a square matrix ([27, Theorem 2.13]). Note that the QR decomposition of matrix A is considered as follows:

$$A_{(M+1)\times(n+1)} = Q_{(M+1)\times(n+1)}R_{(n+1)\times(n+1)},$$

where Q and R are the orthogonal matrix and square matrix, respectively ([27, Theorem 2.12]).

Remark 1. *If we use the first kind of Chebyshev polynomials, then the Chebyshev points are*

$$x_i := \cos\left(\frac{(2i+1)\pi}{2M+2}\right), \quad 0 \le i \le M.$$

If we use the second kind of Chebyshev polynomials, then the Chebyshev points are

$$x_i := \cos\left(\frac{i\pi}{M}\right), \quad 0 \le i \le M.$$

Note that the weights w_i's are omitted when the non-square system (5) is obtained from (3). It shows that system (5) is independent of the weights w_i and only depends on Chebyshev points x_i.

3 Convergence analysis of the method

This section proves the convergence rate of the proposed method. First, we establish that the minimum norm of the residual function approaches zero as the degree n of the approximate solution increases.

Subsequently, we show that the approximate solution converges to the exact solution. To do this, we consider

$$u(x) = \sum_{i=0}^{\infty} a_i T_i(x),$$

as Chebyshev expansion of the exact solution $u(x)$ and

$$\tilde{u}_n(x) = \sum_{i=0}^{n} a_i T_i(x), \tag{7}$$

where T_i is the first kind of Chebyshev polynomials. Thus

$$\|u - \tilde{u}_n\|_{L_w^2}^2 = \| \sum_{i=n+1}^{\infty} a_i T_i \|_{L_w^2}^2 = \sum_{i=n+1}^{\infty} |a_i|^2, \tag{8}$$

since $\{T_i\}_{i=0}^{\infty}$ are orthogonal polynomials in L_w^2, the last equality follows directly from Parseval's identity. According to [30], the following inequality can be proven for some natural number k and constant C:

$$\sum_{i=n+1}^{\infty} |a_i|^2 \leq \frac{C}{(n-k)^{2k+1}}. \tag{9}$$

The detailed proof of inequality (9) is provided in the Appendix. Therefore, equality (8) and inequality (9) imply that

$$\|u - \tilde{u}_n\|_{L_w^2}^2 \to 0 \qquad as \qquad n \to \infty, \tag{10}$$

which shows that

$$\|u - \tilde{u}_n\|_{L_w^2}^2 = O(\frac{1}{n^{2k+1}}). \tag{11}$$

We now establish the following theorem, demonstrating that the norm of the residual function converges to zero as the degree n of the approximate solution increases.

Theorem 1. *Let u be the exact solution of Fredholm integral equation (1) and integral operator $Ku = \int_{-1}^{1} k(x,t)u(t)dt$, be a bounded operator. Then*

$$\min_{c_0,\ldots,c_n} \|R(c_0,\ldots,c_n)\|_{L_w^2}^2 \to 0 \qquad as \qquad n \to \infty. \tag{12}$$

Proof. Let Π_n be the space of polynomials of degree at most n. Thus

$$\min_{c_0,\ldots,c_n} \|R(c_0,\ldots,c_n)\|_{L_w^2}^2 = \min_{v_n \in \Pi_n} \|v_n - f - \lambda K v_n\|_{L_w^2}^2 \leq \|\tilde{u}_n - f - \lambda K \tilde{u}_n\|_{L_w^2}^2,$$

where the function $\tilde{u}_n$ is defined as in (7). Hence

$$\begin{aligned} \min_{c_0,\ldots,c_n} \|R(c_0,\ldots,c_n)\|_{L_w^2}^2 &\leq \|\tilde{u}_n - f - \lambda K \tilde{u}_n\|_{L_w^2}^2 \\ &= \|\tilde{u}_n - u + \lambda K u - \lambda K \tilde{u}_n\|_{L_w^2}^2 \end{aligned}$$

$$\begin{aligned}
&\leq \left(\|\tilde{u}_n - u\|_{L^2_w} + \|\lambda K(u-\tilde{u}_n)\| \right)^2 \\
&\leq 2 \left(\|\tilde{u}_n - u\|^2_{L^2_w} + \|\lambda K(u-\tilde{u}_n)\|^2_{L^2_w} \right) \\
&\leq 2 \left(\|\tilde{u}_n - u\|^2_{L^2_w} + \lambda^2 \|K\|^2 \|u-\tilde{u}_n\|^2_{L^2_w} \right) \\
&= 2 \left(1 + \lambda^2 \|K\|^2 \|\right) \|u-\tilde{u}_n\|^2_{L^2_w}. \qquad (13)
\end{aligned}$$

The first equality follows from the fact that $u = f + \lambda K u$. The first and second inequalities are derived from the triangle inequality and the fact that $(A+B)^2 \leq 2(A^2+B^2)$, respectively. The last inequality stems from the boundedness of the integral operator. By the convergence in (10) and inequality (13), the following inequality is derived, which directly implies (12):

$$0 \leq \min_{c_0,\ldots,c_n} \|R(c_0,\ldots,c_n)\|^2_{L^2_w} \leq 2\left(1+\lambda^2\|K\|^2\|\right)\|u-\tilde{u}_n\|^2_{L^2_w} \to 0 \qquad as \qquad n \to \infty,$$

which completes the proof of the theorem. □

Remark 2. *The previous theorem and* (10) -(11) *yield the convergence rate of minimum of the norm of the residue function,*

$$\min_{c_0,\ldots,c_n} \|R(c_0,\ldots,c_n)\|^2_{L^2_w} = O(\frac{1}{n^{2k+1}}).$$

The next theorem proves the convergence rate of the approximate solution to the exact solution. We need to introduce the definition of projection from $L^2_w[-1,1]$ to subspace Π_n as follows. Since every polynomial in Π_n can be written as $\sum_{i=0}^n c_i T_i$, for some coefficients $c_0,\ldots,c_n$. For every $f \in L^2_w[-1,1]$, one can find the optimal set of coefficients $(c_0^*,\ldots,c_n^*)$ that solves the minimization problem:

$$\min_{c_0,\ldots,c_n} \|\sum_{i=0}^n c_i T_i - f\|^2_{L^2_w} = \|\sum_{i=0}^n c_i^* T_i - f\|^2_{L^2_w}. \qquad (14)$$

Because $L^2_w[-1,1]$ is a Hilbert space and Π_n is a closed subspace, such a unique set of coefficients that minimizes this distance is guaranteed to exist. The resulting polynomial, $\sum_{i=0}^n c_i^* T_i$, is the best polynomial approximation to f.

Definition 1. *Let P_n be a linear transformation $P_n : L^2_w[-1,1] \to \Pi_n$, for every $f \in L^2_w[-1,1]$, the projection of f is denoted by $P_n f := Q_n = \sum_{i=0}^n c_i^* T_i$, such that the minimization problem* (14) *attains the minimum at $(c_0^*,\ldots,c_n^*)$.*

It is clear that $P_n Q_n = Q_n$, then

$$P_n^2 f = P_n(P_n f) = P_n Q_n = Q_n = P_n f,$$

which shows that P_n is a projection ($P_n^2 = P_n$).

The following theorem ([19, Theorem 12.1.2]) proves that the approximate solution converges to the exact solution under some assumptions.

Theorem 2 ([19]). *Assume $K : V \to V$ is bounded, with V a Banach space; and assume $\lambda - K : V \to V$ is one-to-one and onto. Further assume*

$$\|K - P_n K\| \to 0 \quad \text{as } n \to \infty,$$

for projection P_n from V to a finite dimensional subspace of V. Then for all sufficiently large n, say $n \geq N$, the operator $(\lambda - P_n K)^{-1}$ exists as a bounded operator from V to V. Moreover, it is uniformly bounded:

$$\sup_{n \geq N} \|(\lambda - P_n K)^{-1}\| < \infty.$$

Also, the following two-sided error estimate holds:

$$\frac{|\lambda|}{\|\lambda - P_n K\|} \|u - P_n u\| \leq \|u - u_n\| \leq |\lambda| \|(\lambda - P_n K)^{-1}\| \|u - P_n u\|, \tag{15}$$

for the approximate solution u_n with n sufficiently large and the exact solution u.

Thus $\|u - u_n\|$ converges to zero at exactly the same speed as $\|u - P_n u\|$.

Proof. See [19, Theorem 12.1.2]. □

Corollary 1. *The previous theorem and order of convergence (10)-(11) conclude that the approximate solution converges to the exact solution of order $O(n^{-k-\frac{1}{2}})$.*

Remark 3. *Note that, $\|u - P_n u\| = \|\sum_{i=n+1}^{\infty} a_i T_i\|$ converges to zero, at least order of $O(n^{-k-\frac{1}{2}})$ and in the special cases the convergence rate is more than $O(n^{-k-\frac{1}{2}})$. Therefore, the approximate solution converges to the exact solution, at least of order $O(n^{-k-\frac{1}{2}})$. See Remark 4 in the Appendix for further details.*

4 Numerical experiments

In this section, numerical examples demonstrate that computing the least squares solution using QR decomposition method is significantly faster than the traditional zero-gradient method. The provided numerical examples taken from [32] are utilized to perform a comparative analysis of the three methods. Computations were performed using MAPLE 18 on a system with an Intel® Core™ i7-10750H processor (2.60 GHz) and 16 GB RAM.

Example 1. In this example we consider the kernel function in the integral equation (1) as

$$k(x,t) := e^{x+t},$$

with $\lambda = 1$ and the exact solution given by

$$u(x) := e^x + \cos(x).$$

The corresponding right-hand side function $f(x)$ is given by

$$f(x) = 0.5\Big((\cos(1) - \sin(1))e^{x-1} - (\cos(1) + \sin(1))e^{x+1} + 2\cos(x) + 2e^x + e^{x-2} - e^{x+2}\Big).$$

Table 1 shows that the QR method yields significantly lower errors compared to the zero-gradient method for $n = 16, 32$. Moreover, its computational time is substantially shorter than that of the zero-gradient method and Galerkin method.

Table 1: Absolute Error and CPU time for three methods on Examples 1-4

		Zero-Gradient method		QR decomposition method $M = n+15$		Galerkin method	
Example.	N	CPU time	Absolute Error	CPU time	Absolute Error	CPU time	Absolute Error
1	4	0.39s	2.0 e-2	0.03s	1.0 e-2	0.41s	1.5 e-2
	8	5.31s	1.0 e-6	0.12s	4.0 e-7	6.5s	5.1 e-7
	16	63.9s	5.0 e-8	0.42s	1.3 e-18	72.5s	8.0 e-17
	32	484s	6.0 e-12	0.92s	8.0 e-22	510.5s	2.2 e-21
2	4	0.015s	3.5 e-2	0.015s	2.0 e-2	0.10s	2.8 e-2
	8	0.061s	6.0 e-4	0.030s	2.0 e-4	0.4s	3.5 e-4
	16	0.891s	3.0 e-7	0.079s	1.0 e-7	1.8s	8.9 e-7
	32	1206s	1.2 e-9	0.437s	4.0 e-13	1375s	2.5 e-12
3	4	0.125s	9.0 e-3	0.032s	4.5 e-3	0.18s	5.0 e-3
	8	23.41s	4.5 e-7	0.203s	2.0 e-7	30.55s	2.8 e-7
	16	1704s	1.0 e-9	0.750s	1.0 e-17	2012s	9.2 e-16
	32	Fail	.....	5.140s	2.5 e-19	Fail	.....
4	4	0.2s	6.0 e-2	0.07s	6.0 e-2	0.17s	6.0 e-2
	8	0.71s	2.5 e-3	0.12s	1.5 e-3	0.59s	1.6 e-3
	16	9.57s	3.5 e-6	0.86s	1.3 e-6	12.0s	1.5 e-6
	32	Fail	.....	5.140s	2.5 e-12	Fail	.....

Example 2. Consider the kernel $k(x,t) := x$ in the integral equation (1), and $\lambda = 1$ with the exact solution given by

$$u(x) := x \cdot \arctan(x)$$

and the corresponding right-hand side

$$f(x) = 0.5 \cdot x \cdot (2 - \pi + 2 \cdot \arctan(x)).$$

Table 1 demonstrates that the QR method produces significantly lower errors compared to the zero-gradient method for $n = 16, 32$. Additionally, its CPU time is considerably shorter than that of the zero-gradient method and Galerkin method.

Example 3. Consider the integral equation (1) with the kernel function defined by

$$k(x,t) := \sin(x-t), \qquad \& \qquad \lambda = 4,$$

and the inhomogeneous term

$$f(x) = \cos(x) - 4 \cdot \sin(x) + \cos(x+2) - \cos(x-2),$$

The exact solution is given by

$$u(x) := \cos(x).$$

Table 1 shows that for $n = 16$, the error produced by the QR method is significantly lower than that of the zero-gradient method. Moreover, the computational time of the zero-gradient method exceeds that

more than 2000 times longer than that of the QR method. Also, for $n = 32$, the zero-gradient method and Galerkin method do not achieve the approximate solution, even after over 30 minutes of computational effort, whereas the QR method successfully computes the solution in just 5 seconds.

Example 4. Consider

$$k(x,t) := e^{\arctan(t)},$$

as kernel in the integral equation (1), and $\lambda = 1$ with the exact solution,

$$u(x) := \frac{1}{1+x^2},$$

and inhomogeneous term

$$f(x) = \frac{1}{1+x^2} + e^{\frac{-\pi}{4}} - e^{\frac{\pi}{4}}.$$

As shown in Table 1, the CPU time of the zero-gradient method is more than eleven times longer than that of the QR decomposition method. Furthermore, in the case $n = 32$, despite over 30 minutes of computational effort, the zero-gradient method and Galerkin method do not achieve the approximate solution, while the QR decomposition method successfully obtains the result within 5 seconds.

The well-known Galerkin method is also tested, but it is very time-consuming and could not compete with the QR decomposition approach. In the third and fourth examples with $n = 32$, the zero-gradient method and Galerkin method do not achieve the approximate solution after 2000 seconds of computation, while the QR decomposition method successfully computes an approximate solution in just a few seconds of CPU time. Across all four examples, for $n = 4, 8, 16$, the QR decomposition method yields a smaller absolute error compared to the zero-gradient method. Additionally, the execution time of the QR decomposition method is substantially lower than that of the zero-gradient method and Galerkin method. The QR decomposition method demonstrates superior efficiency and accuracy compared to the zero-gradient method and Galerkin method particularly for $n = 16$ and $n = 32$.

5 Conclusion

This paper presents a numerical method for solving Fredholm integral equations of the second kind using the least squares approach combined with QR decomposition. The proposed method minimizes the norm of the residual function by employing Gaussian quadrature for numerical integration and QR decomposition for efficient computation. The convergence of both the norm of residual function and the approximate solution to the exact solution is rigorously analyzed, demonstrating the method's strong theoretical foundation. A key advantage of this approach is that it avoids the use of partial derivatives and requires fewer computations than traditional zero-gradient methods, resulting in significantly faster execution. Numerical examples clearly illustrate the efficiency of the proposed method. For instance, in the third and fourth examples with $n = 32$, the zero-gradient method and Galerkin method do not achieve the approximate solution, even after over 30 minutes of computational effort, whereas the QR decomposition method finds the solution in just a few seconds. Looking forward, the computational efficiency of this method makes it a strong candidate for real-world inverse problems. For example, in medical imaging and signal processing, the deconvoluting of signals to restore blurry images can be formulated as a Fredholm equation where solution speed is critical. Furthermore, its potential application in physics, such as modeling radiative transfer in atmospheric science, could enable more complex and large-scale simulations where traditional solvers are prohibitively slow.

Appendix

To prove inequality (9) we apply the following theorem [30].
Theorem. If $f, f', \ldots, f^{(k-1)}$ are absolutely continuous on $[-1,1]$ and if

$$\int_{-1}^{1} (1-x^2)^{\frac{-1}{2}} |f^{(k)}(x)| dx = L < \infty,$$

for some $k \geq 0$, then for each $j \geq k+1$,

$$|a_j| \leq \frac{2L}{\pi j(j-1)\cdots(j-k)}. \tag{16}$$

In addition, if f is analytic with $|f(z)| \leq M$ in the region bounded by the ellipse with foci ± 1 and major and minor semiaxis lengths summing to $\rho > 1$, then for each $j \geq 0$,

$$|a_j| \leq \frac{2M}{\rho^j}. \tag{17}$$

Inequality (16) in the above theorem implies the following inequality:

$$\sum_{j=n+1}^{\infty} |a_j|^2 \leq \sum_{j=n+1}^{\infty} \frac{4L^2}{\pi^2 j^2 (j-1)^2 \cdots (j-k)^2} \leq \frac{4L^2}{\pi^2} \sum_{j=n+1}^{\infty} \frac{1}{(j-k)^{2k+2}},$$

for $n > k$. It is clear that

$$\sum_{j=n+1}^{\infty} \frac{1}{(j-k)^{2k+2}} \leq \int_{n}^{\infty} \frac{dx}{(x-k)^{2k+2}} = \frac{1}{(2k+1)(n-k)^{2k+1}},$$

which proves (9).

Remark 4. *In the case* (17)*, the following inequality holds:*

$$\sum_{j=n+1}^{\infty} |a_j|^2 \leq 4M^2 \sum_{j=n+1}^{\infty} (\frac{1}{\rho^2})^j = \frac{4M^2}{\rho^2 - 1} (\frac{1}{\rho^2})^n.$$

In the case (16)*, the approximate solution converges to the exact solution of order* $O(n^{-k-\frac{1}{2}})$ *and in the case* (17)*, the approximate solution converges to the exact solution of order* $O(\rho^{-n})$*, that is more than order* $O(n^{-k-\frac{1}{2}})$*. Therefore, the approximate solution converges to the exact solution, at least of order* $O(n^{-k-\frac{1}{2}})$*, which explains Remark 3.*

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Conflicts of interest

The authors declare that there are no conflicts of interest.

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