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Advanced bounds for eigenvalues in spectral fuzzy graph theory

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Abstract. This paper introduces advanced eigenvalue bounds for Spectral Fuzzy Graphs (SFGs), a subclass of fuzzy graphs with symmetric adjacency matrices enhancing the precision of spectral graph analysis. Leveraging the Rayleigh quotient and the Perron-Frobenius theorem, we establish novel upper and lower bounds for the largest and smallest eigenvalues of fuzzy adjacency matrices. Specific results include eigenvalue bounds for complete and bipartite fuzzy graphs, as well as the demonstration of eigenvalue stability under graph perturbations and unions. A numerical example based on a protein interaction network illustrates the practical applicability of the proposed methods, demonstrating improved accuracy in analyzing network resilience and connectivity.

Keywords: Fuzzy graph, adjacency eigenvalue, Laplacian eigenvalue, spectral radius, energy bounds.
AMS Subject Classification 2010: 05C50, 05C72, 58C40.

1 Introduction

Graph theory, a mathematical framework that models relationships between objects, has its origins in the pioneering work of Euler on the Seven Bridges of Konigsberg [15]. Over the decades, graph theory has evolved into a central discipline, finding applications in diverse fields such as biology, chemistry, computer science, and social networks. The core components encompassing of vertices and edges forms a versatile foundation in the development of algebraic graph theory [6, 21]. With its growth, spectral

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graph theory [45] emerged as a significant branch, focusing on the interplay between graph structure and the eigenvalues of associated matrices [11, 12]. The problem posed by Von Collatz and Sinogowitz to characterize the issue of nullity greater than zero [46] set the main frame for spectral graph theory. This branch has offered profound insights into network dynamics, graph connectivity, and optimization problems [7, 26]. The study of spectral properties [3], such as the algebraic connectivity introduced by Fiedler [16] and bounds on eigenvalues [13, 32], has enabled advancements in understanding the robustness of networks.

The application of spectral graph theory to practical problems has led to a deeper exploration of eigenvalues [41] and their implications. Researchers have derived bounds for eigenvalues of adjacency matrices [5,25], analyzed graph energies [22,23], and studied spectral radii [19,48]. Powers [42] laid the foundation by applying Gershgorin's Theorem to derive basic eigenvalue bounds for real symmetric matrices in 1989. Diaconis and Stroock [14] introduced geometric techniques for bounding eigenvalues in reversible Markov chains, linking spectral theory with probabilistic methods. In 2008, Godsil and Newman [20] connected eigenvalue bounds with combinatorial structures such as independent sets. Later the year, Mohar [38] focused on inequalities for the sum of the largest k eigenvalues and Kumar [30] refined upper bounds for the spectral radius by incorporating minimum and maximum degree constraints. In 2010, Adiga and Swamy [1] derived bounds for the smallest eigenvalue associated with vertex degrees. Sorgun [44] explored Laplacian eigenvalue bounds in the context of weighted graphs in 2013. These efforts have paved the way for analyzing real-world networks with unprecedented mathematical precision. However, traditional graph theory assumes a binary relationship between vertices, which often fails to capture the inherent ambiguity or fuzziness present in real-world systems.

The concept of fuzziness, introduced by Zadeh [51], provides a mathematical approach to represent uncertainty and imprecision. Fuzzy set theory has been widely applied in decision-making, control systems, and data analysis. Its extension to graph theory has led to the development of fuzzy graphs, a framework that accommodates partial relationships between vertices [4,43]. This notion has been further refined through studies on fuzzy adjacency matrices [31], node and arc connectivity [36], and types of arcs in fuzzy graphs [37]. These advancements have opened new avenues for representing complex systems [29] where relationships are not strictly binary.

1.1 Importance of spectral fuzzy graphs

The extension of spectral methods to fuzzy graphs has significantly broadened their applicability in modeling and analyzing real-world systems characterized by uncertainty and imprecision. This convergence has given rise to the emergent field of spectral fuzzy graph theory. In a foundational contribution, Anjali and Mathew (2013) investigated the energy of fuzzy graphs, defining it as the sum of the absolute values of the eigenvalues of the fuzzy adjacency matrix [40]. Their work addressed the limitations inherent in binary representations of complex networks [8, 18]. Further developments include the work of Vimala and Jayalakshmi, who in 2016 established upper and lower bounds for fuzzy graph energy in terms of vertex degree and membership values [31]. Kalpana et al. contributed to this trajectory by exploring connectedness energy in fuzzy graphs [28], thereby enriching the understanding of network dynamics in uncertain environments. More recently, Al-Hawary and Al-Shalaldeh (2023) characterized matrix representations associated with specific fuzzy graph operations [2] and conducted a comprehensive study on their spectral energy. Buvaneswari et al. in 2024 explored operations on spectral fuzzy graphs [9], contributing foundational transformations within the spectral domain. Subsequently, Cai et al. (2025)

provided characterizations and bounds for degree-based energies, enriching the spectral analysis of fuzzy graphs [10].

The study of fuzzy eigenvalues and their bounds has been explored in recent works [2, 35, 49]. The spectral radius and energy in fuzzy graphs have also garnered significant attention, with applications in molecular chemistry and network resilience [40, 47, 50]. For instance, eigenvalue bounds in fuzzy graphs provide insights into network robustness [27, 34], while the fuzzy adjacency matrix framework aids in modeling uncertain systems [17, 39]. Moreover, advancements in the spectral properties of fuzzy graphs have demonstrated their potential in solving real-world problems, such as optimizing network designs [33] and predicting system failures [12, 24]. Theoretical developments, such as fuzzy Zagreb indices [27] and fuzzy Laplacian energy [23], require further refinement to address large-scale systems.

Spectral fuzzy graphs offer a robust analytical framework for interpreting systems where interactions are ambiguous, variable, or dynamically evolving. In biological networks, such as protein- protein interaction (PPI) systems, spectral fuzzy methods facilitate the identification of proteins that exert disproportionate influence on disease pathways. In the domain of cybersecurity, fuzzy spectral techniques enhance anomaly detection by revealing irregularities in uncertain or imprecise network topologies. Similarly, social networks benefit from fuzzy graph models that capture heterogeneity in influence, interaction intensity, and community structure. Through spectral analysis, these graphs yield critical insights into the resilience, stability, and structural integrity of systems embedded in uncertain contexts.

1.2 Motivation and problem statement

Despite the increasing attention directed toward fuzzy graphs and spectral techniques, much of the existing literature remains confined to elementary eigenvalue bounds or narrowly scoped applications. Traditional spectral estimates frequently lack the granularity necessary to capture the structural complexity of fuzzy graphs, particularly in configurations such as bipartite, complete, or highly irregular networks. Furthermore, the stability and behavior of eigenvalues under perturbations, an essential consideration in dynamic or noise-prone systems remain inadequately addressed, thereby constraining the broader applicability of spectral fuzzy methods. By deriving tighter bounds and explicitly addressing eigenvalue stability, this paper bridges these gaps, making significant contributions to both the theoretical advancement of spectral fuzzy graph theory and its practical applications in network analysis.

1.3 Novelty and key contributions

The present work addresses these gaps through the following contributions:

- Establishes sharper bounds for the eigenvalues of fuzzy graphs by incorporating structural and membership-function information into spectral estimates.

- Analyzes the stability of fuzzy graph eigenvalues under structural perturbations, providing insight into their robustness in uncertain environments.

- Demonstrates the proposed techniques on a PPI network associated with Systemic Lupus Erythematosus (SLE), showcasing the utility of spectral fuzzy analysis in biomedical network modeling.

Table 1: Notations and symbols used in the paper.

Symbol	Definition
$G = (V, \sigma, \mu)$	Fuzzy graph with vertex set V, vertex membership σ, and edge membership μ
μ_{ij}	Membership value of an edge (v_i, v_j)
A	Fuzzy adjacency matrix of G
D	Fuzzy degree matrix of G
λ_i	i^{th} eigen value of the adjacency matrix
λ_1, λ_n	Largest and smallest eigenvalue of G
S_{ij}	Fuzzy degree of Separation
D_i	Gershgorin disc
$\|A\|_F$	Frobenius norm of $A = [a_{ij}] \in \mathbb{R}^{p \times q}$
$\Gamma(G)$	Fuzzy spectrum of G
$d(v_i)$	Fuzzy degree of vertex v_i: $\sum \mu_{ij}$
$\rho(G)$	Spectral radius of G
Δ	Maximum degree in G

- Provides a detailed numerical analysis and visual interpretation of the spectral behavior of the network, highlighting the expected ranges of key structural properties as inferred from the total network energy and its underlying spectral components.

1.4 Organization of the paper

This paper is structured as follows. Section 2 outlines theoretical preliminaries and definitions relevant to spectral fuzzy graphs. Section 3 defines spectral bound and fuzzy degree of separation. Section 4 presents key theorems with illustrative examples, focusing on eigenvalue bounds and spectral properties. Section 5 applies the proposed methods to a fuzzy SLE PPI network, computing eigenvalues, maximum degree, and network energy. Section 6 discusses the results, emphasizing implications for disease modeling. Finally, Section 7 concludes the paper and outlines directions for future research.

1.5 Scope and Notation

While the theory of fuzzy graphs accommodates asymmetric or directed edges, this paper focuses exclusively on **Spectral Fuzzy Graphs (SFGs)**, fuzzy graphs whose adjacency matrices are symmetric, ensuring all eigenvalues are real and the matrix is diagonalizable. This restriction enables rigorous spectral analysis and guarantees the applicability of linear algebraic techniques. Henceforth, the term "fuzzy graph" will refer specifically to an SFG unless otherwise stated. All definitions, theorems, and examples presented in this work are applicable solely within this spectral subclass.

The key notations and symbols used throughout this paper are summarized in Table 1.

2 Preliminaries

Definition 1 ([39]). *A fuzzy graph $G = (V, \sigma, \mu)$ is a triplet consisting of a non-empty set V together with a pair of functions $\sigma : V \longrightarrow [0,1]$ which is a fuzzy vertex set and $\mu : V \times V \longrightarrow [0,1]$ which is a fuzzy edge set such that $\mu_{ij} \leq \sigma_i \wedge \sigma_j$ for all $i, j \in V$.*

Definition 2 ([40]). *The adjacency matrix A of a fuzzy graph $G = (V, \sigma, \mu)$ is an $n \times n$ matrix defined as $A = [a_{ij}]$ where $a_{ij} = \mu_{ij}$. The eigenvalues of A are denoted by λ_i where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$.*

Definition 3 ([7]). *For a square matrix M, the multiset of eigenvalues of M is called the spectrum of M and is denoted by $\Gamma(G) = \{\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \ldots, \lambda_p^{(m_p)}\}$, where each λ_i is a distinct eigenvalue of M with multiplicity m_i, for all $i = 1, 2, \ldots, p$.*

Definition 4 ([40]). *Let G be a fuzzy graph and A be its adjacency matrix. The eigenvalues of A are the eigenvalues of G. The adjacency eigenvalues along with their algebraic multiplicities collectively constitute the fuzzy spectrum $\Gamma(G)$.*

Definition 5 ([21]). *The spectral radius $\rho(G)$ of a fuzzy graph G with adjacency matrix A is the largest absolute value of the eigenvalues of A. It is given by $\rho(G) = max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}$, where λ_i are the eigenvalues of A.*

Note: In this section and beyond, the authors focus exclusively on SFGs that are simple, connected, undirected fuzzy graphs whose adjacency matrices are symmetric. This ensures that all eigenvalues are real and that the matrices are fully diagonalizable, which serves as the foundation for the spectral results presented throughout the paper.

3 Spectral bound and fuzzy degree of separation

Definition 6 (Spectral Bound of a Fuzzy Graph). *The spectral bound of a fuzzy graph G is the upper bound on the largest eigenvalue $\lambda_1(G)$ of its adjacency matrix $A(G)$ in terms of the row sums of the adjacency matrix and is defined as*

$$\lambda_1(G) \leq \max_i \sum_{j=1}^{n} \mu_{ij}.$$

Definition 7 (Fuzzy Degree of Separation). *The fuzzy degree of separation between two vertices v_i and v_j in a fuzzy graph G is the absolute difference between the degrees of the vertices v_i and v_j. It is defined as:*

$$S_{ij} = \left| \sum_{k=1}^{n} \mu_{ik} - \sum_{k=1}^{n} \mu_{jk} \right|.$$

4 Eigenvalue bounds for spectral fuzzy graphs

This section presents theorems establishing eigenvalue bounds for fuzzy graphs, including upper and lower bounds for the largest and smallest eigenvalues, analyzing the impact of perturbations and graph unions and deriving specific bounds for complete and bipartite fuzzy graph structures.

Lemma 1. *Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ be a real symmetric matrix with non-negative entries and $a_{ii} = 0$ for all v_i. Then every eigenvalue $\lambda \in \mathbb{R}$ of A satisfies*

$$|\lambda| \leq \max_{1\leq i\leq n} \sum_{j\neq i} a_{ij}.$$

Proof. By the Gershgorin Circle theorem, each eigenvalue λ of a matrix A lies in at least one of the Gershgorin discs defined by

$$D_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j\neq i} |a_{ij}| \right\}, \quad 1 \leq i \leq n.$$

Since $a_{ii} = 0$ and $a_{ij} \geq 0$, each disc D_i is centered at 0 with radius $R_i = \sum_{j\neq i} a_{ij}$. Hence, all eigenvalues lie in the union of these discs and

$$|\lambda| \leq \max_{1\leq i\leq n} R_i = \max_{1\leq i\leq n} \sum_{j\neq i} a_{ij}. \qquad \square$$

Theorem 1. *Let G be a fuzzy graph with adjacency matrix $A(G)$. The largest eigenvalue λ_1 satisfies*

$$\lambda_1(G) \leq \max_i \sum_j \mu_{ij},$$

where μ_{ij} represents the membership value associated with the edge between vertices v_i and v_j.

Proof. Let $A(G) = [\mu_{ij}]$ be the adjacency matrix of the graph G, where $\mu_{ij} \geq 0$, and $\mu_{ii} = 0$ as G is a simple fuzzy graph. Since $A(G)$ is symmetric, all its eigenvalues are real.

By the Gershgorin Circle theorem, let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$. Then every eigenvalue λ of A lies in atleast one of the Gershgorin disks D_i, given by

$$D_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \right\}, \quad i = 1, 2, \ldots, n.$$

For each row i, the corresponding disk D_i is centered at the diagonal entry a_{ii} with radius equal to the sum of the absolute values of the off-diagonal entries in that row. Consequently, the spectrum of A is contained in the union $\bigcup_{i=1}^{n} D_i$.

For the matrix $A(G)$, $a_{ii} = 0$, and since all $\mu_{ij} \geq 0$, the disks are centered at the origin with radius

$$R_i = \sum_{j\neq i} \mu_{ij} = \sum_j \mu_{ij}.$$

Therefore, all eigenvalues lie in the union of disks centered at 0 with radius $\sum_j \mu_{ij}$. This implies

$$|\lambda_i| \leq \max_i \sum_j \mu_{ij}.$$

In particular, the largest eigenvalue in magnitude, $\lambda_1(G)$ satisfies

$$\lambda_1(G) \leq \max_i \sum_j \mu_{ij}. \qquad \square$$

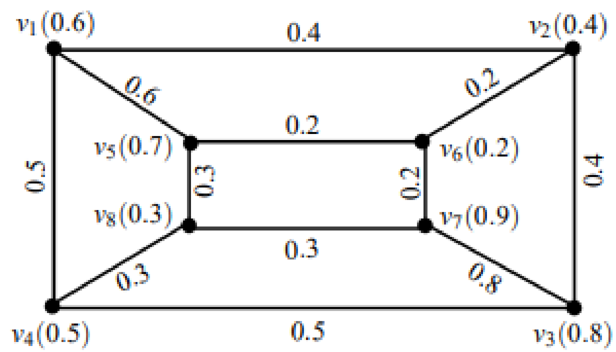

Figure 1: Fuzzy graph G that satisfies $\lambda_1(G) \leq \max_i \sum_{j=1}^{n} \mu_{ij}$.

The result above characterizes the spectral radius of the fuzzy adjacency matrix as bounded above by the maximum row sum. The bound derived is specific to fuzzy graphs and directly relates to their structure by incorporating row sums of the fuzzy adjacency matrix. Unlike general bounds depending on global graph properties (e.g., Frobenius norm or total edge weight), this bound is computationally efficient and provides tighter estimates in cases where row sums dominate.

Example 1. Consider a fuzzy graph G shown in Figure 1 with eight vertices and the fuzzy adjacency matrix as given below:

$$A(G) = \begin{bmatrix} 0 & 0.4 & 0 & 0.5 & 0.6 & 0 & 0 & 0 \\ 0.4 & 0 & 0.4 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0.4 & 0 & 0.5 & 0 & 0 & 0.8 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0.3 \\ 0.6 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0.3 \\ 0 & 0.2 & 0 & 0 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0.2 & 0 & 0.3 \\ 0 & 0 & 0 & 0.3 & 0.3 & 0 & 0.3 & 0 \end{bmatrix}.$$

The row sums are $r_1 = 1.5$, $r_2 = 1.0$, $r_3 = 1.7$, $r_4 = 1.3$, $r_5 = 1.1$, $r_6 = 0.6$, $r_7 = 1.3$, $r_8 = 0.9$ with a maximum of 1.7. The eigenvalues are found as

$$\lambda_i = \{1.2732, 0.6934, 0.2547, 0.2081, -0.2081, -0.2547, -0.6934, -1.2732\}.$$

The theorem holds since $\lambda_1 \approx 1.27 < 1.7$.

Lemma 2. *Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric matrix. Then, for any unit vector $x \in \mathbb{R}^n$, the largest eigenvalue of A satisfies*

$$\lambda_1(A) \geq x^\top Ax.$$

Proof. Since A is real symmetric, it admits a full set of orthonormal eigenvectors and all its eigenvalues are real. The Rayleigh quotient for any non-zero vector $x \in \mathbb{R}^n$ is defined as

$$R(x) = \frac{x^\top Ax}{x^\top x}.$$

For a unit vector x (i.e., $\|x\|_2 = 1$), we have $R(x) = x^\top Ax$. The Rayleigh-Ritz theorem guarantees that the maximum value of this quotient is the largest eigenvalue. Thus

$$\lambda_1(A) = \max_{\|x\|=1} x^\top Ax.$$

Hence, for any unit vector x, $\lambda_1(A) \geq x^\top Ax$. □

Theorem 2. *Let G be a fuzzy graph with $n \geq 2$ vertices and adjacency matrix $A(G) = [\mu_{ij}]_{n\times n}$, where $\mu_{ij} \geq 0$ for all $v_i, v_j \in V$. Also, let*

$$e = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ij},$$

denote the total fuzzy edge weight of G, and let $\lambda_1(G)$ be the largest eigenvalue of $A(G)$. Then

$$\lambda_1(G) \geq \frac{2e}{n}.$$

Proof. Since $A(G)$ is a symmetric matrix with non-negative entries, all its eigenvalues are real. The Rayleigh quotient for any non-zero vector $x \in \mathbb{R}^n$ is defined by

$$R(x) = \frac{x^\top A(G)x}{x^\top x},$$

and satisfies

$$\lambda_1(G) = \max_{x\neq 0} R(x).$$

In particular, for any unit vector $x \in \mathbb{R}^n$ with $\|x\|_2 = 1$, it follows that $\lambda_1(G) \geq x^\top A(G)x$.

Consider the vector $x = \frac{1}{\sqrt{n}}\mathbf{1}$, where $\mathbf{1} \in \mathbb{R}^n$ is the all-ones vector. Clearly, $\|x\|_2 = 1$. Then

$$x^\top A(G)x = \left(\frac{1}{\sqrt{n}}\mathbf{1}\right)^\top A(G)\left(\frac{1}{\sqrt{n}}\mathbf{1}\right) = \frac{1}{n}\mathbf{1}^\top A(G)\mathbf{1}.$$

Since $\mathbf{1}^\top A(G)\mathbf{1} = \sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ij}$, we obtain

$$x^\top A(G)x = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ij} = \frac{2e}{n}.$$

Therefore, by the Rayleigh quotient bound, we have

$$\lambda_1(G) \geq \frac{2e}{n}.$$

as claimed. □

Example 2. Consider a fuzzy graph G as shown in Figure 2 with six vertices and adjacency matrix as follows:

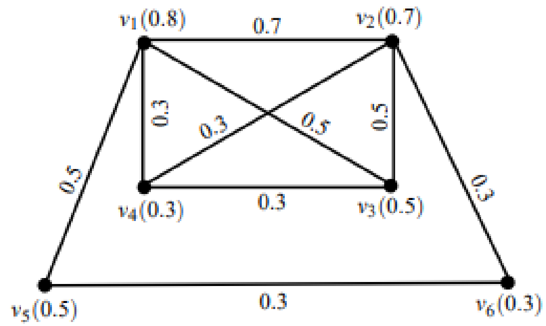

Figure 2: Fuzzy graph G with six vertices that satisfies $\lambda_1(G) \geq \frac{2e}{n}$.

$$A(G) = \begin{bmatrix} 0 & 0.7 & 0.5 & 0.3 & 0.5 & 0 \\ 0.7 & 0 & 0.5 & 0.3 & 0 & 0.3 \\ 0.5 & 0.5 & 0 & 0.3 & 0 & 0 \\ 0.3 & 0.3 & 0.3 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0.3 \\ 0 & 0.3 & 0 & 0 & 0.3 & 0 \end{bmatrix}.$$

The total fuzzy edge weight is $e = \frac{1}{2}(2+1.8+1.3+0.9+0.8+0.6) = 3.7$ and the number of vertices is $n = 6$. The eigenvalues of $A(G)$ are found approximately as

$$\lambda_i = \{1.4352, 0.3106, -0.0597, -0.2302, -0.4965, -0.9595\}.$$

The theorem states that $\lambda_1(G) \geq \frac{2e}{n} = \frac{2(3.7)}{6} \approx 1.233$. The calculated $\lambda_1 \approx 1.4352$ satisfies this inequality.

Remark 1. *The bound*

$$\lambda_1(G) \geq \frac{2e}{n},$$

becomes tight when the fuzzy edge weights are uniformly distributed across all vertices. For instance, this occurs in a complete fuzzy graph where all edge weights are equal. In such cases, the average row sum coincides with the maximum row sum, and the largest eigenvalue approximates the average row sum, making the inequality sharp.

Theorem 3. *Let G be a fuzzy graph with n vertices and adjacency matrix $A(G) = [a_{ij}] \in \mathbb{R}^{n\times n}$, where $a_{ij} \geq 0$, $a_{ii} = 0$, and $a_{ij} = a_{ji}$. Let $\Delta = \max_{1\leq i\leq n} \sum_{j\neq i} a_{ij}$ be the maximum degree of G. Then the smallest eigenvalue $\lambda_n(G)$ of $A(G)$ satisfies*

$$\lambda_n(G) \geq -\Delta.$$

Proof. The adjacency matrix $A(G)$ is real and symmetric, so all its eigenvalues are real. Since G is simple, the diagonal entries satisfy $a_{ii} = 0$, and the off-diagonal entries $a_{ij} \geq 0$ represent fuzzy edge weights.

By the Gershgorin Circle theorem, every eigenvalue λ of $A(G)$ lies within at least one of the Gershgorin discs

$$D_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j\neq i} |a_{ij}| \right\}.$$

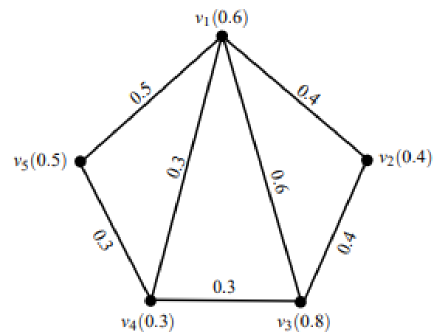

Figure 3: Fuzzy graph G with five vertices that satisfies $\lambda_n(G) \geq -\Delta$.

In this case, $a_{ii} = 0$ and $a_{ij} \geq 0$, so each disc is centered at the origin with radius equal to the degree

$$d(v_i) = \sum_{j \neq i} a_{ij}.$$

Hence, all eigenvalues of $A(G)$ lie within the union of discs centered at 0 with radii $d(v_i)$, that is

$$|\lambda| \leq \max_{1 \leq i \leq n} d(v_i) = \Delta.$$

Therefore, the smallest eigenvalue $\lambda_n(G)$, being real, satisfies

$$\lambda_n(G) \geq -\Delta.$$

as claimed. □

Corollary 1 (Spectral Interval Bound)**.** *Let G be a fuzzy graph with maximum degree Δ. Then all eigenvalues $\lambda_i(G)$ of the adjacency matrix $A(G)$ satisfy*

$$|\lambda_i(G)| \leq \Delta, \quad \textit{for all } 1 \leq i \leq n.$$

Proof. Let $A(G) = [\mu_{ij}]$ be the fuzzy adjacency matrix. Since $A(G)$ is symmetric with $\mu_{ii} = 0$, Lemma 1 applies. The degree of vertex v_i is defined as $d(v_i) = \sum_{j \neq i} \mu_{ij}$, and by assumption, $d(v_i) \leq \Delta$ for all v_i. Thus

$$|\lambda_i(G)| \leq \max_i d(v_i) \leq \Delta.$$ □

Example 3. Consider a fuzzy graph G with five vertices as shown in Figure 3. Its adjacency matrix $A(G)$ is given by:

$$A(G) = \begin{bmatrix} 0 & 0.4 & 0.6 & 0.3 & 0.5 \\ 0.4 & 0 & 0.4 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0.3 & 0 \\ 0.3 & 0 & 0.3 & 0 & 0.3 \\ 0.5 & 0 & 0 & 0.3 & 0 \end{bmatrix}.$$

The degree $d(v_i)$ of each vertex v_i is the sum of the fuzzy weights of its incident edges $d(v_1) = 1.8; d(v_2) = 0.8; d(v_3) = 1.3; d(v_4) = 0.9; d(v_5) = 0.8$. The maximum degree is $\Delta = \max\{d(v_1), d(v_2), d(v_3), d(v_4)\} = 1.8$.

Using numerical computation, the eigenvalues of $A(G)$ are approximately:

$$\lambda_i = \{1.2103, 0.2255, -0.1692, -0.4698, -0.7968\}.$$

The smallest eigenvalue is $\lambda_4 \approx -0.7968$. Thus, $-0.7968 \geq -1.8$ and the inequality stated in theorem $\lambda_n(G) \geq -\Delta$ holds true.

Lemma 3. *Let G be a fuzzy graph with adjacency matrix $A(G) = [\mu_{ij}]$. Suppose $A(G_\varepsilon) = [\mu_{ij} + \delta_{ij}]$ is a perturbed adjacency matrix such that* $\max_{i,j} |\delta_{ij}| < \varepsilon$ *for some* $\varepsilon > 0$. *Then for each eigenvalue λ_i of $A(G)$,*

$$\lim_{\varepsilon \to 0} |\lambda_i(G_\varepsilon) - \lambda_i(G)| = 0.$$

Proof. The adjacency matrix $A(G)$ is real symmetric, hence all its eigenvalues are real. Let $E = A(G_\varepsilon) - A(G) = [\delta_{ij}]$. Since $\max_{i,j} |\delta_{ij}| < \varepsilon$, we have $\|E\|_2 \leq n\varepsilon$. By Weyl's inequality for symmetric matrices,

$$|\lambda_i(G_\varepsilon) - \lambda_i(G)| \leq \|E\|_2 \leq n\varepsilon.$$

Thus, letting $\varepsilon \to 0$ gives $\lim_{\varepsilon \to 0} |\lambda_i(G_\varepsilon) - \lambda_i(G)| = 0$. □

Theorem 4. *Let G be a fuzzy graph with adjacency matrix $A(G) = [\mu_{ij}]$. Suppose the adjacency matrix is perturbed to obtain $A(G_\varepsilon) = [\mu_{ij} + \delta_{ij}]$, where the perturbation satisfies* $\max_{i,j} |\delta_{ij}| < \varepsilon$ *for some* $\varepsilon > 0$. *Then, the eigenvalues $\lambda_i(G_\varepsilon)$ of the perturbed matrix $A(G_\varepsilon)$ depend continuously on ε, and*

$$\lim_{\varepsilon \to 0} |\lambda_i(G_\varepsilon) - \lambda_i(G)| = 0,$$

for each $i = 1, 2, \ldots, n$.

Proof. Let $A = A(G)$ be the original adjacency matrix, and let $E = [\delta_{ij}]$ be the perturbation matrix such that $A(G_\varepsilon) = A + E$. Since A is real symmetric, it has real eigenvalues and a complete set of orthonormal eigenvectors.

Also, since $\|E\|_2 < \varepsilon$ (where $\|\cdot\|_2$ denotes the spectral norm), the matrix $A + E$ remains symmetric. By the classical Weyl's inequality (matrix perturbation theory for Hermitian matrices), the eigenvalues of $A + E$ satisfy,

$$|\lambda_i(A + E) - \lambda_i(A)| \leq \|E\|_2.$$

Therefore, for each i,

$$|\lambda_i(G_\varepsilon) - \lambda_i(G)| \leq \|E\|_2 < \varepsilon.$$

Taking the limit as $\varepsilon \to 0$,

$$\lim_{\varepsilon \to 0} |\lambda_i(G_\varepsilon) - \lambda_i(G)| = 0,$$

which confirms that each eigenvalue varies continuously with respect to small perturbations in the adjacency matrix. □

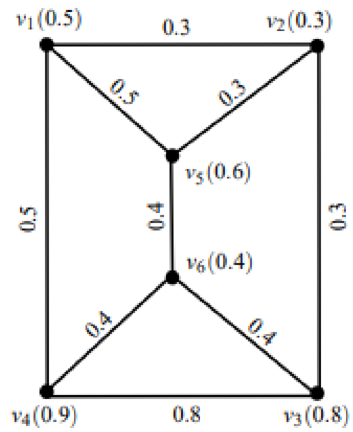

Figure 4: Fuzzy graph G demonstrating perturbation.

Example 4. Consider a fuzzy graph as shown below in Figure 4. The adjacency matrix is given by:

$$A(G) = \begin{bmatrix} 0 & 0.3 & 0 & 0.5 & 0.5 & 0 \\ 0.3 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0.3 & 0 & 0.8 & 0 & 0.4 \\ 0.5 & 0 & 0.8 & 0 & 0 & 0.4 \\ 0.5 & 0.3 & 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0.4 & 0.4 & 0.4 & 0 \end{bmatrix}.$$

Using numerical computation, the eigenvalues of $A(G)$ are found to be approximately:

$$\lambda_i = \{1.3597, 0.4986, 0.0288, -0.0843, -0.7437, -1.0591\}.$$

We introduce a symmetric perturbation matrix E, with entries in $[-0.05, 0.05]$, preserving symmetry and ensuring A' remains real and symmetric:

$$E = \begin{bmatrix} 0 & 0.01 & 0 & -0.03 & 0.02 & 0 \\ 0.01 & 0 & -0.01 & 0 & 0.03 & 0 \\ 0 & -0.01 & 0 & 0.02 & 0 & -0.04 \\ -0.03 & 0 & 0.02 & 0 & 0 & 0.01 \\ 0.02 & 0.03 & 0 & 0 & 0 & -0.02 \\ 0 & 0 & -0.04 & 0.01 & -0.02 & 0 \end{bmatrix}.$$

The perturbed matrix is $A(G_\varepsilon) = A(G) + E$. Numerical computation yields the eigenvalues of $A(G_\varepsilon)$ as approximately:

$$\lambda_i = \{1.3636, 0.5694, 0.0018, -0.1204, -0.7417, -1.0727\}.$$

The differences between corresponding eigenvalues of $A(G)$ and $A(G_\varepsilon)$ are small, illustrating the continuous dependence of eigenvalues on the membership values. Smaller $\|E\|_2$ would lead to even smaller differences.

Lemma 4. *Let $A_1 \in \mathbb{R}^{n_1 \times n_1}$ and $A_2 \in \mathbb{R}^{n_2 \times n_2}$ be symmetric matrices. Then the eigenvalues of the block diagonal matrix*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)},$$

are the union of the eigenvalues of A_1 and A_2.

Proof. Let $\lambda \in \mathbb{R}$. Consider the characteristic polynomial of A:

$$\det(A - \lambda I_{n_1+n_2}) = \det\left(\begin{bmatrix} A_1 - \lambda I_{n_1} & 0 \\ 0 & A_2 - \lambda I_{n_2} \end{bmatrix}\right) = \det(A_1 - \lambda I_{n_1}) \cdot \det(A_2 - \lambda I_{n_2}).$$

Hence, λ is an eigenvalue of A if and only if it is an eigenvalue of A_1 or A_2. The result follows. □

Theorem 5. *Let G_1 and G_2 be fuzzy graphs with disjoint vertex sets and adjacency matrices $A(G_1) \in \mathbb{R}^{n_1 \times n_1}$, $A(G_2) \in \mathbb{R}^{n_2 \times n_2}$. Let $G = G_1 \cup G_2$ denote their union. Then*

$$\lambda_{\max}(G) = \max\{\lambda_{\max}(G_1), \lambda_{\max}(G_2)\}, \quad \lambda_{\min}(G) = \min\{\lambda_{\min}(G_1), \lambda_{\min}(G_2)\}.$$

Proof. Since the vertex sets of G_1 and G_2 are disjoint, the adjacency matrix of G is the block diagonal matrix:

$$A(G) = \begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix}.$$

By the Lemma 4, the spectrum of $A(G)$ is the union of the spectra of $A(G_1)$ and $A(G_2)$. Hence, the largest and smallest eigenvalues of $A(G)$ are the maximum and minimum, respectively, of those from $A(G_1)$ and $A(G_2)$, which proves the result. □

Example 5. Let G_1 be a fuzzy graph with adjacency matrix:

$$A(G_1) = \begin{bmatrix} 0 & 0.8 & 0.4 \\ 0.8 & 0 & 0.5 \\ 0.4 & 0.5 & 0 \end{bmatrix}.$$

The eigenvalues of $A(G_1)$ are approximately $\lambda_1(G_1) = 1.1523$, $\lambda_2(G_1) = -0.3433$, and $\lambda_3(G_1) = -0.8090$.

Let G_2 be a fuzzy graph with adjacency matrix:

$$A(G_2) = \begin{bmatrix} 0 & 0.6 & 0.3 \\ 0.6 & 0 & 0.4 \\ 0.3 & 0.4 & 0 \end{bmatrix}.$$

The eigenvalues of $A(G_2)$ are approximately $\lambda_1(G_2) = 0.8796$, $\lambda_2(G_2) = -0.2674$, and $\lambda_3(G_2) = -0.6122$.

Then the adjacency matrix of $G = G_1 \cup G_2$ is

$$A(G) = \begin{bmatrix} 0 & 0.8 & 0.4 & 0 & 0 & 0 \\ 0.8 & 0 & 0.5 & 0 & 0 & 0 \\ 0.4 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.3 \\ 0 & 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0.3 & 0.4 & 0 \end{bmatrix}.$$

The eigenvalues of $A(G)$ are approximately:

$$\{1.1523, 0.8796, -0.2674, -0.3433, -0.6122, -0.8090\}.$$

Thus, $\lambda_{\max}(G) = \max\{1.1523, 0.8796\} = 1.1523$, $\lambda_{\min}(G) = \min\{-0.2674, -0.3433, -0.6122, -0.8090\} = -0.8090$.

Theorem 6. *Let K_n be a complete fuzzy graph on $n \geq 2$ vertices, with adjacency matrix $A(K_n) = [a_{ij}] \in \mathbb{R}^{n\times n}$, where $a_{ij} \geq 0$ for $i \neq j$, and $a_{ii} = 0$. Define $\mu := \max_{1\leq i<j\leq n} a_{ij}$. Then,*

$$\lambda_1(K_n) \leq (n-1)\mu.$$

Proof. Since K_n is complete and fuzzy, all off-diagonal entries of $A(K_n)$ are non-negative and bounded above by μ, and all diagonal entries are zero. For each row i,

$$\sum_{j\neq i} a_{ij} \leq (n-1)\mu.$$

By Theorem 1, the largest eigenvalue satisfies

$$\lambda_1(K_n) \leq \max_i \sum_{j\neq i} a_{ij} \leq (n-1)\mu.$$

Alternatively, by the Rayleigh quotient for any unit vector v,

$$\lambda_1(K_n) = \max_{v\neq 0} \frac{v^T A(K_n) v}{v^T v}.$$

Choose $v = \frac{1}{\sqrt{2}}(1, 1, 0, \ldots, 0)^T$, which is supported on two vertices. Then

$$v^T A(K_n) v = \frac{1}{2}(a_{12} + a_{21}) = a_{12} \leq \mu.$$

Therefore, this test vector gives $\lambda_1(K_n) \geq a_{12}$, but the upper bound remains $\lambda_1(K_n) \leq (n-1)\mu$. This confirms the bound. □

Example 6. Let K_5 be a complete fuzzy graph with $\sigma_i = 1$ for all vertices v_i as in Figure 5 with the following adjacency matrix $A(K_5)$:

$$A(K_5) = \begin{bmatrix} 0 & 0.8 & 0.7 & 0.6 & 0.9 \\ 0.8 & 0 & 0.5 & 0.4 & 0.7 \\ 0.7 & 0.5 & 0 & 0.9 & 0.6 \\ 0.6 & 0.4 & 0.9 & 0 & 0.8 \\ 0.9 & 0.7 & 0.6 & 0.8 & 0 \end{bmatrix}.$$

The maximum membership value is $\max_{1\leq i<j\leq 5}\{a_{ij}\} = 0.9$. According to the theorem, $(n-1)\mu = (5-1)(0.9) = 3.6$. The eigenvalues are found to be $\lambda_i = \{2.774, -0.198, -0.677, -0.842, -1.055\}$ implying $\lambda_1(K_n) \leq (n-1)\mu$.

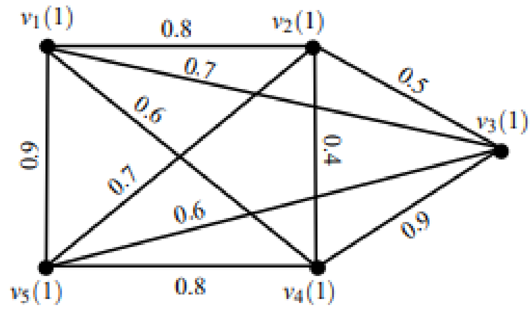

Figure 5: Complete fuzzy graph K_5

Corollary 2. *Let K_n be a complete fuzzy graph on $n \geq 2$ vertices, and suppose the maximum edge weight is μ, attained uniformly in atleast one row of the adjacency matrix $A(K_n)$, i.e.,*

$$\sum_{j \neq i} a_{ij} = (n-1)\mu \quad \text{for some } i.$$

Then the largest eigenvalue satisfies

$$\lambda_1(K_n) = (n-1)\mu.$$

Proof. From the Theorem 6, $\lambda_1(K_n) \leq (n-1)\mu$. By the Perron- Frobenius theorem for non-negative symmetric matrices, the spectral radius $\lambda_1(K_n)$ is equal to the maximum row sum. If there exists a row with sum exactly $(n-1)\mu$, then

$$\lambda_1(K_n) = \max_i \sum_{j \neq i} a_{ij} = (n-1)\mu. \qquad \square$$

Lemma 5. *Let $A \in \mathbb{R}^{p \times q}$ be a non-negative matrix with atmost m non-zero entries (total number of edges), each bounded above by $\mu_{\max}$. Then*

$$\|A\|_F \leq \mu_{\max}\sqrt{m}.$$

Proof. The Frobenius norm of a matrix $A = [a_{ij}] \in \mathbb{R}^{p \times q}$ is defined by

$$\|A\|_F = \sqrt{\sum_{i=1}^{p}\sum_{j=1}^{q} a_{ij}^2}.$$

Suppose A has at most m non-zero entries and that for all i, j, we have $0 \leq a_{ij} \leq \mu_{\max}$. Then

$$\sum_{i,j} a_{ij}^2 \leq m \cdot \mu_{\max}^2,$$

since each non-zero term $a_{ij}^2 \leq \mu_{\max}^2$ and there are atmost m such terms. Taking square roots on both sides yields:

$$\|A\|_F \leq \mu_{\max}\sqrt{m}. \qquad \square$$

Theorem 7. *Let G be a bipartite fuzzy graph with edge membership function $\mu : E(G) \to [0,1]$, where each edge weight satisfies $\mu \leq \mu_{\max}$, and let the number of edges be m. Then the largest eigenvalue $\lambda_1(G)$ of its adjacency matrix A satisfies*

$$\lambda_1(G) \leq \mu_{\max}\sqrt{m}.$$

Proof. Let A be the adjacency matrix of the bipartite fuzzy graph G. Since A is symmetric and non-negative, the Perron-Frobenius theorem ensures that its spectral radius $\rho(A)$ equals its largest eigenvalue:

$$\lambda_1(G) = \rho(A).$$

For any matrix norm $\|\cdot\|$, it holds that $\rho(A) \leq \|A\|$. In particular, the Frobenius norm gives,

$$\rho(A) \leq \|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2}.$$

Since A has atmost m non-zero entries and each $a_{ij} \leq \mu_{\max}$, we have

$$\sum_{i,j} a_{ij}^2 \leq m \cdot \mu_{\max}^2,$$

so

$$\|A\|_F \leq \mu_{\max}\sqrt{m}.$$

Therefore

$$\lambda_1(G) = \rho(A) \leq \|A\|_F \leq \mu_{\max}\sqrt{m}.$$

□

Example 7. Consider a bipartite fuzzy graph G as in Figure 6 with vertex sets $U = \{u_1, u_2, u_3\}$ and $V = \{v_1, v_2\}$. Let the adjacency matrix A be

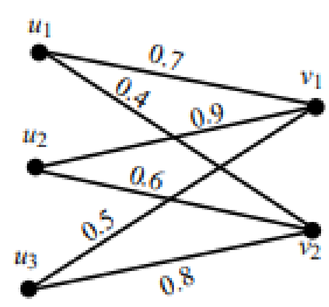

Figure 6: Bipartite fuzzy graph G with U and V.

$$A = \begin{bmatrix} 0.7 & 0.4 \\ 0.9 & 0.6 \\ 0.5 & 0.8 \end{bmatrix}.$$

In this representation

- The maximum edge weight is $\mu_{\max} = 0.9$.
- The number of edges is $m = 6$.
- The largest eigenvalue $\lambda_1(G) = 1.6095$. Although A is not a square matrix, eigenvalues are computed based on singular values. The largest singular value is an upper bound on the magnitude of the largest eigenvalue.

According to the theorem, the largest eigenvalue $\lambda_1(G)$ of the graph is bounded above by,
$\lambda_1(G) \leq \mu_{\max}\sqrt{m}$ implying $1.6095 \leq 2.205$.

Corollary 3. *Let $G = K_{m,n}$ be a complete bipartite fuzzy graph with vertex partitions of sizes m and n, and edge membership function $\mu : E(G) \to [0,1]$, such that $\mu \leq \mu_{\max}$ for all edges. Then the largest eigenvalue $\lambda_1(G)$ of its adjacency matrix satisfies*

$$\lambda_1(G) \leq \mu_{\max}\sqrt{mn}.$$

Proof. In a complete bipartite fuzzy graph $K_{m,n}$, there are exactly $m \times n$ edges. Each edge has weight atmost $\mu_{\max}$, so by applying the Theorem 7 with $m = mn$, we obtain

$$\lambda_1(G) \leq \mu_{\max}\sqrt{mn}.$$

□

Example 8. Consider the complete bipartite fuzzy graph $G = K_{2,3}$, where each edge has a constant membership value $\mu = 0.6$. The adjacency matrix A of G is given by

$$A = \begin{bmatrix} 0 & 0 & 0.6 & 0.6 & 0.6 \\ 0 & 0 & 0.6 & 0.6 & 0.6 \\ 0.6 & 0.6 & 0 & 0 & 0 \\ 0.6 & 0.6 & 0 & 0 & 0 \\ 0.6 & 0.6 & 0 & 0 & 0 \end{bmatrix}.$$

This is a 5×5 symmetric matrix with vertex partitions of sizes $m = 2$ and $n = 3$.

There are $m \cdot n = 2 \cdot 3 = 6$ edges in total, each of weight $\mu_{\max} = 0.6$. By the corollary, the largest eigenvalue $\lambda_1(G)$ satisfies

$$\lambda_1(G) \leq \mu_{\max}\sqrt{mn} = 0.6 \cdot \sqrt{2 \cdot 3} = 0.6 \cdot \sqrt{6} \approx 0.6 \cdot 2.45 = 1.47.$$

Hence, the spectral radius (largest eigenvalue) is bounded above by approximately 1.47.

5 Real world application

5.1 Protein- protein interactions and biological complexity

Proteins are fundamental biological macromolecules composed of chains of amino acids. They perform a vast range of functions within living organisms acting as enzymes, structural scaffolds, signaling molecules, and transporters. However, proteins rarely act alone. Much of their functionality arises through interactions with other proteins, forming dynamic and interconnected systems known as PPI networks. These interactions can be stable or transient, context-dependent, and highly regulated, making them central to the coordination of cellular processes.

5.2 The role of fuzziness in modeling protein interactions

While protein-protein interactions are essential to understanding cellular behavior, capturing them precisely is challenging. Experimental data from high-throughput technologies such as yeast two-hybrid assays or mass spectrometry often contain uncertainty, variability, or incomplete information. As a result, it is rarely possible to state with full certainty whether a given interaction is present, absent, strong, or weak. This inherent ambiguity motivates the use of fuzzy graph models, which allow each interaction to be represented by a degree of membership rather than a binary value. Fuzziness, in this context, quantifies the confidence or reliability of the observed interaction, offering a more faithful representation of biological complexity.

5.3 Relevance of fuzzy graphs in systemic lupus erythematosus

Systemic Lupus Erythematosus (SLE) is a chronic autoimmune disease in which the immune system becomes misdirected and attacks healthy tissues. This misregulation is driven by abnormal protein interactions, particularly among immune signaling pathways. However, due to the heterogeneity of the disease and the variability in patient responses, data on these interactions is often uncertain or incomplete. Fuzzy graphs provide a mathematical framework to handle such imprecision. By incorporating degrees of interaction rather than fixed connections, fuzzy models enable researchers to map SLE-related PPI networks more accurately, identify potential biomarkers, and study the stability and resilience of immune signaling under varying conditions.

5.4 Implementation in PPI networks

This research investigated the protein-protein interaction PPI network associated with SLE, focusing on seven key proteins- AHSG, IGF1, IGF2, IGFBP3, ORM1, ORM2 and SERPINA 1- frequently implicated in the disease. Initial data collection involved retrieving interaction information and associated confidence scores from the UniProt and STRING databases.

While STRING initially provided a complex network encompassing interactions beyond just these seven proteins, the analysis was focused on the interactions among these, as they were considered the most crucial factors in the disease process. Therefore, all interactions involving only these proteins were extracted to simplify the network and concentrate on their direct relationships within the SLE context.

To facilitate analysis, the resulting refined PPI network was modeled as a fuzzy graph, where the confidence scores from the STRING database served as weights, reflecting the uncertainty inherent in the interaction data. This weighted fuzzy graph was then represented as an adjacency matrix, a mathematical structure particularly well- suited for network analysis. All analyses were conducted using the NetworkX library within the Python programming environment.

This adjacency matrix is presented below:

$$A(G) = \begin{bmatrix} 0 & 0 & 0.454 & 0.463 & 0.971 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.499 & 0.999 \\ 0.454 & 0 & 0 & 0.999 & 0.972 & 0 & 0 \\ 0.463 & 0 & 0.999 & 0 & 0.979 & 0 & 0 \\ 0.971 & 0 & 0.972 & 0.979 & 0 & 0 & 0 \\ 0 & 0.499 & 0 & 0 & 0 & 0 & 0.999 \\ 0 & 0.999 & 0 & 0 & 0 & 0.999 & 0 \end{bmatrix}.$$

Each cell within the matrix represents the weighted interaction strength between a pair of proteins, derived directly from the confidence scores provided by STRING after refinement.

Eigenvalue analysis was then performed on the adjacency matrix, offering crucial insights into the overall structure and stability of the refined SLE protein interaction network. The eigenvalues detailed below, reveal key characteristics of the network's connectivity and the influence of individual proteins: $\lambda_1 \approx 2.4590$, $\lambda_2 \approx 1.6842$, $\lambda_3 \approx -0.2723$, $\lambda_4 \approx -0.499$, $\lambda_5 \approx -0.999$, $\lambda_6 \approx -1.1852$, $\lambda_7 \approx -1.1876$.

A visual representation of this simplified network, emphasizing the direct relationships between each pair of proteins based on the refined adjacency matrix, is shown in Figure 7.

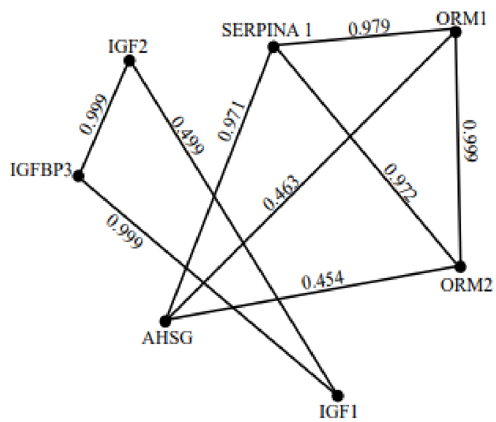

Figure 7: PPI Network of crucial proteins in Systemic Lupus Erythematosus

Further analysis involved calculating the degree centrality of each protein node, a measure of its direct connectivity within the network. Now, we calculate the degrees of each proteins to establish bounds on the energy. The degrees are computed as follows:

Degree of AHSG $= 1.888$; Degree of IGF1 $= 1.498$; Degree of ORM2 $= 2.245$; Degree of ORM1 $= 2.441$; Degree of SERPINA 1 $= 2.922$; Degree of IGF2 $= 1.498$; Degree of IGFBP3 $= 1.998$.

The maximum degree calculated is:

$$\text{Max Degree } (\Delta) = \max\{1.888, 1.498, 2.245, 2.441, 2.922, 1.498, 1.998\} = 2.922.$$

This, combined with the eigenvalues obtained from the adjacency matrix, allowed for the calculation of the network energy - a metric reflecting the overall stability and connectivity of the protein interaction network.

The energy of the fuzzy graph, calculated as the sum of the absolute values of the eigenvalues, is derived as follows:

$$\text{Energy} = |\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4| + |\lambda_5| + |\lambda_6| + |\lambda_7| = 8.2863.$$

Finally, the bounds derived defines the expected range of various network properties based on the network energy and its constituent components are tabulated in Table 2.

These bounds provide a critical framework for interpreting the structure and function of the refined network.

Table 2: Comparison of bounds across different methods

Method	**Δ (Max. Degree)**	**LEUB ($\lambda_1 \leq$)**	**SELB ($\lambda_n \geq$)**	**LELB ($\lambda_1 \geq$)**
Proposed Fuzzy Method	2.922	2.922	-2.922	2.0957
Weighted Graph Analysis	2.922	2.4590	-1.1876	N/A

Note: LEUB = Largest Eigenvalue Upper Bound; SELB = Smallest Eigenvalue Lower Bound; LELB = Largest Eigenvalue Lower Bound; N/A = Not Applicable; Δ represents the Maximum Degree.

6 Results and discussion

1. This study analyzed the SLE- related PPI network of nine proteins (AHSG, IGF1, IGF2, IGFBP3, ORM1, ORM2, VWF, SERPINA1, LEP) using a fuzzy graph representation weighted by STRING confidence scores.

2. Eigenvalue analysis revealed a largest eigenvalue (λ_1) of approximately 2.4590, with other eigenvalues ranging from 1.6842 to -1.1876.

3. The maximum degree centrality was 2.922, resulting in a network energy of 8.2863.

4. These findings, compared to a weighted graph analysis in Table 2 (showing LEUB, SELB, LELB values), offer insights into the structure and stability of the SLE PPI network (Figure 7).

6.1 Comparative analysis with existing bounds

To contextualize the novelty of the proposed eigenvalue bounds, we compare them against existing results from both classical and spectral fuzzy graph theory. Table 3 summarizes the key differences and numerical bounds for the largest and smallest eigenvalues.

Hong (1988) established an early upper bound $\lambda_1 \leq \Delta$, which is simple but often loose. Kumar (2010) proposed a tighter bound $\lambda_1 \leq \sqrt{2m}$ based on the number of edges and degrees, still rooted in crisp graph theory. However, these do not adapt well to fuzzy weighted structures.

Mahato and Chakraverty (2016) introduced a filtering algorithm for fuzzy eigenvalue estimation using interval analysis and perturbation, but their method does not produce explicit analytical bounds. Similarly, Vimala and Jayalakshmi (2016) focused on fuzzy graph representation and energy calculations, without providing spectral bounds for adjacency matrices.

In contrast, our proposed method yields explicit upper and lower bounds for eigenvalues of fuzzy adjacency matrices by leveraging the Rayleigh quotient and Perron-Frobenius theory within a spectral fuzzy graph framework. As seen in Table 3, our approach provides the tightest analytical bounds directly applicable to fuzzy PPI networks.

7 Conclusion

This paper presents novel theorems that advance the understanding of eigenvalue bounds in fuzzy graphs. By leveraging foundational principles from spectral graph theory, such as the Rayleigh quotient and the Perron-Frobenius theorem, new upper and lower bounds for the largest and smallest eigenvalues of the adjacency matrix are established.

Table 3: Numerical comparison of eigenvalue bounds across methods

Method	Upper Bound for λ_1	Lower Bound for λ_n
Hong (1988)	2.922	N/A
Kumar (2010)	3.806	N/A
Mahato & Chakraverty (2016)	2.4590	-2.4590
Vimala & Jayalakshmi (2016)	N/A	N/A
Proposed Method	**2.922**	**-2.922**

The current study is limited to adjacency matrices of spectral fuzzy graphs, where the adjacency matrices are assumed to be symmetric. Extensions to other matrix forms such as the Laplacian and signless Laplacian matrices remain unexplored and represent a natural progression of this work.

Future research could expand this framework to incorporate Laplacian based eigenvalue bounds and investigate the behaviour of spectral fuzzy graphs in dynamic, weighted, or time-varying network settings. Such extensions would further enhance the applicability of fuzzy spectral methods to real world systems where uncertainty and evolution coexist.

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Conflicts of interest

The authors declare that there are no conflicts of interest.

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