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Tau-Collocation method for weakly singular Volterra integral equations and related special cases

Sedaghat Shahmorad[†][*], Mahdi Mostafazadeh[†], Fevzi Erdogan[‡]

[†] *Department of Applied Mathematics, University of Tabriz, Tabriz*
[‡] *Department of Mathematics, Van Yuzuncu Yil University, Van, Turkey*

Email(s): shahmorad@tabrizu.ac.ir, m.mostafazadeh@tabrizu.ac.ir, ferdogan@yyu.edu.tr

Abstract. The present study examines the implementation of the tau-collocation method for solving a class of Volterra integral equations and related cases which their kernels contain (special) weak singularity of type $(x^2-s^2)^{-1/2}$. These types of equations can be written in the form of the so-called *cordial* Volterra integral equations and so inherit their properties. We will recall some conditions on the kernel and forcing function for which the existence and uniqueness of a solution has been proven. Then we will discuss regularity conditions for the solution of same types equations which indicate that unlike the standard Volterra integral equations with singularity of the form $(x-s)^{-\alpha}$, $0<\alpha<1$, these types of equations have regular solutions if the kernel and forcing functions are sufficiently smooth. This property allows us to use the classical Jacobi polynomials as a basis functions for collocation method. For this method, we will first derive a matrix formulation that makes it easy to implement. We will prove convergence of the method by providing an error bound.

Keywords: Tau-collocation method, cordial Volterra integral equations, weak singularity.
AMS Subject Classification 2010: 45D05.

1 Introduction

Throughout this work, we study the numerical solution of two types of weakly singular Volterra integral equations (VIEs). First, we consider a linear Abel–Volterra integral equation of the form

$$y(x)+\int_0^x (x-s)^{\alpha-1}k(x,s)y(s)\,\mathrm{d}s=h(x),\quad x\in\Omega=[0,1], \tag{1}$$

where $0<\alpha\leq 1$, and $h(x)$ and $k(x,s)$ are given smooth functions on Ω.

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For $0 < \alpha < 1$, the solution of this equation typically presents a weak singularity at the origin, even when the non-homogeneous term h is smooth. Consequently, spectral methods based on standard polynomial basis such as Chebyshev, Legendre, or Jacobi polynomials yield numerical solutions with low accuracy due to their inability to capture the singular behavior effectively.

In the second part, we consider a first–kind VIE with a weakly singular kernel of the form $(x^\gamma - s^\gamma)^{-\alpha}$, where $\gamma \geq 1$ and $0 < \alpha < 1$. This form was explored in a seminal paper by K. E. Atkinson in 1974 [2].

Atkinson's work motivated Vainikko to establish a new class of VIEs, now called Cordial VIEs (CVIEs). In two fundamental papers published in 2009 and 2010 [14, 15], Vainikko discussed the compactness properties of the cordial integral operator

$$\mathscr{V}u(x) = \int_0^x x^{-1}\varphi(x/s)k(x,s)u(s)\,\mathrm{d}s, \quad \varphi \in L^1(0,1),$$

and also explored numerical solutions to second-kind CVIEs of the form

$$u(x) = f(x) + \mathscr{V}u(x).$$

The CVIEs are related to a range of other Volterra equations. For example, VIEs with a boundary weakly singular integral operator, i.e.,

$$\mathscr{H}u(x) = \int_0^x x^{-\gamma}s^{\gamma-1}k(x,s)u(s)\,\mathrm{d}s, \quad 0 < \gamma < 1,$$

are related to CVIEs with the L^1 core $\varphi(x) = x^{\gamma-1}$. Similarly, the so–called third-kind VIEs,

$$x^\alpha u(x) = f(x) + \int_0^x (x-s)^{\alpha-1}k(x,s)u(s)\,\mathrm{d}s, \quad 0 < \alpha < 1,$$

can be expressed in terms of cordial operators as

$$u(x) = x^{-\alpha}f(x) + \mathscr{V}u(x),$$

with the core $\varphi(x) = (1-x)^{\alpha-1}$.

In a part of this study, we apply the tau–collocation method to VIEs that have weak singularities of the type $(x^2 - s^2)^{-1/2}$. Despite their weakly singular kernels, these equations can, under certain conditions, possess smooth solutions. Therefore, the use of standard Jacobi polynomial bases for the approximation space appears justified. Consequently, we use these polynomials and analyze the error bounds and convergence properties of our method.

Therefore, we motivated to propose the following in this paper:

- develop the tau-method in terms of fractional Jacobi polynomials (Müntz–Jacobi polynomials) as basis functions, which provide superior approximation properties for functions with singularities at the origin.

- construct a new matrix formulation in terms of these fractional polynomials (Theorem 1).

- use special collocation points to remove reminder term.

2 Preliminaries

2.1 Jacobi polynomials

We begin by recalling the definition of Jacobi polynomials, denoted by $P_n^{\nu,\zeta}(x)$, which are orthogonal with respect to the weight function $\omega^{(\nu,\zeta)}(x) = (1-x)^{\nu}(1+x)^{\zeta}$ on the interval $[-1,1]$. That is, they satisfy the orthogonality condition

$$\int_{-1}^{1} P_n^{\nu,\zeta}(x) P_l^{\nu,\zeta}(x)\omega^{(\nu,\zeta)}(x)\,\mathrm{d}x = \tilde{z}_n^{(\nu,\zeta)}\delta_{ln}, \quad l,n = 0,1,2,\ldots,$$

where δ_{ln} is the Kronecker delta function, and

$$\tilde{z}_n^{(\nu,\zeta)} = \|P_n^{\nu,\zeta}\|^2_{L^2_{\omega^{\nu,\zeta}}} = \frac{2^{\nu+\zeta+1}\Gamma(n+\nu+1)\Gamma(n+\zeta+1)}{\Gamma(n+\nu+\zeta+1)(2n+\nu+\zeta+1)n!}.$$

These polynomials have the explicit form

$$P_n^{\nu,\zeta}(x) = 2^{-n}\sum_{i=0}^{n}\binom{n+\nu}{i}\binom{n+\zeta}{n-i}(x-1)^{n-i}(x+1)^{i}. \tag{2}$$

For more details, we refer the interested reader to [13] .

2.2 Fractional Jacobi polynomials

Let $J_n^{\nu,\zeta}(x) := P_n^{\nu,\zeta}(2x-1)$, for $\nu,\zeta > -1$, denote the shifted Jacobi polynomials of degree n on the interval Ω. Then for $0 < \beta \leq 1$, the fractional Jacobi polynomials, as introduced in [3, 8], are defined as

$$J_i^{\nu,\zeta,\beta}(x) = J_i^{\nu,\zeta}(x^{\beta}). \tag{3}$$

These fractional polynomials satisfy an orthogonality condition in the form

$$\int_0^1 J_l^{\nu,\zeta,\beta}(x) J_n^{\nu,\zeta,\beta}(x)\chi^{\nu,\zeta,\beta}(x)\,\mathrm{d}x = \lambda_i^{(\nu,\zeta)}\delta_{ln}, \tag{4}$$

where $\chi^{\nu,\zeta,\beta}(x) = \beta(1-x^{\beta})^{\nu}x^{(\zeta+1)\beta-1}$ and

$$\lambda_i^{(\nu,\zeta)} = \frac{\Gamma(i+\nu+1)\Gamma(i+\zeta+1)}{(2i+\nu+\zeta+1)i!\Gamma(i+\nu+\zeta+1)}. \tag{5}$$

Regarding (2) and (3), an explicit formula for $J_i^{\nu,\zeta,\beta}(x)$ is derived as follows [4]:

$$J_i^{\nu,\zeta,\beta}(x) = \sum_{\ell=0}^{i}\underbrace{\frac{(-1)^{i-\ell}\Gamma(i+\ell+\nu+\zeta+1)\Gamma(i+\zeta+1)}{\Gamma(i+\nu+\zeta+1)\Gamma(\ell+\zeta+1)(i-\ell)!\ell!}}_{=:h_{\ell}^{\alpha,\beta,i}}x^{\ell\beta}. \tag{6}$$

Let us define the fractional interpolation operator, I_n^{β}, by

$$I_n^{\beta}(u;\nu,\zeta)(x) = \sum_{l=0}^{n}\mathscr{L}_l^{(\nu,\zeta,\beta)}(x)u(x_l),$$

where the fractional Lagrange basis polynomials $\mathscr{L}_i^{(\nu,\zeta,\beta)}$ are defined as follows:

$$\mathscr{L}_i^{(\nu,\zeta,\beta)}(x) := \prod_{l=0, l\neq i}^{n} \frac{x^\beta - x_l^\beta}{x_i^\beta - x_l^\beta}, \quad x \in \Omega, \quad i = 0,1,\ldots,n,$$

with the interpolation nodes $\{x_l\}$, $l = 0,1,\ldots,n$, as the roots of fractional Jacobi polynomial $J_{n+1}^{\nu,\zeta,\beta}(x)$. For $\beta = 1$ we have the classical Lagrange interpolation operator $\left(I_n := I_n^1 \text{ with } \mathscr{L}_i^{(\nu,\zeta)}(x) := \mathscr{L}_i^{(\nu,\zeta,1)}(x)\right)$.

For $\nu = 0$ and $\zeta = \frac{1}{\beta} - 1$, we have a special class of these fractional polynomials [8], called "Müntz-Jacobi polynomials", i.e.,

$$J_i^\beta(x) := J_i^{0,\frac{1}{\beta}-1,\beta}(x) = \sum_{\ell=0}^{i} \frac{(-1)^{i-\ell}\Gamma(i+\ell+\frac{1}{\beta})}{\Gamma(\ell+\frac{1}{\beta})(i-\ell)!\ell!} x^{\ell\beta}, \tag{7}$$

which satisfy the orthogonality condition

$$\int_0^1 J_i^\beta(x) J_j^\beta(x)\,\mathrm{d}x = \frac{1}{(2i\beta+1)}\delta_{ij}, \quad 0 < \beta \leq 1. \tag{8}$$

The Müntz-Jacobi polynomials will be used as our basis functions in the next section.

3 The tau–collocation method

Here, we formulate the operational approach of the Tau-Collocation Method (TCM) to solve the integral equation (1). We first prove the following theorem to write the integral part of Eq. (1) in a matrix–vector multiplication form.

Theorem 1. *Let the approximate solution of Eq. (1) in terms of the Müntz–Jacobi polynomials be presented in the form of:*

$$y_N(x) = \sum_{i=0}^{N} a_i J_i^\beta(x) = Y_N \Phi(x) = Y_N \Phi X_x, \quad 0 < \beta \leq 1, \tag{9}$$

where

$$Y_N := [a_0, a_1, a_2, \cdots, a_N, 0, \ldots], \;\; \Phi(x) := [J_0^\beta(x), J_1^\beta(x), \ldots]^T, \;\; X_x := [1, x^\beta, \ldots]^T \tag{10}$$

and Φ is a non-singular lower triangular matrix. Then, we have

$$\int_0^x (x-s)^{\alpha-1} k(x,s) y_N(s)\,\mathrm{d}s = Y_N \mathscr{W}(x),$$

where

$$\mathscr{W}(x) := \sum_{j=0}^{N} \omega_j x^\alpha k(x, xu_j)\Phi(xu_j)$$

and u_j are the roots of Jacobi polynomial $J_{N+1}^{\alpha-1,0}(x)$ with the associated Gauss quadrature weights

$$\omega_j = \frac{2N+\alpha+1}{2(N+\alpha) J_N^{\alpha-1,0}(u_j)\partial_x J_{N+1}^{\alpha-1,0}(u_j)}.$$

Proof. We have

$$\int_0^x (x-s)^{\alpha-1} k(x,s) y_N(s)\,\mathrm{d}s = Y_N \int_0^x (x-s)^{\alpha-1} k(x,s)\Phi(s)\,\mathrm{d}s$$
$$= Y_N x^{\alpha} \int_0^1 (1-u)^{\alpha-1} k(x,xu)\Phi(xu)\,\mathrm{d}u. \tag{11}$$

Applying the Gauss-Jacobi quadrature rule to the right-hand side integral in (11), yields the approximation

$$\int_0^1 (1-u)^{\alpha-1} k(x,xu)\Phi(xu)\,\mathrm{d}u \simeq \sum_{j=0}^{N} \omega_j k(x,xu_j)\Phi(xu_j). \tag{12}$$

Substituting (12) in Eq. (11), implies

$$\int_0^x (x-s)^{\alpha-1} k(x,s) y_N(s)\,\mathrm{d}s = Y_N \sum_{j=0}^{N} x^{\alpha} \omega_j k(x,xu_j)\Phi(xu_j), \tag{13}$$

which completes the proof. □

Now, using the matrix formulations from Theorems 1 in Eq. (1), we get a perturbed equation (tau equation) as follows (note that $y_N(x)$ is an approximation to $y(x)$) [3]:

$$Y_N\Big(\Phi(x) - \mathscr{W}(x)\Big) = h(x) + \left(\sum_{i=0}^{\infty} \widehat{\tau}_i t^{i\beta}\right) J_{N+1}^{\beta}(x). \tag{14}$$

Collocating Eq. (14) at $x_\ell,\ \ell = 0,\ldots,N$, being the roots of the $J_{N+1}^{\theta}(x)$, yields

$$Y_N \varpi(x_\ell) = h(x_\ell) + \underbrace{\left(\sum_{i=0}^{\infty} \widehat{\tau}_i x_\ell^{i\beta}\right) J_{N+1}^{\beta}(x_\ell)}_{=0}, \tag{15}$$

where $\varpi(x) := \Phi(x) - \mathscr{W}(x)$. From (15) a system of linear algebraic equations is constructed in the following form:

$$Y_N \underbrace{[\varpi(x_0)|\varpi(x_1)|\ldots|\varpi(x_N)]}_{G} = \underbrace{[h(x_0), h(x_1), \ldots, h(x_N)]}_{H}, \tag{16}$$

in which G is an $(N+1)\times(N+1)$ matrix and H is an $N+1$ vector. The unknown vector Y_N is then determined by solving system (16).

The convergence analysis of this method using Müntz-Jacobi polynomials as basis functions has been investigated in [3]. For other variants of Jacobi polynomials, convergence results can be found in [6, 9]. For an application of the tau method, see, for instance, [1], where the authors have used this method to solve a special fractional delay differential equation.

Example 1. Consider a weakly singular VIE in the following form:

$$y(x) = \sqrt{x} + \frac{\pi}{2}x^3 + \frac{3\pi}{4}x + \frac{5\pi}{16} - \int_0^x \frac{(x+s)^2 y(s)\,\mathrm{d}s}{\sqrt{x-s}}, \quad x \in \Omega,$$

with the given exact solution $y(x) = \sqrt{x}$.

Table 1: The values of E_{max}

N	E_{max}
5	7.5233e-05
10	1.3615e-05
20	2.0378e-06
30	6.4179e-07

We applied our method to this equation and obtained maximum absolute errors for $N = 5, 10, 20, 30$ as shown in Table 1, where E_{max} is defined by $E_{max} = \max_{x\in\Omega} |y(x) - y_N(x)|$ for the exact and approximate solutions $y(x)$ and $y_N(x)$, respectively.

4 The CVIEs

In this section, we describe application of TCM to VIEs of the form

$$y(x) = f(x) + \int_0^x (x^2 - s^2)^{-1/2} k(x,s) y(s) \mathrm{d}s, \quad x \in \Omega, \tag{17}$$

that have certain weak singularities in their kernel functions. These types of equations arise in some mathematical modeling processes and can be written in the form

$$y(x) = f(x) + \int_0^x x^{-1} \psi(s/x) k(x,s) y(s)\, \mathrm{d}s, \tag{18}$$

where $\psi(t) = (1 - t^2)^{-1/2}$. This form is called a CVIE, as described in the second part of introduction section. We first derive matrix formulation for the Jacobi spectral TCM and then we recall some regularity properties of Eq. (17). Finally, we obtain error bound in L^∞–norm for the proposed method.

To solve Eq. (17) by TCM, we apply the variable transformation $s = \frac{s^2}{x^2}$, to reduce it to the following equivalent form:

$$y(x) = f(x) + \frac{1}{2} \int_0^1 (1-s)^{-1/2} s^{-1/2} k(x, xs^{1/2}) y(xs^{1/2})\, \mathrm{d}s.$$

The goal of the TCM is to obtain an approximate solution $y_N(x) = \sum_{i=0}^{N} y_i^N J_i^{\alpha,\beta}(x)$ such that

$$y_N(x_l) = f(x_l) + \frac{1}{2} \int_0^1 (1-s)^{-1/2} s^{-1/2} k^N(x_l, x_l s^{1/2}) y_N(x_l s^{1/2})\, \mathrm{d}s, \tag{19}$$

where $\{x_l\}, l = 0, 1, \ldots, N$ are the roots of the shifted Jacobi polynomial $J_{N+1}^{\alpha,\beta}(x)$ and $k^N(\cdot,\cdot)$ denotes the bivariate interpolation of the kernel $k(\cdot,\cdot)$ with respect to the nodes $\{x_l\}$, i.e.,

$$\begin{aligned} k^N(x,s) := I_{N,N}(k_{ij}; \alpha, \beta)(x,s) &= \sum_{\nu_0,\nu_1=0}^{N} k(x_{\nu_0}, s_{\nu_1}) \mathscr{L}_{\nu_0}^{(\alpha,\beta)}(x) \mathscr{L}_{\nu_1}^{(\alpha,\beta)}(s) \\ &= \sum_{\nu_0,\nu_1=0}^{N} \bar{k}^{\nu_0,\nu_1} J_{\nu_0}^{\alpha,\beta}(x) J_{\nu_1}^{\alpha,\beta}(s). \end{aligned}$$

Then, Eq. (19) can be reduced to the following system of algebraic equations:

$$
\begin{aligned}
y_N(x_l) =& f(x_l) + \frac{1}{2} \sum_{\nu_0,\nu_1,\nu_2=0}^{N} \sum_{\sigma_0=0}^{\nu_1} \sum_{\sigma_1=0}^{\nu_2} \bar{k}^{\nu_0,\nu_1} h_{\sigma_0}^{\alpha,\beta,\nu_1} h_{\sigma_1}^{\alpha,\beta,\nu_2} J_{\nu_0}^{\alpha,\beta}(x_l) y_{\nu_2}^N x_l^{\sigma_0+\sigma_1} \\
& \times B\left(1/2, \frac{1}{2}(\sigma_0+\sigma_1)+1/2\right),
\end{aligned}
$$

which has the following matrix-vector multiplication form

$$
\Pi_N \bar{Y}^N = \underline{F}^N,
$$

where

$$
\Pi_N = \begin{bmatrix} \underline{\Phi}_N(x_0) - \bar{K}(x_0) \\ \hline \underline{\Phi}_N(x_1) - \bar{K}(x_1) \\ \hline \vdots \\ \hline \underline{\Phi}_N(x_N) - \bar{K}(x_N) \end{bmatrix}_{(N+1)\times(N+1)}, \quad \underline{F}^N = [f(x_0), f(x_1), \ldots, f(x_N)]^T,
$$

for which, we have $\bar{Y}^N = [y_0^N, y_1^N, \ldots, y_N^N]^T$, $\underline{\Phi}_N(x_l) = \left[1, J_1^{\alpha,\beta}(x_l), \ldots, J_N^{\alpha,\beta}(x_l)\right]$ and

$$
\begin{aligned}
\bar{K}(x_l) &= [\eta_{\nu_2}(x_l)]_{\nu_2=0}^N, \\
\eta_{\nu_2}(x_l) :=& \frac{1}{2} \sum_{\nu_0,\nu_1=0}^{N} \sum_{\sigma_0=0}^{\nu_1} \sum_{\sigma_1=0}^{\nu_2} \bar{k}^{\nu_0,\nu_1} h_{\sigma_0}^{\alpha,\beta,\nu_1} h_{\sigma_1}^{\alpha,\beta,\nu_2} J_{\nu_0}^{\alpha,\beta}(x_l) x_l^{\sigma_0+\sigma_1} \\
& \times B\left(1/2, \frac{1}{2}(\sigma_0+\sigma_1)+1/2\right).
\end{aligned}
$$

We now present the fundamental existence, uniqueness, and regularity results for Eq. (17).

Theorem 2 ([5])**.** *Let $f \in C^m(\Omega)$ and $k \in C^m(D)$ with $m \geq 0$, $D := \{(x,s) :\ 0 \leq s \leq x \leq 1\}$ and $k(0,0) = 0$. Then Eq. (17) has a unique solution $y \in C^m(\Omega)$.*

4.1 Error bounds and convergence

We state a bound for the Lebesgue constant by the following lemma (see [10]).

Lemma 1. *For the Jacobi interpolation operator $I_n(\cdot;\alpha,\beta)$ we have*

$$
\|I_n\|_\infty \leq \begin{cases} O(\log n), & -1 < \alpha, \beta \leq -1/2, \\ O(n^{\max(\alpha,\beta)+1/2}), & \textit{otherwise (o.w.)}. \end{cases}
$$

The following lemma, provides an error bound for the Lagrange interpolation operator I_n^λ, which shows that the convergence rate depends on the smoothness index m and the Jacobi parameters α and β.

Lemma 2 ([7])**.** *If $v \in H_m^{(\alpha,\beta,\lambda)}(\Omega)$, where*

$$H_m^{(\alpha,\beta,\lambda)}(\Omega) = \{v(x) \in L^2_{\chi^{\alpha,\beta,\lambda}}(\Omega) : \partial_x^k v(x^{1/\lambda}) \in L^2_{\chi^{\alpha,\beta}}(\Omega), k = 0,1,\ldots,m\} \quad 0 < \lambda < 1,$$

then

$$\|v - I_n^\lambda(v;\alpha,\beta)\|_\infty \le C \begin{cases} n^{\frac{3}{4}-m}\|\partial_x^m v(x^{1/\lambda})\|_{L^2_{\chi^{\alpha,\beta}}}, & \alpha=\beta=0,\\ n^{\frac{1}{2}-m}\|\partial_x^m v(x^{1/\lambda})\|_{L^2_{\chi^{\alpha,\beta}}}, & \alpha=\beta=-\frac{1}{2},\\ (1+\log n)n^{\frac{1}{2}-m}\|\partial_x^m v(x^{1/\lambda})\|_{L^2_{\chi^{\alpha,\beta}}}, & -1<\alpha,\beta<-\frac{1}{2},\\ (1+n^{\max(\alpha,\beta)+\frac{1}{2}})n^{\frac{1}{2}-m}\|\partial_x^m v(x^{1/\lambda})\|_{L^2_{\chi^{\alpha,\beta}}}, & o.w. \end{cases}$$

in which the constant $C > 0$ is independent of n, and $\chi^{\alpha,\beta} := \chi^{\alpha,\beta,1}$. The classical case of this lemma is recovered when $\lambda = 1$.

We denote by $C^{m,\tau}(\Omega)$ the space of all functions whose m-th derivatives are Hölder continuous of order τ, equipped with the standard norm $\|\cdot\|_{m,\tau}$ (see [13]).

Lemma 3 ([12])**.** *For each integer $m \ge 0$ and $0 \le \tau \le 1$, there exist a constant $C = C(m,\tau) > 0$ such that for any functions $v \in C^{m,\tau}(\Omega)$, there is an operator $\mathscr{W}_n : C^{m,\tau}(\Omega) \to \mathscr{P}_n$ for which we have*

$$\|v - \mathscr{W}_n v\|_\infty \le C n^{-(m+\tau)}\|v\|_{m,\tau},$$

where $\mathscr{P}_n$ stands for the space of all polynomials with degree not exceeding n.

The next lemma, presents a Grönwall type inequality related to CVIEs.

Lemma 4 (Grönwall inequality for CVIEs, see [5])**.** *Let $f, g \in C(\Omega)$ be non-negative functions with $g(x)$ additionally assumed to be non-decreasing. Suppose that for all $x \in \Omega$ we have*

$$f(x) \le g(x) + M\int_0^x x^{-1}\psi(s/x)f(s)\,\mathrm{d}s,$$

where $M > 0$, $\psi(x) \ge 0$ for $x \in (0,1)$, $\psi \in L^1(0,1)$ and $M\int_0^1 \psi(s)\,\mathrm{d}s < 1$. Then the following bound holds:

$$f(x) \le \frac{g(x)}{1 - M\int_0^1 \psi(s)\,\mathrm{d}s}, \quad x \in \Omega.$$

We define the operator $\mathscr{H} : C^m(\Omega) \to C^m(\Omega)$ (see [15]) as

$$\mathscr{H}\phi(x) = \int_0^x x^{-1}\psi(s/x)k(x,s)\phi(s)\,\mathrm{d}s,$$

where $\psi(t) = (1-t^2)^{-1/2}$. This operator is compact if $k(0,0) = 0$ and non-compact if $k(0,0) \ne 0$. Since

$$\int_0^1 (1-s^2)^{\bar{\mu}}\,\mathrm{d}s = \frac{\sqrt{\pi}}{2}\frac{\Gamma(1+\bar{\mu})}{\Gamma(\bar{\mu}+\frac{3}{2})}, \quad -1 < \bar{\mu}, \tag{20}$$

$\psi \in L^p(\Omega)$ for $1 < p < 2$.

Lemma 5. Assume that the following hypotheses are true for $0<\mu\le 1$:
I: $k\in C(D)$ and $k(0,0)=0$,
II: $|k(x_1,s)-k(x_2,s)|\le L_1|x_1-x_2|^{\mu}$, $L_1>0$, and $x_1,x_2\in\Omega$,
III: $|k(x,s)|\le 2L_2x^{\mu}$ for $(x,s)\in D$ and $L_2>0$.
Then $\|\mathscr{H}v\|_{0,\tau}\le C\|v\|_\infty$, for $0<\tau<\rho\le\min\{1/2,\mu\}$ and $v\in C(\Omega)$.

Proof. It is sufficient to show that

$$\frac{|\mathscr{H}v(x_2)-\mathscr{H}v(x_1)|}{|x_2-x_1|^{\tau}}\le C\|v\|_\infty,\quad x_1,x_2\in\Omega, x_1<x_2.$$

From [15] we know that $|\mathscr{H}v(x_2)-\mathscr{H}v(x_1)|\le C|x_2-x_1|^{\rho}\|v\|_\infty$ for $x_1,x_2\in(0,1]$ and $x_1<x_2$. Thus, the desired result is obtained when $0<x_1<x_2\le 1$. For the case $x_1=0<x_2\le 1$, we have from hypothesis (**III**) that

$$x_2^{-\tau}|\mathscr{H}v(x_2)|\le \pi L_2x_2^{\mu-\tau}\|v\|_\infty\le C\|v\|_\infty.$$

Therefore, the proof is completed. □

Theorem 3 (Convergence). *Let* $E^N(x)=y(x)-y_N(x)$, $f\in C^m(\Omega)$, $k\in C^m(D), m\ge 1$, $M:=\max_{(x,s)\in D}|k(x,s)|<2/\pi$ *and the hypotheses of previous lemma be fulfilled. Then*

$$\|E^N\|_\infty\le$$
$$C\begin{cases} N^{1/2-m}\log N\Big(\|y\|_\infty\big((\log N)^2\rho^s+\rho^x\log N\big)+\|\partial_x^m y\|_{L^2_{\chi^{\alpha,\beta}}}\Big), & -1<\alpha,\beta<-1/2,\\ N^{1/2-m}\Big(\|y\|_\infty\big(\rho^s(\log N)^2+\rho^x\log N\big)+\|\partial_x^m y\|_{L^2_{\chi^{\alpha,\beta}}}\Big), & \alpha=\beta=-1/2,\\ \eta_N\Big(\|y\|_\infty\big(N^{1+2\max(\alpha,\beta)}\rho^s+N^{\max(\alpha,\beta)+1/2}\rho^x\big)+\|\partial_x^m y\|_{L^2_{\chi^{\alpha,\beta}}}\Big), & -1/2<\max(\alpha,\beta)<-1/3,\end{cases}$$

where $\rho^s:=\sup_{0\le x\le 1}\|\partial_s^m k(\cdot,s)\|_{L^2_{\chi^{\alpha,\beta}}}$, $\rho^x:=\sup_{0\le s\le 1}\|\partial_x^m k(x,\cdot)\|_{L^2_{\chi^{\alpha,\beta}}}$, *and* $\eta_N:=N^{1+\max(\alpha,\beta)-m}$.

Proof. It is easy to show that

$$\begin{aligned}|E^N(x)|\le& M\int_0^x(x^2-s^2)^{-1/2}|E^N(s)|\,\mathrm{d}s+\big|\mathscr{H}E^N(x)-I_N(\mathscr{H}E^N;\alpha,\beta)(x)\big|\\&+\big|I_N(\mathscr{H}y_N-\mathscr{H}^Ny_N;\alpha,\beta)(x)\big|+|y(x)-I_N(y;\alpha,\beta)(x)|,\end{aligned}$$

which gives us

$$\begin{aligned}|E^N(x)|\le& M\int_0^x(x^2-s^2)^{-1/2}|E^N(s)|\,\mathrm{d}s+\big\|\mathscr{H}E^N-I_N(\mathscr{H}E^N;\alpha,\beta)\big\|_\infty\\&+\big\|I_N(\mathscr{H}y_N-\mathscr{H}^Ny_N;\alpha,\beta)\big\|_\infty+\|y-I_N(y;\alpha,\beta)\|_\infty,\end{aligned}$$

where

$$\mathscr{H}^N\phi(x)=\int_0^x x^{-1}\psi(s/x)k^N(x,s)\phi(s)\,\mathrm{d}s. \tag{21}$$

Thus, with the aid of Lemma 4, we conclude that

$$\|E^N\|_\infty\le C\left(\big\|\mathscr{H}E^N-I_N(\mathscr{H}E^N;\alpha,\beta)\big\|_\infty+\big\|I_N(\mathscr{H}y_N-\mathscr{H}^Ny_N;\alpha,\beta)\big\|_\infty+\|y-I_N(y;\alpha,\beta)\|_\infty\right).$$

Finally, the proposed result is obtained using Lemmas 1, 2, 3 and 5. □

Theorem 4 (Stability)**.** *Let the hypotheses of Theorem 3 hold. Assume further that the free function f is approximated by a function with an error term $\hat{f} \in C(\Omega) \cap H_1^{(\alpha,\beta,1)}(\Omega)$, which leads to an approximate solution y_N with a corresponding error ε_N. This error can be controlled as follows (i.e., the method is stable for CVIEs):*

$$\|\varepsilon_N\|_\infty \leq C\|\hat{f}\|_\infty + C \begin{cases} N^{-1/2}\log N\|\partial_x \hat{f}\|_{L^2_{\chi^{\alpha,\beta}}}, & -1 < \alpha, \beta < -1/2, \\ N^{-1/2}\|\partial_x \hat{f}\|_{L^2_{\chi^{\alpha,\beta}}}, & \alpha = \beta = -1/2, \\ N^{\max(\alpha,\beta)}\|\partial_x \hat{f}\|_{L^2_{\chi^{\alpha,\beta}}}, & -1/2 < \max(\alpha,\beta) < -1/3, \end{cases}$$

where $C > 0$ is a constant independent of N.

Proof. The approximate solution y_N with its corresponding error satisfies the equation

$$y_N(x_l) + \varepsilon_N(x_l) = f(x_l) + \hat{f}(x_l) + \int_0^{x_l} (x_l^2 - s^2)^{-1/2} k^N(x_l, s)(y_N(s) + \varepsilon_N(s))\mathrm{d}s. \tag{22}$$

Moreover, for $\hat{f} \equiv 0$, we know that

$$y_N(x_l) = f(x_l) + \int_0^{x_l} (x_l^2 - s^2)^{-1/2} k^N(x_l, s) y_N(s)\mathrm{d}s, \tag{23}$$

where $\{x_l\}, l = 0, 1, \ldots, N$ are the roots of the shifted Jacobi polynomial $J_{N+1}^{\alpha,\beta}(x)$. Subtracting (23) from (22), we obtain

$$\varepsilon_N(x_l) = \hat{f}(x_l) + \mathscr{H}^N \varepsilon_N(x_l).$$

Multiplying both sides of the above equation by $\mathscr{L}_i$ and summing over i from 0 to N, and noting that $\varepsilon_N(x)$ is a polynomial of degree at most N, we deduce

$$\varepsilon_N(x) = I_N(\hat{f}; \alpha, \beta)(x) + I_N(\mathscr{H}^N \varepsilon_N; \alpha, \beta)(x).$$

By adding and diminishing some terms, this can be rewritten as

$$\begin{aligned} \varepsilon_N(x) =& \mathscr{H}\varepsilon_N(x) + \hat{f}(x) + I_N(\hat{f}; \alpha, \beta)(x) - \hat{f}(x) + \mathscr{H}^N \varepsilon_N(x) \\ &- \mathscr{H}\varepsilon_N(x) + I_N(\mathscr{H}^N \varepsilon_N; \alpha, \beta)(x) - \mathscr{H}^N \varepsilon_N(x). \end{aligned} \tag{24}$$

Since

$$\|I_N(\mathscr{H}^N \varepsilon_N; \alpha, \beta) - \mathscr{H}^N \varepsilon_N\|_\infty \leq (1 + \|I_N\|_\infty)\|\mathscr{H}\varepsilon_N - \mathscr{H}^N \varepsilon_N\|_\infty + \|\mathscr{H}\varepsilon_N - I_N(\mathscr{H}\varepsilon_N; \alpha, \beta)\|_\infty,$$

it follows from (24) that

$$\begin{aligned} |\varepsilon_N(x)| \leq& M \int_0^x (x^2 - s^2)^{-1/2} |\varepsilon_N(s)|\,\mathrm{d}s + \|\hat{f}\|_\infty + \|I_N(\hat{f}; \alpha, \beta) - \hat{f}\|_\infty \\ &+ (2 + \|I_N\|_\infty)\|\mathscr{H}\varepsilon_N - \mathscr{H}^N \varepsilon_N\|_\infty + \|\mathscr{H}\varepsilon_N - I_N(\mathscr{H}\varepsilon_N; \alpha, \beta)\|_\infty. \end{aligned}$$

Therefore, applying Lemma 4, we obtain

$$\begin{aligned} \|\varepsilon_N\|_\infty \leq& C(\|\hat{f}\|_\infty + \|I_N(\hat{f}; \alpha, \beta) - \hat{f}\|_\infty + (2 + \|I_N\|_\infty)\|\mathscr{H}\varepsilon_N \\ &- \mathscr{H}^N \varepsilon_N\|_\infty + \|\mathscr{H}\varepsilon_N - I_N(\mathscr{H}\varepsilon_N; \alpha, \beta)\|_\infty). \end{aligned} \tag{25}$$

Now, the right hand side of (25) is bounded as follows. Letting $\bar{\mu} = -1/2$ in (20), we have

$$\|\mathscr{H}\varepsilon_N - \mathscr{H}^N\varepsilon_N\|_\infty \le \frac{\pi}{2}\|k - k^N\|_\infty\|\varepsilon_N\|_\infty,$$

hence by Lemmas 1 and 2, we get

$$(2 + \|I_N\|_\infty)\|\mathscr{H}\varepsilon_N - \mathscr{H}^N\varepsilon_N\|_\infty \le$$
$$C\begin{cases} N^{1/2-m}(\log N)^2\Big(\|\varepsilon_N\|_\infty\big(\log N\rho^s + \rho^x\big)\Big), & -1 < \alpha, \beta < -1/2,\\ N^{1/2-m}\log N\Big(\|\varepsilon_N\|_\infty\big(\rho^s\log N + \rho^x\big)\Big), & \alpha = \beta = -1/2,\\ \eta_N\Big(\|\varepsilon_N\|_\infty\big(N^{1+2\max(\alpha,\beta)}\rho^s + \rho^x\big)\Big), & -1/2 < \max(\alpha,\beta) < -1/3. \end{cases} \tag{26}$$

On the other hand, since $\mathscr{H}\varepsilon_N \in C^{0,\nu}(\Omega)$ for all $\nu \in \Omega$, there exists an operator $\mathscr{W}_N : C^{0,\nu}(\Omega) \to \mathscr{P}_N$ such that

$$\begin{aligned} \|\mathscr{H}\varepsilon_N - I_N(\mathscr{H}\varepsilon_N;\alpha,\beta)\|_\infty &= \|(I - I_N)\mathscr{H}\varepsilon_N\|_\infty = \|(I - I_N)(\mathscr{H}\varepsilon_N - \mathscr{W}_N\mathscr{H}\varepsilon_N)\|_\infty\\ &\le (1 + \|I_N\|_\infty)N^{-\nu}\|\mathscr{H}\varepsilon_N\|_{0,\nu} \le C(1 + \|I_N\|_\infty)N^{-\nu}\|\varepsilon_N\|_\infty\\ &\le C\begin{cases} N^{-\nu}\log N\|\varepsilon_N\|_\infty, & -1 < \alpha, \beta \le -1/2,\\ N^{\max(\alpha,\beta)-\nu+1/2}\|\varepsilon_N\|_\infty, & -1/2 < \max(\alpha,\beta) < -1/3, \end{cases} \end{aligned} \tag{27}$$

where I stands for identity operator and in the last case we assume that $\max(\alpha,\beta) + 1/2 < \nu$. For the second inequality we used Lemma 5. Hence, considering for instance the case $-1 < \alpha, \beta < -1/2$, we conclude from (25), (26) and (27) that

$$(1 - N^{1/2-m}(\log N)^2(\log N\rho^s + \rho^x) - N^{-\nu}\log N)\|\varepsilon_N\|_\infty \le C(\|\hat{f}\|_\infty + \|I_N(\hat{f};\alpha,\beta) - \hat{f}\|_\infty).$$

Consequently, for sufficiently large N we have

$$\|\varepsilon_N\|_\infty \le C(\|\hat{f}\|_\infty + \|I_N(\hat{f};\alpha,\beta) - \hat{f}\|_\infty).$$

This completes the proof (using Lemma 2) for this case. The argument for the other cases is analogous. □

4.2 Error estimation

When the exact solution is not known, by the following process we can find an approximation to the absolute error. Since $y_N(x)$ is an approximation to $y(x)$, it satisfies the perturbed equation

$$y_N(x) = f(x) + \int_0^x (x^2 - s^2)^{-1/2}k(x,s)y_N(s)\mathrm{d}s + H_N(x), \quad x \in \Omega, \tag{28}$$

where $H_N(x)$ is the perturbation term that can be approximated as follows:

$$H_N(x) \approx y_N(x) - \left(f(x) + \int_0^x (x^2 - s^2)^{-1/2}k^N(x,s)y_N(s)\mathrm{d}s\right).$$

Subtracting (28) from (17) yields

$$E^N(x) = -H_N(x) + \int_0^x (x^2 - s^2)^{-1/2} k(x,s) E^N(s)\mathrm{d}s, \quad x \in \Omega. \tag{29}$$

This is Eq. (17) for which $E^N(x)$ and $-H_N(x)$ are in the roles of $y(x)$ and $f(x)$, respectively. Therefore, it can be solved by the same way as we did for (17). We denote its approximate solution by $E^{N,M}(x)$, and refer to it as the error estimation (see Example 3).

Example 2. Assume that in the VIE (17), we have

$$k(x,s) = \frac{4}{3\pi} x e^{x+s-2},$$
$$f(x) = e^x - \frac{2}{3} x e^{x-2} \left((MB(2x) + ST(2x)) \right),$$

and the exact solution is $y(x) = e^x$. Here, **MB(x)** denotes the modified Bessel function of the first kind (see e.g. [11]) satisfying the differential equation

$$x^2 u''(x) + x u'(x) - (x^2 + y^2) u(x) = 0,$$

and **ST(x)** is the modified Struve function of the first kind (see [11]), which satisfies the equation

$$x^2 u''(x) + x u'(x) - (x^2 + y^2) u(x) = \frac{4(x/2)^{y+1}}{\sqrt{\pi}\Gamma(y + 1/2)}.$$

Numerical results for this example are presented in Table 2. The L^∞-norm of the error E^N, reported in this table for three different parameter pairs (α, β), confirms that the error converges to zero as N grows.

Example 3. Consider the following equation

$$y(x) = e^x + \frac{3}{2\pi} \int_0^x (x^2 - s^2)^{-1/2} x s y(s)\mathrm{d}s, \quad x \in \Omega.$$

For this equation the exact solution is unknown, so we are unable to calculate the exact absolute error. Therefore, we use TCM and report the maximum absolute value of the error as an estimate in Table 3. In this table, we present the maximum absolute values of the error estimation $E^{N,M}$ for different choices of (α, β). The results clearly confirm the numerical convergence of the proposed method.

5 Application of the method

This section demonstrates the application of TCM to a linear system of weakly singular VIEs and a system of single-term fractional differential equations, providing a foundation for extending this method to more general systems.

Table 2: Numerical results of Example 2 ($\|E^N\|_\infty$)

N	$(\alpha,\beta)=(\frac{-2}{3},\frac{-3}{4})$	$(\alpha,\beta)=(\frac{-9}{10},\frac{-9}{10})$	$(\alpha,\beta)=(\frac{-7}{9},\frac{-1}{2})$
4	3.52925e-05	4.10686e-05	3.32552e-05
6	5.06255e-08	5.87477e-08	4.89540e-08
8	4.32564e-11	4.99859e-11	4.24051e-11
10	2.42762e-14	2.80128e-14	2.40798e-14
12	9.65710e-18	1.11221e-17	9.64593e-18
14	2.85789e-21	3.28676e-21	2.87074e-21
16	6.54196e-25	7.51300e-25	6.59643e-25

Table 3: Error estimation results of Example 3 ($\|E^{N,M}\|_\infty$)

(N,M)	$(\alpha,\beta)=(\frac{-2}{3},\frac{-3}{4})$	$(\alpha,\beta)=(\frac{-9}{10},\frac{-9}{10})$	$(\alpha,\beta)=(\frac{-7}{9},\frac{-1}{2})$
(2,3)	9.94161e-02	1.22184e-01	9.02386e-02
(4,5)	2.52662e-03	3.00957e-03	2.42614e-03
(6,7)	5.37246e-05	6.32069e-05	5.20937e-05
(8,9)	9.94757e-07	1.14857e-06	9.57215e-07
(10,11)	1.61644e-08	1.85519e-08	1.55525e-08
(12,13)	2.37072e-10	2.71469e-10	2.28554e-10

Example 4. Consider a system of the weakly singular VIEs

$$\begin{cases} y_1(t)+\int_0^t (t-s)^{-1/2}y_1(s)\,\mathrm{d}s-\dfrac{1}{2}\int_0^t (t-s)^{-1/2}ty_2(s)\,\mathrm{d}s=f_1(t),\\ y_2(t)+\dfrac{1}{3}\int_0^t (t-s)^{-1/2}t^{1/2}y_1(s)\,\mathrm{d}s+\dfrac{1}{3}\int_0^t (t-s)^{-1/2}y_2(s)\,\mathrm{d}s=f_2(t), \end{cases} \tag{30}$$

where the right-hand side and exact solution vectors specified as follows:

$$\mathbf{f}(t)=\begin{bmatrix} t^{\sqrt{3}}+B(\frac{1}{2},\sqrt{3}+1)t^{\frac{1}{2}+\sqrt{3}}-\frac{1}{2}B(\frac{1}{2},\sqrt{5}+1)t^{\frac{3}{2}+\sqrt{5}}\\ t^{\sqrt{5}}+\frac{1}{3}B(\frac{1}{2},\sqrt{3}+1)t^{\sqrt{3}+1}+\frac{1}{3}B(\frac{1}{2},\sqrt{5}+1)t^{\frac{1}{2}+\sqrt{5}} \end{bmatrix},\quad \mathbf{y}(t)=\begin{bmatrix} t^{\sqrt{3}}\\ t^{\sqrt{5}} \end{bmatrix}$$

For $N=5,10,15$, the L^2-norm of errors are reported in Table 4. In this case, $y_{1,N}$ and $y_{2,N}$ represent approximations of y_1 and y_2, respectively, and the results show the accuracy of our method.

Example 5. Consider a system of fractional differential equations

$$\begin{cases} D^{\alpha}_{*0}y_1(t)=2y_1(t)-y_2(t),\\ D^{\alpha}_{*0}y_2(t)=4y_1(t)-3y_2(t),\\ y_1(0)=1.2,\quad y_2(0)=4.2, \end{cases} \tag{31}$$

with the exact solutions

$$y_1(t)=\frac{1}{5}E_\alpha(t^\alpha)+E_\alpha(-2t^\alpha),\ \ y_2(t)=\frac{1}{5}E_\alpha(t^\alpha)+4E_\alpha(-2t^\alpha),$$

Table 4: Errors in L^2-norm

	$N=5$	$N=10$	$N=15$
$\|\|y_1-y_{1,N}\|\|_{L^2}$	$1.328885e-03$	$5.614582e-06$	$3.238418e-07$
$\|\|y_2-y_{2,N}\|\|_{L^2}$	$4.433615e-03$	$1.188122e-06$	$2.647999e-08$

Table 5: Errors in L^2-norm for Example 5 with proposed method for $\alpha=\frac{1}{2}$

	$N=5$	$N=10$	$N=15$	$N=20$
$\|\|y_1-y_{1,N}\|\|_{L^2}$	6.7972e-04	1.3622e-09	1.9572e-16	2.8693e-23
$\|\|y_2-y_{2,N}\|\|_{L^2}$	2.7796e-03	5.4449e-09	7.8288e-16	4.2498e-23

where $E_\alpha(t)$ denotes the Mittag-Leffler function of one parameter.

A system of Abel-Volterra integral equations that is equivalent to the system (31), is given by

$$\begin{cases} y_1(t) = 1.2 + \frac{2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y_1(s)\,\mathrm{d}s - \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y_1(s)\,\mathrm{d}s, \\ y_2(t) = 4.2 + \dfrac{4}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y_1(s)\,\mathrm{d}s - \dfrac{3}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y_1(s)\,\mathrm{d}s. \end{cases}$$

We report numerical results of this system in Table 5. The results in this table show L^2 convergence of the proposed method.

6 Conclusion

This paper presents an application of the tau-collocation method for solving weakly singular and cordial Volterra integral equations. A convergence and stability analysis has been discussed specifically for the cordial case (Theorems 3 and 4). Numerical examples for both types of equations are provided to validate the theoretical results and demonstrate the method's numerical accuracy. Finally, we solved systems of weakly singular integral equations and fractional differential equations to show the method's applicability to these more general cases.

Conflicts of interest

The authors declare that there are no conflicts of interest.

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