Journal of Mathematical Modeling
Vol. 14, No. 1, 2026, pp. 115-135. Research Article

JMM

A theoretical study for an air pollution model as a free boundary problem

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Abstract. This paper presents a theoretical investigation of an air pollution model formulated as a free boundary problem. The study examines the dynamics of pollutant dispersion in the atmosphere, where the boundaries of the polluted region are not fixed but evolve over time. Using advanced mathematical techniques and partial differential equations, we construct a model that accounts for various factors influencing air pollution, including emission sources, meteorological conditions, and chemical reactions. The incorporation of a free boundary framework provides a more realistic representation of pollutant spread and environmental interactions. To analyze the model, we employ the Friedman-Rubinstein integral representation method and apply the Banach contraction theorem to solve an equivalent nonlinear Volterra integral equation.

Keywords: Air pollution, free boundary problem, Volterra's integral equations, Green's function.
AMS Subject Classification 2010: 35R35, 34A12, 35A01.

1 Introduction

The impact of air pollution on human life is particularly evident in densely populated and industrialized regions. Due to the rapid development of many countries, air pollution has become a global concern. Various international and regional organizations have conducted a range of experiments to prevent the expansion of pollution, leading to significant outcomes and the establishment of effective urban air quality standards. Urban air pollution is intensified by anthropogenic factors such as population growth, rural-to-urban migration, industrial expansion, vehicular traffic, and geographical factors including topography, location, and temperature inversions. Air pollution poses a serious threat to both human health and the environment, as industrial emissions release harmful substances into the atmosphere [2, 18].

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Received: 05 January 2025/ Revised: 10 September 2025/ Accepted: 19 September 2025
DOI: 10.22124/jmm.2025.29478.2618

The control and study of air pollution have become increasingly crucial. In general, atmospheric diffusion refers to the behavior of gases and particles in turbulent flow environments [2, 13]. Two fundamental modeling approaches are widely used to describe atmospheric diffusion: the Eulerian and Lagrangian frameworks. The Eulerian approach describes species behavior relative to a fixed pollution sources and is commonly applied to model heat and mass transfer phenomena [2, 3, 8]. In contrast, the Lagrangian approach describes concentration changes along moving pollution sources and provides alternative mathematical formulations [4]. Both frameworks offer valid descriptions of turbulent diffusion [4, 17].

Various models have been proposed to describe air pollution dispersion, including regression models [8], box models [8, 15], Eulerian models [12, 17], Lagrangian models [5, 10, 12, 17], Gaussian models [6], and hybrid approaches [9]. Each model has specific advantages. For example, in Eulerian diffusion models, the pollution source is fixed in space, whereas in Lagrangian models, it moves with a certain velocity. In Gaussian models, the average concentration of a pollutant emitted from a point source follows a Gaussian distribution. Although this assumption is strictly valid only under homogeneous turbulence and stationary conditions, it forms the basis for deriving equations in more general atmospheric cases.

In this paper, we present a theoretical study of air pollution modeling formulated as a free boundary problem. This approach does not conflict with existing models but rather offers a more robust theoretical foundation. We reformulate the model as an equivalent nonlinear Volterra integral equation and prove the existence and uniqueness of its solution using the Banach fixed-point theorem.

The advantage of the free boundary formulation lies in its ability to estimate pollutant concentrations and the dynamic location of the pollution front over time. This is achieved by considering pollutant production sources and local wind velocities, thereby enabling predictive insights to mitigate potential atmospheric pollution crises.

The outline of this paper is organized as follows: In Section 2, we consider the mathematical model of air pollution as a fixed boundary problem. In Section 3, we derive the free boundary for the proposed model. In Section 4, we reduce the problem to an equivalent problem of solving a nonlinear Volterra-type integral equation. In Section 5, we prove the local (in time) existence and uniqueness of the solution for the one-dimensional free boundary model.

2 Mathematical model

Oyjinda et al. [14] conducted numerical simulations to analyze air pollution measurements near industrial zones. Their study simulated air pollution control strategies to achieve desired pollutant concentration levels. Monitoring stations were installed to record concentration data of air pollutants. The numerical experiments considered different scenarios, including both normal and controlled emission conditions. The concentration of air pollutants was approximated using an explicit finite difference method.
The governing equations of the system were formulated as follows:

$$\frac{\partial c}{\partial t}+v_x\frac{\partial c}{\partial x}+v_z\frac{\partial c}{\partial z}=D_x\frac{\partial^2 c}{\partial x^2}+D_z\frac{\partial^2 c}{\partial z^2}+S(x,z,t), \tag{1}$$

where $c=c(x,z,t)$ denotes the concentration of the air pollutant at the spatial-temporal point (x,z,t) measured in kg/m^3. The values v_x and v_z represent the wind velocity components in the x- and z-directions, respectively, measured in meters per second (m/s). The parameters D_x and D_z denote the

diffusion coefficients in the x- and z-directions, respectively, with units of (m^2/s). The term S refers to the pollutant source term representing the growth rate of pollutant due to emission sources, with units of (sec^{-1}).
Accordingly, the following initial and boundary conditions are assumed:

$$c(x,z,0) = f(x,z), \qquad x > 0, \quad z > 0, \tag{2}$$

$$\frac{\partial c}{\partial x}(0,z,t) = \frac{\partial c}{\partial x}(L,z,t) = 0, \qquad 0 \le z \le H, \quad t > 0, \tag{3}$$

$$\frac{\partial c}{\partial z}(x,0,t) = v_d c, \qquad 0 \le x \le L, \quad t > 0, \tag{4}$$

$$\frac{\partial c}{\partial z}(x,H,t) = 0, \qquad 0 \le x \le L, \quad t > 0, \tag{5}$$

for all $t > 0$, where L is the length of the domain in the x- direction, H is the height of the atmospheric inversion layer, and v_d denotes the dry deposition velocity of the primary pollutant (in (m/s)).

The researchers in [14] analyzed air pollution near industrial zones through numerical simulations using an explicit finite difference technique. Their analysis was based on the assumption that the domain has a fixed boundary at $x = L$. However, a more realistic representation of the system should consider it as a free boundary problem.

3 Motivation of the free boundary model

In this section, we outline the derivation of a free boundary formulation for the model (1)-(5), inspired by the approach in [1]. Due to the dynamical behavior of pollutant dispersion in the atmosphere, the boundaries of the affected region are not fixed but evolve over time. By employing advanced mathematical techniques and partial differential equations, our model incorporates key factors influencing air pollution, including emission sources, meteorological conditions, and chemical interactions. The free boundary framework provides a more accurate representation of how pollution spreads and evolves over time. Suppose the polluted atmospheric region is defined as $\Omega = ((x,z,t) \mid 0 \le z \le H,\ 0 \le x < \phi(z,t) 0 < t < T)$, where $x = \phi(z,t)$ denotes the moving surface between the polluted and unpolluted parts of the atmosphere. Therefore, to obtain a more realistic model, we are motivated to reformulate the problem (1)-(5) as follows:

$$\frac{\partial c}{\partial t} + v_x \frac{\partial c}{\partial x} + v_z \frac{\partial c}{\partial z} = D\frac{\partial^2 c}{\partial x^2} + D\frac{\partial^2 c}{\partial z^2} + S(x,z,t), \qquad (x,z,t) \in \Omega, \tag{6}$$

$$c(x,z,0) = f(x,z), \qquad 0 \le z \le H,\ 0 \le x < R(z), \tag{7}$$

$$\frac{\partial c}{\partial x}(0,z,t) = 0, \qquad 0 \le z \le H,\ 0 < t < T, \tag{8}$$

$$c(x,z,t)\Big|_{x=\phi(z,t)} = 0, \qquad 0 < t < T,$$

$$\frac{\partial c}{\partial z}(x,0,t) = v_d c, \qquad 0 \le x < \phi(0,t),\ \ 0 < t < T, \tag{9}$$

$$\frac{\partial c}{\partial z}(x,H,t) = 0, \qquad 0 < t < T, \tag{10}$$

$$< \nabla\Phi, \nabla c > \Big|_{x=\phi(z,t)} = k\frac{\partial \Phi}{\partial t}, \qquad 0 < t < T, \tag{11}$$

$$\phi(z,0) = R(z), \tag{12}$$

where $\Phi(x,z,t) = x - \phi(z,t) = 0$ represents the moving boundary.

In the next section, we prove the existence of the solution of the reformulated problem by applying the Banach fixed-point theorem.

4 Reduction to an integral equation

To simplify the analysis, we consider the following one-dimensional formulation. We aim to prove the existence and uniqueness of the solution to the following problem:

$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} = \frac{\partial^2 c}{\partial x^2} + S(x,t), \qquad 0 < t < T,\ \ 0 < x < \phi(t), \tag{13}$$

$$c(x,0) = h(x), \tag{14}$$

$$c(\phi(t),t) = 0, \qquad 0 < t < T, \tag{15}$$

$$c(0,t) = g(t) \qquad 0 < t < T, \tag{16}$$

$$\phi(0) = b > 0, \tag{17}$$

$$c_x(\phi(t),t) = -\phi'(t), \qquad 0 < t < T. \tag{18}$$

Here, the free boundary $x = \phi(t)$ is unknown and must be determined simultaneously with the concentration $c(x,t)$.

Definition 1. We say that the pair $c(x,t), \phi(t)$ constitutes a solution to the problem (13)-(18) for all $t < \gamma$ $(0 < \gamma \le \infty)$, if the following conditions are satisfied:

1. $\frac{\partial^2 c}{\partial x^2}$, $\frac{\partial c}{\partial t}$ and $\frac{\partial c}{\partial x}$ are continuous for $0 < x < \phi(t)$, $0 < t < \gamma$.

2. $c(x,t)$ and $\frac{\partial c}{\partial x}$ are continuous for $0 \le x \le \phi(t)$, $0 < t < \gamma$.

3. $c(x,t)$ is continuous at $t = 0$ for $0 < x \le b$ and $0 \le \liminf c(x,t) \le \limsup c(x,t) < \infty$ as $t \to 0$ and $x \to 0$.

4. $\phi(t)$ is continuously differentiable for $0 < t < \gamma$.

5. All equations (13)-(18) are satisfied.

Theorem 1. *Assume that $h(x)$ for $0 \le x \le b$, $g(t)$ for $0 \le t < \infty$, and $S(x,t)$ for $0 \le x \le \phi(t)$, $0 < t < \infty$, are continuously differentiable functions. Then there exists a unique solution $c(x,t), \phi(t)$ to the system (13)-(18) for all $t < \infty$.*

The proof is given in this section and in the next one. In this section we reduce the problem (13)-(18) to an equivalent problem of solving a nonlinear Volterra-type integral equation for $c_x(\phi(t),t)$. In reducing the problem (13)-(18) to an equivalent problem of solving a nonlinear Volterra-type integral equation to a problem of solving an integral equation, we shall make use of the following lemma.

Lemma 1. *Let $\rho(t)$, $(0 \le t \le \sigma)$, be a continuous function, and let $\phi(t)$, $(0 \le t \le \sigma)$, satisfy a Lipschitz condition. Then, for every $(0 < t \le \sigma)$, the following holds:*

$$\lim_{x \to \phi(t)} \frac{\partial}{\partial x} \int_0^t \rho(\tau) K(x,t,\phi(\tau),\tau) d\tau = \frac{\rho(t)}{2} + \int_0^t \rho(\tau) \left[\frac{\partial K(x,t,\phi(\tau),\tau)}{\partial x} \right]_{x=\phi(t)} d\tau, \tag{19}$$

where

$$K(x,t,\xi,\tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} \exp\left(-\frac{(x-\xi-v(t-\tau))^2}{4(t-\tau)} \right). \tag{20}$$

Proof. Following the approach in [7], we begin by proving that for any fixed positive $\delta < t$, the integral

$$\begin{aligned} I = & \int_{t-\delta}^t \frac{x-\phi(\tau)-v(t-\tau)}{2(t-\tau)} K(x,t,\phi(\tau),\tau) d\tau \\ & - \int_{t-\delta}^t \frac{\phi(t)-\phi(\tau)-v(t-\tau)}{2(t-\tau)} K(\phi(t),t,\phi(\tau),\tau) d\tau. \end{aligned} \tag{21}$$

satisfies the relation

$$\limsup_{x \to \phi(t)^-} |I + \frac{1}{2}| \le A\sqrt{\delta}. \tag{22}$$

Throughout this proof, we denote by A a generic constant that is independent of x, t, and δ (though it may depend on γ).
Let us write $I = I_1 + I_2$, where

$$\begin{aligned} I_1 &= \int_{t-\delta}^t \frac{x-\phi(t)}{2(t-\tau)} K(x,t,\phi(\tau),\tau) d\tau, \\ I_2 &= \int_{t-\delta}^t \frac{\phi(t)-\phi(\tau)-v(t-\tau)}{2(t-\tau)} \Big[K(x,t,\phi(\tau),\tau) - K(\phi(t),t,\phi(\tau),\tau) \Big] d\tau. \end{aligned}$$

Since, by assumption, $|\phi(t)-\phi(\tau)| \le A|t-\tau|$, we obtain

$$\begin{aligned} |I_2| \le & \int_{t-\delta}^t \frac{|\phi(t)-\phi(\tau)-v(t-\tau)|}{4\sqrt{\pi}(t-\tau)\sqrt{t-\tau}} \Big[exp\left(-\frac{(x-\phi(\tau)-v(t-\tau))^2}{4(t-\tau)} \right) \\ & + exp\left(-\frac{(\phi(t)-\phi(\tau)-v(t-\tau))^2}{4(t-\tau)} \right) \Big] d\tau \\ \le & (A+v) \int_{t-\delta}^t \frac{d\tau}{\sqrt{t-\tau}} \le A\sqrt{\delta}. \end{aligned} \tag{23}$$

To evaluate I_1, we define

$$J = \int_{t-\delta}^t \frac{x-\phi(t)}{2(t-\tau)} K(x,t,\phi(t),\tau) d\tau, \tag{24}$$

then, we can write

$$J - I_1 = \int_{t-\delta}^{t} \frac{x-\phi(t)}{2(t-\tau)} K(x,t,\phi(t),\tau) \times \Big[1 - \underbrace{\exp\big[-\frac{(x-\phi(\tau)-v(t-\tau))^2-(x-\phi(t)-v(t-\tau))^2}{4(t-\tau)}\big]}_{F}\Big] d\tau. \tag{25}$$

The F is bounded, as shown in

$$\begin{aligned} F &\leq \frac{1}{4(t-\tau)} |\phi(t)-\phi(\tau)| \Big(|x-\phi(\tau)-v(t-\tau)| + |x-\phi(t)-v(t-\tau)| \Big) \\ &\leq A \Big(|x-\phi(\tau)-v(t-\tau)| + |x-\phi(t)-v(t-\tau)| \Big) \\ &\leq A \Big(|x-\phi(\tau)-v(t-\tau)+\phi(t)-\phi(t)| + |x-\phi(t)-v(t-\tau)| \Big) \\ &\leq A \Big(|\phi(t)-\phi(\tau)| + |x-\phi(t)-v(t-\tau)| \Big). \end{aligned}$$

Since it is sufficient to prove (22) for sufficiently small δ, and since $x \to \phi(t)$, we may assume the right-hand side of the last inequality is less than 1. Substituting this into (25), using the elementary inequality $ye^{-y} \leq const.$, for $y \geq 0$, and noting that $x - \phi(t) < 0$, we can find

$$|J - I_1| \leq A \int_{t-\delta}^{t} \frac{d\tau}{\sqrt{t-\tau}} + A \int_{t-\delta}^{t} \sqrt{t-\tau}\, d\tau \leq A\delta\sqrt{\delta}, \tag{26}$$

and

$$\begin{aligned} J &= \int_{t-\delta}^{t} \frac{x-\phi(t)}{2(t-\tau)} K(x,t,\phi(t),\tau) d\tau \\ &= \int_{t-\delta}^{t} \frac{x-\phi(t)}{4(t-\tau)\sqrt{\pi(t-\tau)}} \exp\{-\frac{(x-\phi(t)-v(t-\tau))^2}{4(t-\tau)}\} d\tau \\ &= \int_{t-\delta}^{t} \frac{x-\phi(t)}{4(t-\tau)\sqrt{\pi(t-\tau)}} \exp\{-\frac{(x-\phi(t))^2 - 2v(x-\phi(t))(t-\tau) + v^2(t-\tau)^2}{4(t-\tau)}\} d\tau \\ &= \int_{t-\delta}^{t} \frac{x-\phi(t)}{4(t-\tau)\sqrt{\pi(t-\tau)}} \exp\{-\frac{(x-\phi(t))^2}{4(t-\tau)}\} \exp\{\frac{v(x-\phi(t))}{2}\} \exp\{-\frac{v^2(t-\tau)^2}{4}\} d\tau \\ &\leq J_1, \end{aligned}$$

where

$$J_1 = \int_{t-\delta}^{t} \frac{x-\phi(t)}{4(t-\tau)\sqrt{\pi(t-\tau)}} \exp\{-\frac{(x-\phi(t))^2}{4(t-\tau)}\} d\tau. \tag{27}$$

Now, for J_1, we apply the substitution $u = \frac{t-\tau}{(x-\phi(t))^2}$ in equation (27). Noting again that $x - \phi(t) < 0$, we obtain

$$J_1 = \int_{t-\delta}^{t} \frac{x-\phi(t)}{4(t-\tau)\sqrt{\pi(t-\tau)}} \exp\{-\frac{(x-\phi(t))^2}{4(t-\tau)}\} d\tau = \frac{-1}{4\pi} \int_{0}^{\delta'} u^{\frac{-3}{2}} \exp\{\frac{-1}{4u}\} du, \tag{28}$$

where $\delta' = \frac{\delta}{(x-\phi(t))^2}$ as $x \to \phi(t)$, $\delta' \to \infty$, and con sequently $J_1 \to \frac{-1}{2}$.

Also, for every positive integer N we can write

$$\begin{aligned}
J_2 =& \int_{t-\delta}^{t} \frac{x-\phi(t)}{4(t-\tau)\sqrt{\pi(t-\tau)}} \exp\{-\frac{(x-\phi(t))^2}{4(t-\tau)}\} \exp\{\frac{v(x-\phi(t))}{2}\} \\
&\times \sum_{i=0}^{N} \frac{(-1)^i}{i!} \left(\frac{v^2}{4}\right)^i (t-\tau)^{2i} d\tau \\
\leq& \int_{t-\delta}^{t} \frac{x-\phi(t)}{4(t-\tau)\sqrt{\pi(t-\tau)}} \exp\{-\frac{(x-\phi(t))^2}{4(t-\tau)}\} \exp\{\frac{v(x-\phi(t))}{2}\} \\
&\times \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{v^2}{4}\right)^i (t-\tau)^{2i} d\tau \\
=& J \leq J_1,
\end{aligned} \tag{29}$$

and deduce

$$J_2 \leq J \leq J_1.$$

For J_2, we can obtain

$$\begin{aligned}
J_2 =& \exp\frac{v(x-\phi(t))}{2} \sum_{i=0}^{N} \frac{(-1)^i}{i!} \left(\frac{v^2}{4}\right)^i \int_{t-\delta}^{t} \frac{x-\phi(t)}{4(t-\tau)\sqrt{\pi(t-\tau)}} \exp - \frac{(x-\phi(t))^2}{4(t-\tau)} (t-\tau)^i d\tau \\
=& \exp\frac{v(x-\phi(t))}{2} \int_{t-\delta}^{t} \frac{x-\phi(t)}{4(t-\tau)\sqrt{\pi(t-\tau)}} \exp - \frac{(x-\phi(t))^2}{4(t-\tau)} d\tau \\
&- \frac{v^2}{4} \exp\frac{v(x-\phi(t))}{2} \int_{t-\delta}^{t} \frac{x-\phi(t)}{4\sqrt{\pi(t-\tau)}} \exp - \frac{(x-\phi(t))^2}{4(t-\tau)} d\tau \\
&+ \frac{(-1)^2}{2!} \left(\frac{v^2}{4}\right)^2 \exp\frac{v(x-\phi(t))}{2} \int_{t-\delta}^{t} \frac{(x-\phi(t))\sqrt{(t-\tau)}}{4\sqrt{\pi}} \exp - \frac{(x-\phi(t))^2}{4(t-\tau)} d\tau + \ldots \\
&+ \frac{(-1)^N}{N!} \left(\frac{v^2}{4}\right)^N \exp\frac{v(x-\phi(t))}{2} \int_{t-\delta}^{t} \frac{x-\phi(t)}{4(t-\tau)\sqrt{\pi(t-\tau)}} \exp - \frac{(x-\phi(t))^2}{4(t-\tau)} (t-\tau)^N d\tau \\
=& J_1 \exp\frac{v(x-\phi(t))}{2} - \frac{v^2(x-\phi(t))^2}{16\sqrt{\pi}} \Gamma(\frac{1}{2}) + \frac{v^4(x-\phi(t))^3}{64\sqrt{\pi}} \Gamma(\frac{3}{2}) + \ldots \\
&+ \frac{v^{2N}(x-\phi(t))^{N+1}}{4^{(N+1)}\sqrt{\pi}} \Gamma(\frac{N+1}{2}).
\end{aligned}$$

As $x \to \phi(t)$, $\delta' \to \infty$, and consequently $J_2 \to \frac{-1}{2}$. Hence, by the squeeze theorem, we conclude that

$$J \to \frac{-1}{2}. \tag{30}$$

Combining this result with equations (23) and (26), and recalling that $I = I_1 + I_2$, equation (22) follows. From (26) and (28), we obtain

$$|I_1| \leq A. \tag{31}$$

Using the Lipschitz continuity of $\phi(t)$ (i.e $|\phi(t)-\phi(\tau)| \le \alpha|t-\tau|$), we then obtain

$$\int_{t-\delta}^{t} \frac{|\phi(t)-\phi(\tau)-v(t-\tau)|}{2(t-\tau)} K(\phi(t),t,\phi(\tau),\tau)d\tau \le A. \tag{32}$$

Finally, we have

$$\int_{t-\delta}^{t} \frac{|x-\phi(\tau)-v(t-\tau)|}{2(t-\tau)} K(x,t,\phi(\tau),\tau)d\tau \le A. \tag{33}$$

The proof follows by writing

$$\frac{|x-\phi(\tau)-v(t-\tau)|}{2(t-\tau)}K \le -\frac{x-\phi(t)-v(t-\tau)}{2(t-\tau)}K + \frac{|\phi(t)-\phi(\tau)-v(t-\tau)|}{2(t-\tau)}K,$$

equation (31) and the Lipschitz continuity of $\phi(t)$.

To complete the proof of the lemma using (22), (32), and (33), we define

$$\begin{aligned} L_1 = &\int_{t-\delta}^{t} \rho(\tau)\frac{x-\phi(\tau)-v(t-\tau)}{2(t-\tau)} K(x,t,\phi(\tau),\tau)d\tau \\ &- \int_{t-\delta}^{t} \rho(\tau)\frac{\phi(t)-\phi(\tau)-v(t-\tau)}{2(t-\tau)} K(\phi(t),t,\phi(\tau),\tau)d\tau, \end{aligned} \tag{34}$$

and claim that

$$\limsup_{x\to\phi(t)} |L_1 + \frac{\rho(t)}{2}| \le A\sqrt{\delta} + A\ l.u.b_{t-\delta\le\tau\le t}|\rho(t)-\rho(\tau)|. \tag{35}$$

This follows by writing in equation (34), $\rho(\tau) = \rho(t) + [\rho(\tau)-\rho(t)]$, and applying (22), (32), and (33). Note that the function

$$\begin{aligned} L_2 = &\int_{0}^{t-\delta} \rho(\tau)\frac{x-\phi(\tau)-v(t-\tau)}{2(t-\tau)} K(x,t,\phi(\tau),\tau)d\tau \\ &- \int_{0}^{t-\delta} \rho(\tau)\frac{\phi(t)-\phi(\tau)-v(t-\tau)}{2(t-\tau)} K(\phi(t),t,\phi(\tau),\tau)d\tau, \end{aligned}$$

satisfies $\lim_{x\to\phi(t)} L_2 = 0$. Combining this observation with (22) we obtain

$$\limsup_{x\to\phi(t)} |L_1 + L_2 + \frac{\rho(t)}{2}| \le A\sqrt{\delta} + A\ l.u.b_{t-\delta\le\tau\le t}|\rho(t)-\rho(\tau)|.$$

Since the left-hand side is independent of δ, and the right-hand side can be made arbitrarily small for sufficiently small δ, we obtain the desired jump relation (equation (19))

$$\limsup_{x\to\phi(t)} |L_1 + L_2 + \frac{\rho(t)}{2}| = 0. \qquad \square$$

We have the following equivalence for the existence of solutions to the problem (13)-(18). To this end, we employ the Friedman-Rubinstein's method [7, 11, 16] and show that the problem (13)-(18) is equivalent to the Volterra-type integral equation.

Theorem 2. *The solution of the problem* (13)-(18) *is*

$$\begin{aligned} c(x,t) = &\int_0^b G(x,t,\xi,0)h(\xi)d\xi + \int_0^t G(x,t,\phi(\tau),\tau)w(\tau)d\tau \\ &+ \int_0^t G_\xi(x,t,0,\tau)g(\tau)d\tau + \int_0^t \int_0^{\phi(t)} G(x,t,\xi,\tau)s(\xi,\tau)d\xi d\tau, \end{aligned} \tag{36}$$

$$\phi(t) = b - \int_0^t w(\tau)d\tau. \tag{37}$$

where the function $w \in C^0[0,T]$ *defined by*

$$w(t) = c_x(\phi(t),t), \tag{38}$$

must satisfy the following nonlinear Volterra integral equation

$$\begin{aligned} w(t) =&2\int_0^b G_x(\phi(t),t,\xi,0)h(\xi)d\xi + 2\int_0^t G_x(\phi(t),t,\phi(\tau),\tau)w(\tau)d\tau \\ &+2\int_0^t G_{\xi x}(\phi(t),t,0,\tau)g(\tau)d\tau + 2\int_0^t \int_0^{\phi(t)} G_x(\phi(t),t,\xi,\tau)S(\xi,\tau)d\xi d\tau, \end{aligned} \tag{39}$$

where G *is the Green function and* K *is the fundamental solution, defined respectively by*

$$\begin{aligned} G(x,t,\xi,\tau) &= K(x,t,\xi,\tau) - \exp(xv)K(-x,t,\xi,\tau), \\ K(x,t,\xi,\tau) &= \frac{H(t-\tau)}{2\sqrt{\pi(t-\tau)}} exp\left(-\frac{(x-\xi-v(t-\tau))^2}{4(t-\tau)}\right). \end{aligned} \tag{40}$$

Proof. Suppose that $c(x,t), \phi(t)$ constitute a solution of (13)-(18). We integrate, on the domain $D = ((\xi,\tau)|, \varepsilon < \tau < t-\varepsilon, 0 < \xi < \phi(\tau),)$, the Green identity

$$(Gc_\xi - cG_\xi)_\xi - v(Gc)_\xi - (Gc)_\tau = -GS(x,t). \tag{41}$$

By letting $\varepsilon \longrightarrow 0$, and using (14)-(18), we obtain

$$\begin{aligned} c(x,t) = &\int_0^b G(x,t,\xi,0)h(\xi)d\xi + \int_0^t G(x,t,\phi(\tau),\tau)c_\xi(\phi(\tau),\tau)d\tau \\ &+ \int_0^t G_\xi(x,t,0,\tau)g(\tau)d\tau + \int_0^t \int_0^{\phi(t)} G(x,t,\xi,\tau)s(\xi,\tau)d\xi d\tau. \end{aligned} \tag{42}$$

By differentiating both sides of (42), taking $x \to \phi(t)^-$ and applying jump relation in Lemma 1, we can obtain

$$\begin{aligned} w(t) =&2\int_0^b G_x(\phi(t),t,\xi,0)h(\xi)d\xi + 2\int_0^t G_x(\phi(t),t,\phi(\tau),\tau)w(t)d\tau \\ &+2\int_0^t G_{\xi x}(\phi(t),t,0,\tau)g(\tau)d\tau + 2\int_0^t \int_0^{\phi(t)} G_x(\phi(t),t,\xi,\tau)S(\xi,\tau)d\xi d\tau. \end{aligned} \tag{43}$$

We have thus shown that for any solution $c(x,t),\phi(t)$ of the problem (13)-(18) for $t<\tau$, the function $w(t)$ defined by (38) satisfies the nonlinear Volterra integral equation (39). Conversely, suppose that the function $w(t)$ is a solution of (39), with $\phi(t)$ given by (37). We shall prove that $c(x,t),\phi(t)$ then constitute a solution of (13)-(18), where $c(x,t)$ is defined by (36) with $c_{\xi}(\phi(\tau),\tau)$ replaced by $w(\tau)$. Verification of (13) and initial/boundary conditions (14) and (16)-(18) is straightforward. It remains to verify that $c(\phi(t),t)=0$. In order to prove condition (15) we define $\psi(t)=c(\phi(t),t)$. By integrating the Green's identity (41) over the domain $0<\varepsilon<\tau<t-\varepsilon, 0<\xi<\phi(\tau)$ and letting $\varepsilon \longrightarrow 0$, we obtain

$$\begin{aligned}
c(x,t)=&\int_0^b G(x,t,\xi,0)h(\xi)d\xi+\int_0^t G(x,t,\phi(\tau),\tau)w(\tau)d\tau+\int_0^t G_{\xi}(x,t,0,\tau)g(\tau)d\tau\\
&+\int_0^t\int_0^{\phi(t)} G(x,t,\xi,\tau)S(\xi,\tau)d\xi d\tau-\int_0^t G_{\xi}(x,t,\phi(\tau),\tau)\psi(\tau)d\tau\\
&-v\int_0^t G(x,t,\phi(\tau),\tau)\psi(\tau)d\tau+\int_b^{\phi(t)} G(x,t,\xi,\phi^{-1}(\xi))c(\xi,\phi^{-1}(\xi))d\xi.
\end{aligned}\tag{44}$$

Comparing the resulting integral representation of $c(x,t)$ with the definition of $c(x,t)$ by (36), with $c_{\xi}(\phi(\tau),\tau)=w(\tau)$, we conclude that

$$\int_0^t \psi(\tau)\Big[G_{\xi}(x,t,\phi(\tau),\tau)+vG(x,t,\phi(\tau),\tau)+G(x,t,\phi(\tau),\tau)w(\tau)\Big]d\tau=0.\tag{45}$$

Taking the limit $x\to\phi(t)^-$ and using jump relation from Lemma 1, we find that the function $\Psi(t)=c(\phi(t),t)$ satisfies

$$\frac{\Psi(t)}{2}+\int_0^t \Psi(\tau)\Big[G_{\xi}(\phi(t),t,\phi(\tau),\tau)+vG(\phi(t),t,\phi(\tau),\tau)+G(\phi(t),t,\phi(\tau),\tau)w(\tau)\Big]d\tau=0.\tag{46}$$

As in [3], we can show that

$$\begin{aligned}
|\Psi(t)|&\le A\int_0^t |\Psi(\tau)|\Big|G_{\xi}(\phi(t),t,\phi(\tau),\tau)+vG(\phi(t),t,\phi(\tau),\tau)+w(\tau)G(\phi(t),t,\phi(\tau),\tau)\Big|d\tau\\
&\le A\int_0^t |\Psi(\tau)|\Big|G_{\xi}(\phi(t),t,\phi(\tau),\tau)+(v+c)G(\phi(t),t,\phi(\tau),\tau)\Big|d\tau\\
&\le A\int_0^t |\Psi(\tau)|\Big(\frac{C_2}{\sqrt{t-\tau}}+\frac{C_3}{\sqrt{t-\tau}}+\frac{C_1}{\sqrt{t-\tau}}\Big)d\tau\\
&\le C\int_0^t \frac{|\Psi(\tau)|}{\sqrt{t-\tau}}d\tau\\
&\le C^2\int_0^t \frac{1}{\sqrt{t-\tau}}d\tau\int_0^{\tau}\frac{|\Phi(\eta)|}{\sqrt{\tau-\eta}}d\eta\\
&=C^2\int_0^t |\Psi(\eta)|d\eta\int_{\eta}^t \frac{d\eta}{[(t-\tau)(\tau-\eta)]^{\frac{1}{2}}}\\
&=\pi C^2\int_0^t |\Psi(\eta)|d\eta,
\end{aligned}\tag{47}$$

where $C=C(t)$. Hence, using the Gronwall's inequality, we conclude $\Psi(t)=0$, and thus $c(\phi(t),t)=0$ holds. □

5 Existence and uniqueness of solution

We now apply the Banach fixed point theorem to prove the local existence and uniqueness of solution $w(t)$ to the nonlinear Volterra integral equation (39), where σ, is a sufficiently small positive number $0 \leq \sigma \leq T$ to be determined. Consider the Banach space

$$C_{R,\sigma} = \left(w \middle| w : [0,\sigma] \to \mathbb{R} \; w \in C[0,\sigma] \, ||w||_\sigma = max_{t\in[0,\sigma]}|w(t)| \leq R \right).$$

We define the map $T : C_{R,\sigma} \to C_{R,\sigma}$ by

$$\begin{aligned} T(w)(t) =& 2\int_0^b G_x(\phi(t),t,\xi,0)h(\xi)d\xi + 2\int_0^t G_x(\phi(t),t,\phi(\tau),\tau)w(\tau)d\tau \\ &+ 2\int_0^t G_{\xi x}(\phi(t),t,0,\tau)g(\tau)d\tau + 2\int_0^t\int_0^{\phi(t)} G_x(\phi(t),t,\xi,\tau)S(\xi,\tau)d\xi d\tau \\ =& H_1 + H_2 + H_3 + H_4. \end{aligned} \tag{48}$$

Theorem 3. *Let $h(x)$ for $0 \leq x \leq b$, $g(t)$ for $0 \leq t < \infty$, and $S(x,t)$ for $0 \leq x \leq \phi(t)$ and $0 < t < \infty$ be continuously differentiable functions. Then, the map $T : C_{R,\sigma} \to C_{R,\sigma}$ is well-defined and is a contraction mapping provided that R and σ satisfy the following inequalities:*

$$R \geq ||h^{'}|| \sum_{i=1}^{3} C_i(\pi,b,e,R,v,||s||,||g||), \tag{49}$$

and

$$\sqrt{\sigma} < \min\left\{ \frac{R}{\sum_{i=1}^{3} C_i(\pi,b,e,R,v,||s||,||g||)} - ||h^{'}||, \frac{1}{\sum_{i=0}^{7}\lambda_i} \right\}. \tag{50}$$

Proof. The boundedness of $||H_i||$ for $i = 1,2,3,4$ is proved as follows. For $||H_1||$, we have

$$\begin{aligned} |H_1| \leq& |\int_0^b h(\xi)G_x(\phi(t),t,\xi,0)d\xi| \\ \leq& |\int_0^\infty \Big(K_\xi(\phi(t),t,\xi,0) + K_\xi(\phi(t),t,-\xi,0)\Big)h(\xi)d\xi| \\ =& |\int_0^\infty \Big(K(\phi(t),t,\xi,0) + K(\phi(t),t,-\xi,0)\Big)h'(\xi)d\xi| \\ \leq& ||h'||\Big(|\int_0^\infty K(\phi(t),t,\xi,0)d\xi| + |\int_0^\infty K(-\phi(t),t,\xi,0)d\xi|\Big) \\ \leq& C_1||h'||. \end{aligned}$$

We note that

$$
\begin{aligned}
|G_x(\phi(t),t,\phi(\tau),\tau)| \leq & |K_x(\phi(t),t,\phi(\tau),\tau)| + v|exp(v.\phi(t))||K(-\phi(t),t,\phi(\tau),\tau)| \\
& + |exp(v\phi(t))||K_x(-\phi(t),t,\phi(\tau),\tau)| \\
\leq & \frac{2|\phi(t)-\phi(\tau)+v(t-\tau)|}{4(t-\tau)} \frac{1}{2\sqrt{\pi(t-\tau)}} |exp - \frac{(\phi(t)-\phi(\tau)+v(t-\tau))^2}{4(t-\tau)}| \\
& + |v||M| \frac{1}{2\sqrt{\pi(t-\tau)}} exp - \frac{(\phi(t)+\phi(\tau)+v(t-\tau))^2}{4(t-\tau)}| \\
& + \frac{2|\phi(t)+\phi(\tau)+v(t-\tau)|}{4(t-\tau)} M \frac{1}{2\sqrt{\pi(t-\tau)}} |exp - \frac{(\phi(t)+\phi(\tau)+v(t-\tau))^2}{4(t-\tau)}| \\
\leq & \frac{R+v}{4\sqrt{\pi(t-\tau)}} + \frac{C}{2\sqrt{\pi(t-\tau)}} + \frac{3bC'}{8\sqrt{D\pi(t-\tau)}} (\frac{4}{eb^2})^{\frac{1}{2}} \leq \frac{C_2}{\sqrt{(t-\tau)}}.
\end{aligned} \tag{51}
$$

Now we are able to estimate H_i for $i = 2,3,4$ as follows

$$
\begin{aligned}
|H_2| = & |\int_0^t G_x(\phi(t),t,\phi(\tau),\tau) w(t) d\tau| \\
\leq & \int_0^t |G_x(\phi(t),t,\phi(\tau),\tau)||w(\tau)| d\tau \\
\leq & R \int_0^t |G_x(\phi(t),t,\phi(\tau),\tau)| d\tau \\
\leq & C_2(b,v,e,\pi,R)\sqrt{\sigma},
\end{aligned}
$$

$$
\begin{aligned}
|H_3| = & |\int_0^t G_{\xi x}(\phi(t),t,0,\tau) g(\tau) d\tau| \\
\leq & \int_0^t |G_{\xi x}(\phi(t),t,0,\tau)||g(\tau)| d\tau \\
\leq & ||g|| \int_0^t |-G_{xx}(\phi(t),t,0,\tau)| d\tau \\
\leq & C_3(b,v,e,\pi,R,||g||)\sqrt{\sigma},
\end{aligned}
$$

$$
\begin{aligned}
|H_4| = & |\int_0^t \int_0^{\phi(t)} G_x(\phi(t),t,\xi,\tau) S(\xi,\tau) d\xi d\tau| \\
= & \int_0^t |\int_0^{\phi(t)} G_x(\phi(t),t,\xi,\tau) S(\xi,\tau) d\xi| d\tau \\
\leq & C_4 ||S|| \int_0^t \frac{\phi(t)}{\sqrt{(t-\tau)}} d\tau \\
\leq & C_4(b,v,e,\pi,R,||S||)\sqrt{\sigma}.
\end{aligned}
$$

Therefore, we can write

$$
\begin{aligned}
||T(w)|| \leq & ||H_1|| + ||H_2|| + ||H_3|| + ||H_4|| \\
\leq & ||h'|| + \sum_{i=1}^{3} C_i(\pi,b,e,R,v,||s||,||g||)\sqrt{\sigma},
\end{aligned}
$$

where the constants C_2, C_3, C_4 are simple combinations of $\pi, e, b, \frac{1}{b}, v, k, M, R$.
If R is now taken to be

$$R \geq ||h^{'}|| \sum_{i=1}^{3} C_i(\pi, b, e, R, v, ||s||, ||g||), \tag{52}$$

then for

$$\sqrt{\sigma} < \frac{R}{\sum_{i=1}^{3} C_i(\pi, b, e, R, v, ||s||, ||g||)} - ||h^{'}||, \tag{53}$$

the ball $C_{R,\sigma}$ is mapped into itself.
Now, we prove that T is a contraction on $C_{R,\sigma}$. Let $U_1 = T(w_1)$ and $U_2 = T(w_2)$, then

$$\begin{aligned}
U_1 - U_2 =& T(w_1) - T(w_2) \\
=& \Big[\int_0^b G_x(\phi_1(t), t, \xi, 0) h(\xi) d\xi - \int_0^b G_x(\phi_2(t), t, \xi, 0) h(\xi) d\xi \Big] \\
&+ \Big[\int_0^t G_x(\phi_1(t), t, \phi_1(\tau), \tau) w_1(\tau) d\tau - \int_0^t G_x(\phi_2(t), t, \phi_2(\tau), \tau) w_2(\tau) d\tau \Big] \\
&+ \Big[\int_0^t G_{\xi x}(\phi_1(t), t, 0, \tau) g(\tau) d\tau - \int_0^t G_{\xi x}(\phi_2(t), t, 0, \tau) g(\tau) d\tau \Big] \\
&+ \Big[\int_0^t \int_0^{\phi_1(t)} G_x(\phi_1(t), t, \phi_1(\tau), \tau) S(\xi, \tau) d\xi d\tau \\
&- \int_0^t \int_0^{\phi_2(t)} G_x(\phi_2(t), t, \phi_2(\tau), \tau) S(\xi, \tau) d\xi d\tau \Big] \\
=& \Delta H_1 + \Delta H_2 + \Delta H_3 + \Delta H_4.
\end{aligned} \tag{54}$$

We can write

$$\begin{aligned}
|G_x(\phi_1(t), t, \xi, 0) - G_x(\phi_2(t), t, \xi, 0)| =& \Big| [K_x(\phi_1(t), t, \xi, 0) - v exp(v.\phi_1(t)) \times K(-\phi_1(t), t, \xi, 0) \\
&- exp(v.\phi_1(t)) K_x(-\phi_1(t), t, \xi, 0)] + [-K_x(\phi_2(t), t, \xi, 0) \\
&+ v exp(v.\phi_2(t)) K(-\phi_2(t), t, \xi, 0) + exp(v.\phi_2(t)) K_x(-\phi_2(t), t, \xi, 0)] \Big| \\
\leq& |K_x(\phi_1(t), t, \xi, 0) - K_x(\phi_2(t), t, \xi, 0)| \\
&+ v |exp(v.\phi_1(t)) K(-\phi_1(t), t, \xi, 0) - exp(v.\phi_2(t)) K(-\phi_2(t), t, \xi, 0)| \\
&+ |exp(v.\phi_1(t)) K_x(-\phi_1(t), t, \xi, 0) - exp(v.\phi_2(t)) K_x(-\phi_2(t), t, \xi, 0)| \\
=& E_1 + v E_2 + M E_3.
\end{aligned} \tag{55}$$

By the mean value theorem, there exists $d_1 = d_1(t)$ between $\phi_1(t)$ and $\phi_2(t)$ such that

$$\begin{aligned}
E_1 &= |K_x(\phi_1(t), t, \xi, 0) - K_x(\phi_2(t), t, \xi, 0)| = |K_{xx}(d_1(t), t, \xi, 0)| |\phi_1(t) - \phi_2(t)| \\
&= |\phi_1(t) - \phi_2(t)| \Big| \frac{-1}{2t} + \frac{(d_1(t) - \xi - vt)^2}{4t^2} \Big| |K(d_1(t), t, \xi, 0)|.
\end{aligned} \tag{56}$$

Using the elementary inequality $ye^{-y} \leq const$ for $y \geq 0$, we find

$$\begin{aligned}
||h||\int_0^b E_1 d\xi \leq & ||h||\frac{\sqrt{t}}{2\sqrt{\pi}}||w_1 - w_2||\int_0^\infty (\frac{-1}{2t} + \frac{(d(t)-\xi-vt)^2}{4t^2}) \\
& \times exp(-\frac{(d_1(t)-\xi-vt)^2}{4t})d\xi \\
\leq & ||h||\frac{\sqrt{t}}{2\sqrt{\pi}}||w_1 - w_2||(\frac{d_1(t)-\xi-vt}{2t})exp(-\frac{(d_1(t)-\xi-vt)^2}{4t})\Big|_0^\infty \\
\leq & ||h||\frac{\sqrt{t}}{2\sqrt{\pi}|d_1(t)-vt|}||w_1 - w_2||\frac{(d_1(t)-vt)^2}{2t}exp(-\frac{(d_1(t)-vt)^2}{4t}) \\
\leq & \frac{\sqrt{t}}{4\sqrt{\pi}}||w_1 - w_2||\frac{const.}{|d_1(t)-vt|} \\
\leq & ||h||\frac{\sqrt{t}}{4\sqrt{\pi}}||w_1 - w_2||R_1,
\end{aligned} \tag{57}$$

where

$$\frac{const.}{|d_1(t)-vt|} \leq R_1. \tag{58}$$

By the mean value theorem for E_2, we have

$$\begin{aligned}
E_2 = & |exp(v.\phi_1(t))K(-\phi_1(t),t,\xi,0) - exp(v.\phi_2(t))K(-\phi_2(t),t,\xi,0)| \\
= & \exp(v.\phi_1(t)) \times |K(-\phi_1(t),t,\xi,0) - K(-\phi_2(t),t,\xi,0)| \\
& + |\exp(v.\phi_1(t)) - exp(v.\phi_2(t))|K(-\phi_2(t),t,\xi,0)| \\
\leq & M\frac{c(t)+\xi+vt}{2t}|K(c(t),t,\xi,0)||\phi_1(t)-\phi_2(t)| \\
& + |\exp(v.\phi_1(t)) - \exp(v.\phi_2(t))|K(-\phi_2(t),t,\xi,0)|.
\end{aligned} \tag{59}$$

Then

$$\begin{aligned}
v||h||\int_0^b E_2 d\xi \leq & vM||h||\int_0^b \frac{c(t)+\xi+vt}{2t}|K(c(t),t,\xi,0)||\phi_1(t)-\phi_2(t)|d\xi \\
& + v||h||\int_0^b |\exp(v.\phi_1(t)) - \exp(v.\phi_2(t))|K(-\phi_2(t),t,\xi,0)|d\xi \\
\leq & \frac{vM\sqrt{t}}{2\sqrt{\pi}}||h||||w_1 - w_2||\int_0^\infty \frac{c(t)+\xi+vt}{2t}exp(-\frac{(c(t)+\xi+vt)^2}{4t})d\xi \\
& + v^2||h||\exp(v.c'(t))||w_1 - w_2||t\int_0^\infty |K(-\phi_2(t),t,\xi,0)|d\xi \\
\leq & \frac{vM\sqrt{t}}{2\sqrt{\pi}}||h||||w_1 - w_2||exp(-\frac{(c'(t)+\xi+vt)^2}{4t})\Big|_0^\infty + v^2M||h||||w_1 - w_2||t \\
\leq & \Big(\frac{Mv}{2\sqrt{\pi}} + \frac{v^2M}{2}\sqrt{t}\Big)||h||||w_1 - w_2|\sqrt{t}.
\end{aligned}$$

Finally, for E_3, we have

$$
\begin{aligned}
E_3 \leq & \exp(v.\phi_1(t))\Big|K_x(-\phi_1(t),t,\xi,0)-K_x(-\phi_2(t),t,\xi,0)\Big| \\
& +\Big|\exp(v.\phi_1(t))-\exp(v.\phi_2(t))\Big|K_x(-\phi_2(t),t,\xi,0)| \\
\leq & M|K_{xx}(c''(t),t,\xi,0)||\phi_1(t)-\phi_2(t)| \\
& +vexp(v.c''(t))|K_x(-\phi_2(t),t,\xi,0)||\phi_1(t)-\phi_2(t)| \\
\leq & M|\phi_1(t)-\phi_2(t)|\Big|\frac{-1}{2t}+\frac{(c''(t)+\xi+vt)^2}{4t^2}\Big||K(c''(t),t,\xi,0)| \\
& +Mv|\phi_1(t)-\phi_2(t)||K_\xi(-\phi_2(t),t,\xi,0)|,
\end{aligned}
\tag{60}
$$

where there exists $c(t)$, $c'(t)$ and $c''(t)$ between $\phi_1(t)$ and $\phi_2(t)$. Then

$$
\begin{aligned}
||h||\int_0^b E_3 d\xi \leq & M||h||\frac{\sqrt{t}}{2\sqrt{\pi}}||w_1-w_2||\int_0^\infty(\frac{-1}{2t}+\frac{(c''(t)+\xi+vt)^2}{4t^2}) \\
& exp(-\frac{(c''(t)+\xi+vt)^2}{4t})d\xi \\
& +Mv||h|||\phi_1(t)-\phi_2(t)|\int_0^\infty K_\xi(-\phi_2(t),t,\xi,0)d\xi \\
\leq & M||h||\frac{\sqrt{t}}{2\sqrt{\pi}}||w_1-w_2||(\frac{c''(t)+\xi+vt}{2t})exp(-\frac{(c''(t)+\xi+vt)^2}{4t})\Big|_0^\infty \\
& +Mv||h||\frac{t}{2}||w_1-w_2|| \\
\leq & M||h||\frac{\sqrt{t}}{2\sqrt{\pi}(c''(t)+vt)}||w_1-w_2||(\frac{(c''(t)+vt)^2}{2t})exp(-\frac{(c''(t)+vt)^2}{4t}) \\
& +Mv||h||\frac{t}{2}||w_1-w_2|| \\
\leq & M||h||\frac{\sqrt{t}}{4\sqrt{\pi}}||w_1-w_2||\frac{c}{c''(t)+vt}+Mv||h||\frac{t}{2}||w_1-w_2|| \\
\leq & M\frac{\sqrt{t}}{4\sqrt{\pi}}||w_1-w_2||R_2+Mv\frac{t}{2}||w_1-w_2||.
\end{aligned}
$$

Now we are able to estimate ΔH_1 as follows:

$$
\begin{aligned}
|\Delta H_1| \leq & ||h||\Big(\frac{1}{4\sqrt{\pi}}R_1+\frac{vM}{2\sqrt{\pi}}+\frac{v^2M}{2}\sqrt{t}+\frac{M}{4\sqrt{\pi}}R_2+Mv\frac{\sqrt{t}}{2}\Big)||w_1-w_2||\sqrt{t} \\
\leq & ||h||\Big(\frac{1}{4\sqrt{\pi}}R_1+\frac{vM}{2\sqrt{\pi}}+\frac{v^2M}{2}\sqrt{\sigma}+\frac{M}{4\sqrt{\pi}}R_2+Mv\frac{\sqrt{\sigma}}{2}\Big)||w_1-w_2||\sqrt{\sigma} \\
= & \lambda_0||w_1-w_2||\sqrt{\sigma}.
\end{aligned}
\tag{61}
$$

In the same way, we have

$$
\begin{aligned}
&|G_x(\phi_1(t),t,\phi_1(\tau),\tau)-G_x(\phi_2(t),t,\phi_2(\tau),\tau)| \\
&\qquad=\Big|[K_x(\phi_1(t),t,\phi_1(\tau),\tau)-vexp(v.\phi_1(t))K(-\phi_1(t),t,\phi_1(\tau),\tau) \\
&\qquad\quad-exp(v.\phi_1(t))K_x(-\phi_1(t),t,\phi_1(\tau),\tau)]+[-K_x(\phi_2(t),t,\phi_2(\tau),\tau) \\
&\qquad\quad+vexp(v.\phi_2(t))K(-\phi_2(t),t,\phi_2(\tau),\tau)+exp(v.\phi_2(t))K_x(-\phi_2(t),t,\phi_2(\tau),\tau)]\Big| \\
&\qquad\leq\Big|K_x(\phi_1(t),t,\phi_1(\tau),\tau)-K_x(\phi_2(t),t,\phi_2(\tau),\tau)\Big| \\
&\qquad\quad+v\Big|exp(v.\phi_1(t))K(-\phi_1(t),t,\phi_1(\tau),\tau)-exp(v.\phi_2(t))K(-\phi_2(t),t,\phi_2(\tau),\tau)\Big| \\
&\qquad\quad+\Big|exp(v.\phi_1(t))K_x(-\phi_1(t),t,\phi_1(\tau),\tau)-exp(v.\phi_2(t))K_x(-\phi_2(t),t,\phi_2(\tau),\tau)\Big| \\
&\qquad=E_1+vE_2+E_3.
\end{aligned}
\tag{62}
$$

Define $f_{t,\tau}(x):=exp\Big(-\frac{x^2}{4(t-\tau)}\Big)$. We can write

$$
E_1=\frac{1}{2\sqrt{\pi(t-\tau)}}|f'_{t,\tau}(\phi_1(t)-\phi_1(\tau)-v(t-\tau))-f'_{t,\tau}(\phi_2(t)-\phi_2(\tau)-v(t-\tau))|.
$$

By the mean value theorem, there exists $d_1=d_1(t,\tau)$ between $\phi_1(t)-\phi_1(\tau)$ and $\phi_2(t)-\phi_2(\tau)$ such that

$$
\begin{aligned}
|E_1|=&\frac{1}{2\sqrt{\pi(t-\tau)}}|f''_{t,\tau}(d_1)||\phi_1(t)-\phi_1(\tau)-v(t-\tau)-\phi_2(t)+\phi_2(\tau)+v(t-\tau)| \\
\leq&\frac{1}{2\sqrt{\pi(t-\tau)}}\Big|\frac{-1}{2(t-\tau)}+\frac{d_1^2}{4(t-\tau)^2}\Big|exp(-\frac{d_1^2}{4(t-\tau)}) \\
&\times\Big(|\phi_1(t)-\phi_2(t)|+|\phi_1(\tau)-\phi_2(\tau)|\Big).
\end{aligned}
\tag{63}
$$

Because

$$
|d_1|\leq\max(\,|\phi_i(t)-\phi_i(\tau)-v(t-\tau)|,\quad i=1,2)\;\leq R|t-\tau|+v|t-\tau|,
$$

then

$$
\begin{aligned}
R\int_0^t E_1\leq&Rt||w_1-w_2||\int_0^\infty\frac{1}{2\sqrt{\pi}}\frac{-1}{(t-\tau)^{\frac{3}{2}}}d\tau \\
&+Rt||w_1-w_2||\frac{1}{2\sqrt{\pi}}\int_0^t\frac{R^2(t-\tau)^2+2Rv(t-\tau)^2+v^2(t-\tau)^2}{4(t-\tau)^{\frac{5}{2}}}d\tau \\
\leq&Rt||w_1-w_2||\int_0^\infty\frac{1}{2\sqrt{\pi}}\frac{-1}{(t-\tau)^{\frac{3}{2}}}d\tau+Rt||w_1-w_2||\frac{1}{2\sqrt{\pi}}\int_0^t\frac{(R+v)^2}{\sqrt{t-\tau}}d\tau \\
\leq&\frac{1}{2\sqrt{\pi}}\Big(R+\frac{R(R+v)^2}{4}t\Big)||w_1-w_2||\sqrt{t} \\
\leq&\frac{1}{2\sqrt{\pi}}\Big(R+\frac{R(R+v)^2}{4}\sigma\Big)||w_1-w_2||\sqrt{\sigma}=\lambda_1||w_1-w_2||\sqrt{\sigma}.
\end{aligned}
\tag{64}
$$

For E_2, we have

$$\begin{aligned}
E_2 \leq& \exp(v.\phi_1(t))\Big|K(-\phi_1(t),t,\phi_1(\tau),\tau)-K(-\phi_2(t),t,\phi_2(\tau),\tau)\Big| \\
&+\Big|\exp(v.\phi_1(t))-\exp(v.\phi_2(t))\Big|K(-\phi_2(t),t,\phi_2(\tau),\tau) \\
\leq& M\frac{1}{2\sqrt{\pi(t-\tau)}}|f_{t,\tau}(\phi_1(t)+\phi_1(\tau)+v(t-\tau))-f_{t,\tau}(\phi_2(t)+\phi_2(\tau)+v(t-\tau))| \\
&+\exp(v.c(t))|\phi_1(t)-\phi_2(t)|K(-\phi_2(t),t,\phi_2(\tau),\tau).
\end{aligned}$$

By the mean value theorem, there exists $d_2 = d_2(t,\tau)$ between $\phi_1(t)-\phi_1(\tau)+v(t-\tau)$ and $\phi_2(t)-\phi_2(\tau)+v(t-\tau)$ such that

$$b \leq b+vt \leq \phi_1(t)+\phi_1(\tau)+v(t-\tau) \leq d_2(t,\tau) \leq \phi_2(t)+\phi_2(\tau)+v(t-\tau) \leq 3b+vt.$$

Then

$$\begin{aligned}
E_2 =& \frac{M}{2\sqrt{\pi(t-\tau)}}|f'_{t,\tau}(d_2)||\phi_1(t)+\phi_1(\tau)+v(t-\tau)-\phi_2(t)-\phi_2(\tau)-v(t-\tau)| \\
&+M|\phi_1(t)-\phi_2(t)|K(-\phi_2(t),t,\phi_2(\tau),\tau) \\
\leq& \frac{M}{2\sqrt{\pi(t-\tau)}}|\frac{3b+v(t-\tau)}{2(t-\tau)}exp(-\frac{d_2^{*2}}{4(t-\tau)}) \\
&\times|\Big(|\phi_1(t)-\phi_2(t)|+|\phi_1(\tau)-\phi_2(\tau)|\Big)+... \\
\leq& \frac{M}{4\sqrt{\pi}}|\frac{3b+v(t-\tau)}{(t-\tau)^{\frac{3}{2}}}exp(-\frac{b^2}{4(t-\tau)})||w_1-w_2||t+M||w_1-w_2||tK(-\phi_2(t),t,\phi_2(\tau),\tau) \\
\leq& \Big(\frac{3Mb}{4\sqrt{\pi}}((\frac{6}{eb^2})^{\frac{3}{2}}+\frac{v}{\sqrt{t-\tau}}exp(-\frac{b^2}{4(t-\tau)}))+MK(-\phi_2(t),t,\phi_2(\tau),\tau)\Big)||w_1-w_2||t.
\end{aligned}$$

We deduce that

$$\begin{aligned}
vR\int_0^t E_2 d\tau \leq& \Big(\frac{3vRb}{4\sqrt{\pi}}(\frac{6}{eb^2})^{\frac{3}{2}}t\sqrt{t}+\frac{3Rv^2}{4\sqrt{\pi}}t+\frac{MRvt}{2\sqrt{\pi}}\Big)||w_1-w_2||\sqrt{t} \\
\leq& \Big(\frac{3vRb}{4\sqrt{\pi}}(\frac{6}{eb^2})^{\frac{3}{2}}\sigma\sqrt{\sigma}+\frac{3Rv^2}{4\sqrt{\pi}}\sigma+\frac{MRv\sigma}{2\sqrt{\pi}}\Big)||w_1-w_2||\sqrt{\sigma} \\
=& \lambda_2||w_1-w_2||\sqrt{\sigma}.
\end{aligned} \tag{65}$$

Finally, for E_3, we have

$$\begin{aligned}
E_3 =& \Big|exp(v.\phi_1(t))K_x(-\phi_1(t),t,\phi_1(\tau),\tau)-exp(v.\phi_2(t))K_x(-\phi_2(t),t,\phi_2(\tau),\tau)\Big| \\
\leq& exp(v.\phi_1(t))\Big|K_x(-\phi_1(t),t,\phi_1(\tau),\tau)-K_x(-\phi_2(t),t,\phi_2(\tau),\tau)\Big| \\
&+\Big|exp(v.\phi_1(t))-exp(v.\phi_2(t))\Big||K_x(-\phi_2(t),t,\phi_2(\tau),\tau)|.
\end{aligned}$$

By applying the mean value theorem, there exists $d_3 = d_3(t,\tau)$ between $\phi_1(t)-\phi_1(\tau)+v(t-\tau)$ and $\phi_2(t)-\phi_2(\tau)+v(t-\tau)$ such that

$$b \le b+vt \le \phi_1(t)+\phi_1(\tau)+v(t-\tau) \le d_3(t,\tau) \le \phi_2(t)+\phi_2(\tau)+v(t-\tau) \le 3b+vt.$$

Then

$$\begin{aligned}
E_3 =& \frac{1}{2\sqrt{\pi(t-\tau)}}|f''_{t,\tau}(d_3)||\phi_1(t)-\phi_1(\tau)+v(t-\tau)-\phi_2(t)-\phi_2(\tau)-v(t-\tau)|\\
&+exp(v.d(t))|\phi_1(t)-\phi_2(t)||K_x(-\phi_2(t),t,\phi_2(\tau),\tau)|\\
\le& \frac{1}{2\sqrt{\pi(t-\tau)}}\Big|\frac{-1}{2(t-\tau)}+\frac{d_3^2}{4(t-\tau)^2}\Big|\\
&\times exp(-\frac{d_3^2}{4(t-\tau)})\Big(|\phi_1(t)-\phi_2(t)|+|\phi_1(\tau)-\phi_2(\tau)|\Big)\\
&+Mt||w_1-w_2|||K_x(-\phi_2(t),t,\phi_2(\tau),\tau)|,
\end{aligned}$$

and

$$\begin{aligned}
R\int_0^t E_3 d\tau \le& \frac{R}{4\sqrt{\pi}}(\frac{6}{eb^2})^{\frac{3}{2}}t+\frac{R}{2\sqrt{\pi}}\int_0^t \frac{(3b+v(t-\tau))^2}{4(t-\tau)^{\frac{5}{2}}}exp(-\frac{d_3^2}{4(t-\tau)})||w_1-w_2||td\tau\\
&+\frac{MRt}{\sqrt{\pi}}||w_1-w_2||\int_0^t \frac{\phi_2(t)+\phi_2(\tau)+v(t-\tau)}{2(t-\tau)\sqrt{t-\tau}}exp(-\frac{(\phi_2(t)+\phi_2(\tau)+v(t-\tau))^2}{4(t-\tau)})d\tau\\
\le& \frac{R}{2\sqrt{\pi}}\Big(\frac{1}{2}(\frac{6}{eb^2})^{\frac{3}{2}}t\sqrt{t}+\frac{9b^2}{4}(\frac{10}{eb^2})^{\frac{5}{2}}t\sqrt{t}+\frac{6bv}{4}(\frac{6}{eb^2})^{\frac{3}{2}}t\sqrt{t}+\frac{v^2}{4}t\Big)||w_1-w_2||\sqrt{t}\\
&+\frac{MRt}{\sqrt{\pi}}||w_1-w_2||\Big(\int_0^t \frac{3b}{2}(\frac{6}{eb^2})^{\frac{3}{2}}d\tau+\frac{v}{2}\int_0^t \frac{1}{\sqrt{t-\tau}}d\tau\Big)\\
\le& \frac{R}{2\sqrt{\pi}}\Big(\frac{1}{2}(\frac{6}{eb^2})^{\frac{3}{2}}t\sqrt{t}+\frac{9b^2}{4}(\frac{10}{eb^2})^{\frac{5}{2}}t\sqrt{t}+\frac{6bv}{4}(\frac{6}{eb^2})^{\frac{3}{2}}t\sqrt{t}+\frac{v^2}{4}t\Big)||w_1-w_2||\sqrt{t}\\
&+\frac{MRt}{\sqrt{\pi}}\Big(\frac{3b}{2}(\frac{6}{eb^2})^{\frac{3}{2}}\sqrt{t}+\frac{v}{2}\Big)||w_1-w_2||\sqrt{t}\\
\le& \frac{R}{2\sqrt{\pi}}\Big(\frac{1}{2}(\frac{6}{eb^2})^{\frac{3}{2}}\sigma\sqrt{\sigma}+\frac{9b^2}{4}(\frac{10}{eb^2})^{\frac{5}{2}}\sigma\sqrt{\sigma}+\frac{6bv}{4}(\frac{6}{eb^2})^{\frac{3}{2}}\sigma\sqrt{\sigma}+\frac{v^2}{4}\sigma\\
&+\frac{MR\sigma}{\sqrt{\pi}}\Big(\frac{3b}{2}(\frac{6}{eb^2})^{\frac{3}{2}}\sqrt{\sigma}+\frac{v}{2}\Big)\Big)||w_1-w_2||\sqrt{\sigma}\\
=& \lambda_3||w_1-w_2||\sqrt{\sigma}.
\end{aligned}$$

Therefore, we can estimate ΔH_2 as follows:

$$|\Delta H_2| \le \Big(\lambda_1+\lambda_2+\lambda_3\Big)||w_1-w_2||\sqrt{\sigma}. \tag{66}$$

In the same way, we obtain

$$|\Delta H_3| \le \Big(\lambda_4+\lambda_5+\lambda_6\Big)||w_1-w_2||\sqrt{\sigma} \tag{67}$$

and

$$|\Delta H_4| \leq \lambda_7 ||w_1 - w_2|| \sqrt{\sigma}, \tag{68}$$

where the constants λ_4, λ_5, λ_6, and λ_7 are simple combinations of $\pi, e, b, \frac{1}{b}, v, k, M, R, \sigma$. Therefore, we can write

$$\|T(w_1) - T(w_2)\| \leq \sum_{i=0}^{7} \lambda_i \|w_1 - w_2\| \sqrt{\sigma}.$$

By selecting

$$\sqrt{\sigma} < \frac{1}{\sum_{i=0}^{7} \lambda_i},$$

the map T becomes a contraction mapping on $C_{R,\sigma}$ and therefore it has a unique fixed point. □

Proof of Theorem 1: Taking into account Theorem 2, problem (13)-(18) is equivalent to the problem of finding a continuous solution $w(t)$ for the integral equation (39). Employing Theorem 3, we conclude that T has a unique fixed point $w(t)$ in $C_{R,\sigma}$. The $w(t)$ is then a solution of (39). In view of Theorem 2, we have thus proved the existence of a solution of (13)-(18) for all $t < \sigma$, for some σ sufficiently small (restricted only by (50)). Therefore, the proof of Theorem 1 is completed. □

6 Conclusion

We have proposed a robust mathematical framework for modeling air pollution as a free boundary problem, offering a dynamic and physically consistent representation of pollutant dispersion in the atmosphere. Unlike traditional fixed-boundary models, our approach accounts for the evolving nature of the spatial domain, allowing the pollutant front to move in response to physical processes and boundary interactions. This makes the model particularly well-suited for representing real-world situations, where pollutant plumes propagate dynamically due to variable emission rates, atmospheric transport, and chemical reactions. The theoretical analysis was carried out by reformulating the original partial differential equation with a free boundary into a nonlinear Volterra integral equation using the Friedman–Rubinstein integral representation. This reformulation allowed us to rigorously prove the existence and uniqueness of the solution by applying the Banach fixed-point theorem, under Dirichlet boundary conditions. The resulting framework incorporates key transport phenomena such as advection, diffusion, and reactive mechanisms, together with time-dependent source terms, offering a solid basis for analytical exploration of air pollution dynamics in bounded regions.

While the main objective of this study has been to establish a rigorous framework for proving the existence and uniqueness of the free boundary problem, the proposed method also offers potential for future analytical and computational works. In particular, the extension of the model to include Neumann boundary conditions would enable the treatment of impermeable boundary conditions, broadening the range of physical configurations that can be studied. Another potential direction lies in the application of this framework to multi-dimensional settings or more complex chemical interaction mechanisms.

Acknowledgements

We would like to thank editor and the anonymous referees for their valuable comments and helpful suggestions to improve the earlier version of this paper.

Conflicts of interest

The authors declare that there are no conflicts of interest.

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