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Boundary value problems for singular iterative dynamic equations on time scales

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Abstract. This study explores an iterative system of singular three-point boundary value problems within the context of time scales. The objective is to identify conditions that guarantee the existence of countable positive solutions. The research employs Holder's inequality and Krasnoselskii's cone fixed point theorem, set within a Banach space framework, to derive the necessary criteria. The theoretical findings are illustrated through a practical example, highlighting the sufficiency of the derived conditions for ensuring multiple positive solutions.

Keywords: Existence criteria, iterative BVP, time scale, cone, Banach space.
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1 Introduction

Analyzing dynamic systems requires a framework capable of integrating both continuous and discrete behaviors. Traditionally, this has been achieved through differential equations for continuous systems and difference equations for discrete ones. This research focuses on the theory of time scales, which offers a unified approach to these analyses [7]. Introduced by Stefan Hilger in 1988, time scales allow for the modeling of hybrid systems by considering time as a non-empty, closed subset of the real numbers [18]. This approach enables the study of phenomena that exhibit both continuous and discrete changes. Over recent years, the theory of time scales has greatly enhanced our understanding of boundary value problems (BVPs) [27]. Researchers have employed a variety of mathematical methods, including fixed-point theorems, upper and lower solution techniques, degree theory, and variational methods, to explore the existence and characteristics of solutions to these problems [16, 21, 25, 28].

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Recent advancements in iterative BVPs on time scales include studies on singular multipoint problems [10, 15] and integro-dynamic equations [13]. For singular systems, works such as [24] explore nonlocal initial value problems in Banach spaces, while [14, 26] addresses impulses and nonlocal conditions. Additional contributions include nonlocal initial value problems for first-order dynamic equations [21, 23], discrete Ponzi scheme models via Sturm-Liouville theory [8], and comprehensive overviews in books like [2]. Our approach unifies these by focusing on iterative singular three-point BVPs, establishing countable positive solutions under varied integrability conditions for coefficients.

Compared to prior works such as [10] on singular multipoint BVPs and [15] on p-Laplacian equations, the novelty lies in our iterative framework for singular three-point BVPs, the application of Hölder's inequality for multiple integrability cases, and the proof of countable positive solutions using Krasnoselskii's theorem in a cone setting.

This unified framework transcends theoretical benefits, offering a powerful tool for modeling real-world phenomena across disciplines. Its strength lies in its ability to capture systems exhibiting both continuous and discrete dynamics, a common feature in fields like neural networks, heat transfer, and epidemiology [22]. For instance, models for insect population dynamics or disease propagation necessitate a hybrid approach to accurately represent the interplay between continuous changes (e.g., population growth) and discrete events (e.g., birth events, transmission). The foundational aspects of this approach have been extensively documented in the literature [1, 11, 12]. In this work, we use the mixed delta-nabla derivative operator to perform the analysis without requiring too many jumpers in the computation. Atici and Guseinov [9] were the first authors to propose the idea of equal search for mixed properties (see [4, 6, 19]).

In [10], Bohner and Luo studied the existence of solutions to the following singular second order m-point boundary value problem on time scales:

$$r^{\Delta\nabla}(s) = h(s,r,r^{\Delta}) + e(s), \quad s \in (a,b],$$
$$r^{\Delta}(a) = 0, \;\; r(\sigma(b)) = \sum_{i=1}^{m-2} a_i r(\xi_i).$$

Recently, Dogan [15] developed positive solution by index theory to the following BVP:

$$\big(\phi_p(r^{\Delta}(s))\big)^{\nabla} + \Upsilon(s)h\big(s,r(s)\big) = 0, \; s \in [0,d]_{\mathbb{S}},$$
$$r(0) = \sum_{\ell=1}^{l-2} a_\ell r(\zeta_\ell), \; \phi_p(r^{\Delta}(d)) = \sum_{\ell=1}^{l-2} b_i \phi_p(r^{\Delta}(\zeta_\ell)).$$

Building on recent advancements, we consider a dynamic iterative system subject to two-point boundary conditions and the presence of multiple singularities. Using Krasnoselskii's fixed point theorem within a Banach space framework, we demonstrate the existence of a countable set of positive solutions to the following BVP:

$$\begin{cases} r_i^{\Delta\nabla}(s) + \upsilon(s)h_i\big(r_{i+1}(s)\big) = 0, \; 1 \le i \le j, \; s \in (0,\sigma(1)]_{\mathbb{S}}, \\ r_{j+1}(s) = r_1(s), \; s \in (0,\sigma(1)]_{\mathbb{S}}, \end{cases} \tag{1}$$

$$r_i^{\Delta}(0) = 0, \; a r_i(\sigma(1)) - b r_i(\tau) = 0, \; 1 \le i \le j, \tag{2}$$

where $n \in \mathbb{N}, a, b \in \mathbb{R}^+$, with $b < a$, and $0 < \tau < \sigma(1)/2$, $\upsilon(s) = \prod_{i=1}^{m} \upsilon_i(s)$ and $\upsilon_i(s) \in L_{\nabla}^{p_i}((0,\sigma(1)]_{\mathbb{S}})$ $(p_i \ge 1, \, 1 \le i \le m)$ has a singularity in the interval $(0,\sigma(1)/2]_{\mathbb{S}}$. Using Hölder's inequality alongside

Krasnoselskii's fixed point theorem within a Banach space, we demonstrate the presence of countable positive solutions to (1)–(2). In this study, we operate under the following assumptions:

($\mathtt{C}_1$) $h_i \in \mathrm{C}\big([0,+\infty)\big)$.

($\mathtt{C}_2$) $\{s_d\}_{d=1}^{\infty}$ be a sequence that satisfies the condition $0 < s_{d+1} < s_d < \dfrac{\sigma(1)}{2}$,

$$\lim_{d\to\infty} s_d = s^* < \frac{\sigma(1)}{2}, \lim_{s\to s_d} \upsilon_\gamma(s) = +\infty, \gamma = 1,\dots,m.$$

Moreover, for every γ from 1 to m, there exists a positive constant η_γ satisfying $\upsilon_\gamma(s) > \eta_\gamma$.

2 Preliminaries

In this section, we review key foundational results that will be useful in the subsequent sections. A time scale $\mathbb{S}$ is a nonempty closed subset of $\mathbb{R}$, introduced to unify and generalize results from continuous and discrete analysis. The backward and forward jump operators, ρ and σ, are defined as $\rho(\mathtt{z}) = \sup\{\mathtt{u} \in \mathbb{S} : \mathtt{u} < \mathtt{z}\}$ and $\sigma(\mathtt{z}) = \inf\{\mathtt{u} \in \mathbb{S} : \mathtt{u} > \mathtt{z}\}$, respectively. The forward jump operator identifies the next point in $\mathbb{S}$ larger than $\mathtt{z}$, while the backward jump operator gives the previous point smaller than $\mathtt{z}$. The graininess function, $\mu(\mathtt{z}) = \sigma(\mathtt{z}) - \mathtt{z}$, measures the distance between a point and its forward jump. Intervals on a time scale are defined similarly to real numbers, with the interval $[\mathtt{c}, d]_{\mathbb{S}} = \{\mathtt{z} \in \mathbb{S} : \mathtt{c} \le \mathtt{z} \le d\}$. Points on time scales can be classified based on their relation to the jump operators. A point $\mathtt{z}$ is called right-scattered if $\sigma(\mathtt{z}) > \mathtt{z}$ and left-scattered if $\rho(\mathtt{z}) < \mathtt{z}$. A point is considered right-dense if $\sigma(\mathtt{z}) = \mathtt{z}$, and left-dense if $\rho(\mathtt{z}) = \mathtt{z}$. If a point is scattered both left and right, it is called isolated, and if it is dense on both sides, it is termed dense. The delta derivative, denoted as $\mathtt{h}^{\Delta}(\mathtt{z})$, of a function $\mathtt{h}$ at a point $\mathtt{z} \in \mathbb{S}^{\kappa}$ (the non-maximal set of $\mathbb{S}$) is defined as the number such that for every $\varepsilon > 0$, there exists a neighborhood V of $\mathtt{z}$ where the inequality

$$|\mathtt{h}(\sigma(\mathtt{z})) - \mathtt{h}(\mathtt{u}) - \mathtt{h}^{\Delta}(\mathtt{z})(\sigma(\mathtt{z}) - \mathtt{u})| \le \varepsilon|\sigma(\mathtt{z}) - \mathtt{u}|$$

holds for $\mathtt{u} \in V$. This derivative generalizes the standard derivative to time scales.

Lemma 1. *For any* $\mathrm{G}(s) \in \mathrm{C}_{\mathtt{ld}}((0,\sigma(1)]_{\mathbb{S}}, \mathbb{R})$, *the boundary value problem*

$$r_1^{\Delta\nabla}(s) + \mathrm{G}(s) = 0,\ s \in (0,\sigma(1)]_{\mathbb{S}}, \tag{1}$$

$$r_1^{\Delta}(0) = 0,\ ar_1(\sigma(1)) - br_1(\tau) = 0 \tag{2}$$

has a unique solution

$$r_1(s) = \int_0^{\sigma(1)} M(s,t)\mathrm{G}(t)\nabla t + \frac{b}{a-b}\int_0^{\sigma(1)} M(\tau,t)\mathrm{G}(t)\nabla t, \tag{3}$$

where

$$M(s,t) = \begin{cases} \sigma(1) - s, & if\ \ 0 \le t \le s \le \sigma(1), \\ \sigma(1) - t, & if\ \ 0 \le s \le t \le \sigma(1). \end{cases} \tag{4}$$

Proof. Suppose r_1 is a solution of (1), then

$$\begin{aligned} r_1(s) &= -\int_0^s \int_0^t \mathrm{G}(t_1)\nabla t_1 \Delta t + \mathtt{A}s + \mathtt{B} \\ &= -\int_0^t (s-t)\mathrm{G}(t)\nabla t + \mathtt{A}s + \mathtt{B}, \end{aligned}$$

where $\mathtt{A} = r_1^{\Delta}(0)$ and $\mathtt{B} = r_1(0)$. From (2), we obtain $\mathtt{A} = 0$ and

$$\mathtt{B} = \int_0^{\sigma(1)} (\sigma(1)-t)\mathrm{G}(t)\nabla t + \frac{b}{a} r_1(\tau).$$

So, we get

$$\begin{aligned} r_1(s) &= -\int_0^s (s-t)\mathrm{G}(t)\nabla t + \int_0^{\sigma(1)} (\sigma(1)-t)\mathrm{G}(t)\nabla t + \frac{b}{a} r_1(\tau) \\ &= \int_0^{\sigma(1)} M(s,t)\mathrm{G}(t)\nabla t + \frac{b}{a} r_1(\tau). \end{aligned}$$

Set $s = \tau$ and multiply by $\frac{b}{a}$ in the equation above (2). We obtain

$$r_1(\tau) = \frac{a}{a-b} \int_0^{\sigma(1)} M(\tau,t)\mathrm{G}(t)\nabla t. \tag{5}$$

By substituting (5) into (2), we obtain the desired solution (3). This concludes the proof. □

Lemma 2. *Assume that conditions* $(\mathtt{C}_1)$ *and* $(\mathtt{C}_2)$ *are satisfied. Let* $\nu \in (0, \sigma(1)/2)_{\mathbb{S}}$ *and* $\tau \in [\nu, \sigma(1)-\nu]_{\mathbb{S}}$. *The kernel* $M(s,t)$ *exhibits the subsequent characteristics:*

(i) $0 \le M(s,t) \le M(t,t)$ *with respect to any* $s,t \in [0,\sigma(1)]_{\mathbb{S}}$,

(ii) $\frac{\nu}{\sigma(1)} M(t,t) \le M(s,t)$ *for all* $s \in [\nu, \sigma(1)-\nu]_{\mathbb{S}}$ *and* $t \in [0,\sigma(1)]_{\mathbb{S}}$.

Proof. To see that inequality (i) holds is straightforward. For the proof of (ii), let $s \in [\nu, \sigma(1)-\nu]_{\mathbb{S}}$ and assume $t \le s$. Then

$$\frac{M(s,t)}{M(t,t)} = \frac{\sigma(1)-s}{\sigma(1)-t} \ge \frac{\nu}{\sigma(1)}.$$

For $s \le t$,

$$\frac{M(s,t)}{M(t,t)} = \frac{\sigma(1)-t}{\sigma(1)-t} = 1 \ge \frac{\nu}{\sigma(1)}.$$

This concludes the proof. □

Note that an $j-$tuple $(r_1(s), r_2(s), r_3(s), \ldots, r_j(s))$ is a solution of the iterative boundary value problem (1)–(2) if and only if

$$r_i(s) = \int_0^{\sigma(1)} M(s,t)\upsilon(t)h_i(r_{i+1}(t))\nabla t + \frac{b}{a-b}\int_0^{\sigma(1)} M(\tau,t)\upsilon(t)h_i(r_{i+1}(t))\nabla t,$$

and

$$r_{i+1}(s) = r_1(s),\ s \in (0,\sigma(1)]_{\mathbb{S}},\ 1 \le i \le j.$$

That is

$$\begin{aligned} r_1(s) = &\int_0^{\sigma(1)} M(s,t_1)\upsilon(t_1)h_1\Big[\int_0^{\sigma(1)} M(t_1,t_2)\upsilon(t_2)h_2\Big[\int_0^{\sigma(1)} M(t_2,t_3)\cdots \\ &\times h_{j-1}\Big[\int_0^{\sigma(1)} M(t_{j-1},t_j)\upsilon(t_j)h_j(r_1(t_j))\Delta t_j\Big]\cdots\Delta t_3\Big]\Delta t_2\Big]\Delta t_1 \\ &+\frac{b}{a-b}\int_0^{\sigma(1)} M(\tau,t_1)\upsilon(t_1)h_1\Big[\int_0^{\sigma(1)} M(t_1,t_2)\upsilon(t_2)\cdots \\ &\times h_{j-1}\Big[\int_0^{\sigma(1)} M(t_{j-1},t_j)\upsilon(t_j)h_j(r_1(t_j))\Delta t_j\Big]\cdots\Delta t_3\Big]\Delta t_2\Big]\Delta t_1. \end{aligned}$$

Let $\mathtt{Y}$ be defined as the Banach space $C_{\mathtt{ld}}((0,\sigma(1)]_{\mathbb{S}},\mathbb{R})$ where the norm is given by $\|r\| = \max_{s\in(0,\sigma(1)]_{\mathbb{S}}} |r(s)|$. For $\nu \in (0,\sigma(1)/2)_{\mathbb{S}}$, the cone $\mathrm{H}_\nu \subset \mathtt{Y}$ be defined as

$$\mathrm{H}_\nu = \left\{ r\in\mathtt{Y} : r(s) \text{ is nonnegative and } \min_{s\in[\nu,\sigma(1)-\nu]_{\mathbb{S}}} r(s) \ge \frac{\nu}{\sigma(1)}\|r(s)\| \right\}.$$

For any $r_1 \in \mathrm{H}_\nu$, define an operator $\mho : \mathrm{H}_\nu \to \mathtt{Y}$ by

$$\begin{aligned} (\mho r_1)(s) = &\int_0^{\sigma(1)} M(s,t_1)\upsilon(t_1)h_1\Big[\int_0^{\sigma(1)} M(t_1,t_2)\upsilon(t_2)h_2\Big[\int_0^{\sigma(1)} M(t_2,t_3)\cdots \\ &\times h_{j-1}\Big[\int_0^{\sigma(1)} M(t_{j-1},t_j)\upsilon(t_j)h_j(r_1(t_j))\Delta t_j\Big]\cdots\Delta t_3\Big]\Delta t_2\Big]\Delta t_1 \\ &+\frac{b}{a-b}\int_0^{\sigma(1)} M(\tau,t_1)\upsilon(t_1)h_1\Big[\int_0^{\sigma(1)} M(t_1,t_2)\upsilon(t_2)\cdots \\ &\times h_{j-1}\Big[\int_0^{\sigma(1)} M(t_{j-1},t_j)\upsilon(t_j)h_j(r_1(t_j))\Delta t_j\Big]\cdots\Delta t_3\Big]\Delta t_2\Big]\Delta t_1. \end{aligned}$$

Lemma 3. *Assume that* $(\mathtt{C}_1)$–$(\mathtt{C}_2)$ *hold. Then for each* $\nu \in (0,\sigma(1)/2)_{\mathbb{S}}$, $\mho(\mathrm{H}_\nu) \subset \mathrm{H}_\nu$ *and* $\mho : \mathrm{H}_\nu \to \mathrm{H}_\nu$ *are completely continuous.*

Proof. According to Lemma 2, $M(s,t) \ge 0$ in all cases where $s,t \in (0,\sigma(1)]_{\mathbb{S}}$. So, $(\mho r_1)(s) \ge 0$. Also, for $r_1 \in \mathrm{H}_\nu$ we have

$$\begin{aligned}
\|\mho r_1\| = \max_{s\in(0,\sigma(1)]_{\mathbb{S}}} \int_0^{\sigma(1)} & M(s,t_1)\upsilon(t_1)h_1\left[\int_0^{\sigma(1)} M(t_1,t_2)\upsilon(t_2)\cdots\right.\\
& \left.h_{j-1}\left[\int_0^{\sigma(1)} M(t_{j-1},t_j)\upsilon(t_j)h_j(r_1(t_j))\Delta t_j\right]\cdots\Delta t_3\right]\Delta t_2\Big]\Delta t_1\\
& + \frac{b}{a-b}\int_0^{\sigma(1)} M(\tau,t_1)\upsilon(t_1)h_1\left[\int_0^{\sigma(1)} M(t_1,t_2)\upsilon(t_2)\right.\\
& \left.\cdots h_{j-1}\left[\int_0^{\sigma(1)} M(t_{j-1},t_j)\upsilon(t_j)h_j(r_1(t_j))\Delta t_j\right]\cdots\Delta t_3\right]\Delta t_2\Big]\Delta t_1\\
\le \int_0^{\sigma(1)} & M(t_1,t_1)\upsilon(t_1)h_1\left[\int_0^{\sigma(1)} M(t_1,t_2)\upsilon(t_2)\right.\\
& \left.\cdots h_{j-1}\left[\int_0^{\sigma(1)} M(t_{j-1},t_j)\upsilon(t_j)h_j(r_1(t_j))\Delta t_j\right]\cdots\Delta t_3\right]\Delta t_2\Big]\Delta t_1\\
& + \frac{b}{a-b}\int_0^{\sigma(1)} M(t_1,t_1)\upsilon(t_1)h_1\left[\int_0^{\sigma(1)} M(t_1,t_2)\upsilon(t_2)\right.\\
& \left.\cdots h_{j-1}\left[\int_0^{\sigma(1)} M(t_{j-1},t_j)\upsilon(t_j)h_j(r_1(t_j))\Delta t_j\right]\cdots\Delta t_3\right]\Delta t_2\Big]\Delta t_1.
\end{aligned}$$

Again from Lemma 2, we get

$$\begin{aligned}
\min_{s\in[\nu,\sigma(1)-\nu]_{\mathbb{S}}}\{(\mho r_1)(s)\} \ge \frac{\nu}{\sigma(1)}\Bigg[&\int_0^{\sigma(1)} M(t_1,t_1)\upsilon(t_1)h_1\left[\int_0^{\sigma(1)} M(t_1,t_2)\upsilon(t_2)\right.\\
& \left.\cdots h_{j-1}\left[\int_0^{\sigma(1)} M(t_{j-1},t_j)\upsilon(t_j)h_j(r_1(t_j))\Delta t_j\right]\cdots\Delta t_3\right]\Delta t_2\Big]\Delta t_1\\
& + \frac{b}{a-b}\int_0^{\sigma(1)} M(t_1,t_1)\upsilon(t_1)h_1\left[\int_0^{\sigma(1)} M(t_1,t_2)\upsilon(t_2)\right.\\
& \left.\cdots h_{j-1}\left[\int_0^{\sigma(1)} M(t_{j-1},t_j)\upsilon(t_j)h_j(r_1(t_j))\Delta t_j\right]\cdots\Delta t_3\right]\Delta t_2\Big]\Delta t_1\Bigg].
\end{aligned}$$

From the two inequalities above, it follows that

$$\min_{s\in[\nu,\sigma(1)-\nu]_{\mathbb{S}}}\{(\mho r_1)(s)\} \ge \frac{\nu}{\sigma(1)}\|\mho r_1\|.$$

Therefore, $\mho r_1 \in \mathrm{H}_\nu$, which implies that $\mho(\mathrm{H}_\nu) \subset \mathrm{H}_\nu$. Next, using standard techniques and the Arzela-Ascoli theorem, it can be shown that the operator $\mho$ is completely continuous. This concludes the proof. □

3 Main Results

We will employ the following theorems to prove the existence of countable set of positive solutions for the iterative system defined by BVP (1)-(2).

Theorem 1 ([17])**.** *Let $\mathscr{X}$ be a Banach space and $\mathscr{E}$ be a cone in $\mathscr{X}$. Also, let m_1, m_2 be open sets with $0 \in m_1$ and $\overline{m}_1 \subset m_2$. Define $\mathscr{A} : \mathscr{E} \cap (\overline{m}_2 \setminus m_1) \to \mathscr{E}$ as a completely continuous operator satisfying*

(a) $\|\mathscr{A}v\| \leq \|v\|$, *for* $v \in \mathscr{E} \cap \partial m_1$, *and* $\|\mathscr{A}v\| \geq \|v\|$, *for* $v \in \mathscr{E} \cap \partial m_2$, *or*

(b) $\|\mathscr{A}v\| \geq \|v\|$, *for* $v \in \mathscr{E} \cap \partial m_1$, *and* $\|\mathscr{A}v\| \leq \|v\|$, *for* $v \in \mathscr{E} \cap \partial m_2$.

Then there exists a fixed point of $\mathscr{A}$ in $\mathscr{E} \cap (\overline{m}_2 \setminus m_1)$.

Theorem 2 ([3,5,20])**.** *Let $f \in L^p_\nabla(J)$ with $p > 1$, $g \in L^q_\Delta(J)$ with $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1_\Delta(J)$ and $\|fg\|_{L^1_\Delta} \leq \|f\|_{L^p_\Delta}\|g\|_{L^q_\Delta}$, where*

$$\|f\|_{L^p_\Delta} := \begin{cases} \left[\int_J |f(s)|^p \Delta s\right]^{\frac{1}{p}}, & \text{if } p \in \mathbb{R}, \\ \inf\{M \in \mathbb{R} \mid |f| \leq M \ \Delta\text{-a.e. on } J\}, & \text{if } p = \infty, \end{cases}$$

and $J = [a, b)_{\mathbb{S}}$.

Theorem 3 (Holder's)**.** *Let $f \in L^{p_i}_\Delta(J)$ with $p_i > 1$, for $i = 1, 2, \cdots, n$ and $\sum_{i=1}^n \frac{1}{p_i} = 1$. Then $\prod_{i=1}^n f_i \in L^1_\Delta(J)$ and $\|\prod_{i=1}^n f_i\|_1 \leq \prod_{i=1}^n \|f_i\|_{p_i}$. Further, if $f \in L^1_\Delta(J)$ and $g \in L^\infty_\Delta(J)$. Then $fg \in L^1_\Delta(J)$ and $\|fg\|_1 \leq \|f\|_1\|g\|_\infty$.*

Three cases for $\upsilon_i \in L^{p_i}_\Delta(0,1)_{\mathbb{S}}$ are as follows:

$$\sum_{i=1}^n \frac{1}{p_i} < 1, \ \sum_{i=1}^n \frac{1}{p_i} = 1, \ \sum_{i=1}^n \frac{1}{p_i} > 1.$$

Firstly, we seek countable positive solutions for the case $\displaystyle\sum_{i=1}^n \frac{1}{p_i} < 1$.

Theorem 4. *Suppose $(\mathtt{C}_1)$–$(\mathtt{C}_2)$ hold. Let $\{\nu_d\}_{d=1}^\infty$ be a sequence with $s_{d+1} < \nu_d < s_d$. Let $\{J_d\}_{d=1}^\infty$ and $\{\mathrm{Q}_d\}_{d=1}^\infty$ be such that*

$$J_{d+1} < \frac{\nu_d}{\sigma(1)}\mathrm{Q}_d < \mathrm{Q}_d < \theta\mathrm{Q}_d < J_d \ \ \text{and} \ \ \frac{\nu_d}{\sigma(1)} < \frac{1}{2}, \ d \in \mathbb{N},$$

where

$$\theta = \max\left\{ \left[\frac{\nu_1}{\sigma(1)}\prod_{i=1}^m \eta_i \int_{\nu_1}^{\sigma(1)-\nu_1} M(t,t)\Delta t\right]^{-1}, \left[\frac{b}{a-b}\frac{\nu_1}{\sigma(1)}\prod_{i=1}^m \eta_i \int_{\nu_1}^{\sigma(1)-\nu_1} M(t,t)\nabla t\right]^{-1} \right\}.$$

Also, assume that h_i satisfies

$(\mathtt{C}_3)$ $h_i(r) \leq \dfrac{N_1 J_d}{2}$ $\forall\, s \in (0, \sigma(1)]_{\mathbb{S}}$, $0 \leq r \leq J_d$, *where*

$$N_1 < \min\left\{ \left[\|M\|_{L^q_\nabla}\prod_{i=1}^m \|\upsilon_i\|_{L^{p_i}_\nabla}\right]^{-1}, \left[\frac{b}{a-b}\|M\|_{L^q_\nabla}\prod_{i=1}^m \|\upsilon_i\|_{L^{p_i}_\nabla}\right]^{-1} \right\},$$

$(\mathtt{C}_4)$ $h_i(r) \geq \frac{\theta Q_d}{2}\ \forall\, s \in [\nu_d, \sigma(1)-\nu_d]_{\mathbb{S}},\ \frac{\nu_d}{\sigma(1)} Q_d \leq r \leq Q_d.$

Then the BVP (1)–(2) has countable positive solutions $\{(r_1^{[d]}, r_2^{[d]}, \dots, r_j^{[d]})\}_{d=1}^{\infty}$ with the property that $r_i^{[d]}(s) \geq 0$ over $(0,\sigma(1)]_{\mathbb{S}}$, $i = 1,2,\dots,j$ and $d \in \mathbb{N}$.

Proof. Let

$$q_{1,s} = \{r \in \Upsilon : \|r\| < J_d\},\ q_{2,s} = \{r \in \Upsilon : \|r\| < Q_d\},$$

be open subsets of r. Also, let $\{\nu_d\}_{s=1}^{\infty}$ be as specified earlier. Furthermore, it can be observed that

$$s^* < s_{d+1} < \nu_d < s_d < \frac{\sigma(1)}{2}.$$

Let $s \in \mathbb{N}$, and consider the cone H_{ν_d} constructed as follows:

$$H_{\nu_d} = \Big\{ r \in \Upsilon : r(s) \geq 0, \min_{s \in [\nu_d, \sigma(1)-\nu_d]_{\mathbb{S}}} r(s) \geq \frac{\nu_d}{\sigma(1)} \|r(s)\| \Big\}.$$

Also, let $r_1 \in H_{\nu_d} \cap \partial q_{1,s}$. Then $r_1(t) \leq J_d = \|r_1\|$ with respect to every $t \in (0,\sigma(1)]_{\mathbb{S}}$. Given condition $(\mathtt{C}_3)$ and considering $t_{j-1} \in (0,\sigma(1)]_{\mathbb{S}}$, it follows that

$$\begin{aligned} \int_0^{\sigma(1)} M(t_{j-1},t_j)\upsilon(t_j)h_j(r_1(t_j))\nabla t_j &\leq \int_0^{\sigma(1)} M(t_j,t_j)\upsilon(t_j)h_j(r_1(t_j))\nabla t_j \\ &\leq \frac{N_1 J_d}{2} \int_0^{\sigma(1)} M(t_j,t_j) \prod_{i=1}^{m} \upsilon_i(t_j)\nabla t_j. \end{aligned}$$

Since $\sum_{i=1}^{j} \frac{1}{p_i} < 1$, there exists a $q > 1$ such that $\frac{1}{q} + \sum_{i=1}^{j} \frac{1}{p_i} = 1$. So,

$$\begin{aligned} \int_0^{\sigma(1)} M(t_{j-1},t_j)\,\upsilon(t_j)\,h_j\big(r_1(t_j)\big)\,\Delta t_j &\leq \frac{N_1 J_d}{2} \|M\|_{L_\nabla^q} \left\| \prod_{i=1}^{m} \upsilon_i \right\|_{L_\nabla^{p_i}} \\ &\leq \frac{N_1 J_d}{2} \|M\|_{L_\nabla^q} \prod_{i=1}^{m} \|\upsilon_i\|_{L_\nabla^{p_i}} \\ &\leq \frac{J_d}{2} < J_d. \end{aligned}$$

The same result can be obtained for t_{j-2} belonging to $(0,\sigma(1)]_{\mathbb{S}}$ by following analogous steps:

$$\begin{aligned}
&\int_0^{\sigma(1)} M(t_{j-2},t_{j-1})\,\upsilon(t_{j-1})\,h_{j-1}\left[\int_0^{\sigma(1)} M(t_{j-1},t_j)\,\upsilon(t_j)\,h_j\big(r_1(t_j)\big)\,\nabla t_j\right]\nabla t_{j-1}\\
&\quad\leq \int_0^{\sigma(1)} M(t_{j-2},t_{j-1})\,\upsilon(t_{j-1})\,h_{j-1}(J_d)\,\nabla t_{j-1}\\
&\quad\leq \int_0^{\sigma(1)} M(t_{j-1},t_{j-1})\,\upsilon(t_{j-1})\,h_{j-1}(J_d)\,\nabla t_{j-1}\\
&\quad\leq \frac{N_1 J_d}{2}\int_0^{\sigma(1)} M(t_{j-1},t_{j-1})\prod_{i=1}^{m}\upsilon_i(t_{j-1})\,\nabla t_{j-1}\\
&\quad\leq \frac{N_1 J_d}{2}\,\|M\|_{L^q_\nabla}\prod_{i=1}^{m}\|\upsilon_i\|_{L^{p_i}_\nabla}\\
&\quad\leq \frac{J_d}{2} < J_d.
\end{aligned}$$

By further applying this iterative argument, we obtain

$$\begin{aligned}
&\int_0^{\sigma(1)} M(s,t_1)\upsilon(t_1)h_1\left[\int_0^{\sigma(1)} M(t_1,t_2)\upsilon(t_2)h_2\left[\int_0^{\sigma(1)} M(t_2,t_3)\cdots\right.\right.\\
&\qquad\qquad \left.\left.\times h_{j-1}\left[\int_0^{\sigma(1)} M(t_{j-1},t_j)\upsilon(t_j)h_j(r_1(t_j))\nabla t_j\right]\cdots\nabla t_3\right]\nabla t_2\right]\nabla t_1\\
&\qquad \leq \frac{J_d}{2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{b}{a-b}\int_0^{\sigma(1)} M(\tau,t_1)\,\upsilon(t_1)\,h_1\left[\int_0^{\sigma(1)} M(t_1,t_2)\,\upsilon(t_2)\cdots\right.\\
&\qquad\qquad \left. h_{j-1}\left[\int_0^{\sigma(1)} M(t_{j-1},t_j)\,\upsilon(t_j)\,h_j\big(r_1(t_j)\big)\,\nabla t_j\right]\cdots\nabla t_2\right]\nabla t_1\\
&\qquad \leq \frac{b}{a-b}\int_0^{\sigma(1)} M(t_1,t_1)\,\upsilon(t_1)\,h_1(J_d)\,\nabla t_1\\
&\qquad \leq \frac{b}{a-b}\cdot\frac{N_1 J_d}{2}\,\|M\|_{L^q_\nabla}\prod_{i=1}^{m}\|\upsilon_i\|_{L^{p_i}_\nabla}\\
&\qquad \leq \frac{J_d}{2}.
\end{aligned}$$

Thus, $(\mho r_1)(s) \leq \frac{J_d}{2}+\frac{J_d}{2} = J_d$. Since $J_d = \|r_1\|$ for $r_1 \in \mathrm{H}_{\nu_d}\cap\partial q_{1,s}$, we get

$$\|\mho r_1\| \leq \|r_1\|. \tag{6}$$

Next, let $s\in[\nu_d,\sigma(1)-\nu_d]_{\mathbb{S}}$. Then

$$\mathrm{Q}_d = \|r_1\| \geq r_1(s) \geq \min_{s\in[\nu_d,\sigma(1)-\nu_d]_{\mathbb{S}}} r_1(s) \geq \frac{\nu_d}{\sigma(1)}\,\|r_1\| \geq \frac{\nu_d}{\sigma(1)}\mathrm{Q}_d.$$

By $(\mathtt{C}_4)$ and for $t_{j-1}\in[\nu_d,\sigma(1)-\nu_d]_{\mathbb{S}}$, we have

$$\begin{aligned}\int_0^{\sigma(1)} M(t_{j-1},t_j)\upsilon(t_j)h_j(r_1(t_j))]\nabla t_j &\geq \int_{\nu_d}^{\sigma(1)-\nu_d} M(t_{j-1},t_j)\upsilon(t_j)h_j(r_1(t_j))\nabla t_j\\ &\geq \frac{\nu_d}{\sigma(1)}\frac{\theta \mathrm{Q}_d}{2}\int_{\nu_d}^{\sigma(1)-\nu_d} M(t_j,t_j)\upsilon(t_j))\nabla t_j\\ &\geq \frac{\nu_d}{\sigma(1)}\frac{\theta \mathrm{Q}_d}{2}\int_{\nu_d}^{\sigma(1)-\nu_d} M(t_j,t_j)\prod_{i=1}^{m}\upsilon_i(t_j))\nabla t_j\\ &\geq \frac{\nu_1}{\sigma(1)}\frac{\theta \mathrm{Q}_d}{2}\prod_{i=1}^{m}\eta_i\int_{\nu_1}^{\sigma(1)-\nu_1} M(t_j,t_j)\nabla t_j \geq \frac{\mathrm{Q}_d}{2},\end{aligned}$$

and

$$\begin{aligned}&\frac{b}{a-b}\int_0^{\sigma(1)} M(\tau,t_{_1})\upsilon(t_{_1})h_1\Bigg[\int_0^{\sigma(1)} M(t_{_1},t_{_2})\upsilon(t_{_2})\cdots\\ &\qquad\qquad \times h_{j-1}\Bigg[\int_0^{\sigma(1)} M(t_{j-1},t_j)\upsilon(t_j)h_j(r_1(t_j))\nabla t_j\Bigg]\cdots\nabla t_{_2}\Bigg]\nabla t_{_1}\\ &\qquad\geq \frac{b}{a-b}\frac{\nu_1}{\sigma(1)}\int_0^{\sigma(1)} M(t_{_1},t_{_1})\upsilon(t_{_1})h_1(J_d)\nabla t_{_1}\\ &\qquad\geq \frac{b}{a-b}\frac{\nu_1}{\sigma(1)}\frac{\theta \mathrm{Q}_d}{2}\prod_{i=1}^{m}\eta_i\int_{\nu_1}^{\sigma(1)-\nu_1} M(t_{_1},t_{_1})\nabla t_{_1}.\end{aligned}$$

By further applying this iterative argument, we obtain $(\mho r_1)(s)\geq \frac{\mathrm{Q}_d}{2}+\frac{\mathrm{Q}_d}{2}=\mathrm{Q}_d$. Thus, if $r_1\in \mathrm{H}_{\nu_d}\cap\partial \mathrm{H}_{2,s}$, then

$$\|\mho r_1\|\geq\|r_1\|. \tag{7}$$

It is clear that $0\in q_{2,k}\subset\overline{q}_{2,k}\subset q_{1,k}$. From (6) and (7), it follows from Theorem 1 that the operator $\mho$ has a fixed point $r_1^{[d]}\in \mathrm{H}_{\nu_d}\cap\left(\overline{q}_{1,d}\backslash q_{2,d}\right)$ such that $r_1^{[d]}(s)\geq 0$ on $(0,\sigma(1)]_{\mathbb{S}}$, and $s\in\mathbb{N}$. Next setting $r_{j+1}=r_1$, we obtain countable positive solutions $\{(r_1^{[d]},r_2^{[d]},\ldots,r_j^{[d]})\}_{d=1}^{\infty}$ of (1)-(2) given iteratively by

$$r_i(s)=\int_0^{\sigma(1)} M(s,t)\upsilon(t)h_i(r_{i+1}(t))\nabla t,\ s\in(0,\sigma(1)]_{\mathbb{S}},\ i=n,n-1,\ldots,1.$$

The proof is completed. □

For case $\displaystyle\sum_{i=1}^{m}\frac{1}{p_i}=1$, we have the following theorem.

Theorem 5. *Suppose* $(\mathtt{C}_1)$–$(\mathtt{C}_2)$ *hold. Let* $\{\nu_d\}_{s=1}^{\infty}$ *be a sequence with* $s_{d+1}<\nu_d<s_d$. *Let* $\{J_d\}_{d=1}^{\infty}$ *and* $\{\mathrm{Q}_d\}_{d=1}^{\infty}$ *be such that*

$$J_{d+1}<\frac{\nu_d}{\sigma(1)}\mathrm{Q}_d<\mathrm{Q}_d<\theta\mathrm{Q}_d<J_d \quad \textit{and}\quad \frac{\nu_d}{\sigma(1)}<\frac{1}{2},\ d\in\mathbb{N}.$$

Also, assume that h_i *satisfies* $(\mathtt{C}_4)$ *and*

$(\mathtt{C}_5)$ $h_i(r) \le \dfrac{N_2 J_d}{2}$ $\forall\, s \in (0,\sigma(1)]_{\mathbb{S}}, 0 \le r \le J_d,$

where

$$N_2 < \min\left\{ \left[\|M\|_{L^\infty_\nabla} \prod_{i=1}^m \|\upsilon_i\|_{L^{p_i}_\nabla} \right]^{-1}, \left[\frac{b}{a-b} \|M\|_{L^\infty_\nabla} \prod_{i=1}^m \|\upsilon_i\|_{L^{p_i}_\nabla} \right]^{-1} \right\}.$$

Then the system (1)-(2) *has countable positive solutions, denoted by* $\{(r_1^{[d]}, r_2^{[d]}, \ldots, r_j^{[d]})\}_{d=1}^\infty$ *which satisfies* $r_i^{[d]}(s) \ge 0$ *over* $(0,\sigma(1)]_{\mathbb{S}}$, $i = 1,2,\ldots,j$ *as well as* $d \in \mathbb{N}$.

Proof. Given a specific d, denote $q_{1,d}$ as defined in Theorem 4. Consider the element $r_1 \in \mathrm{H}_{\nu_d} \cap \partial q_{2,d}$. Then $r_1(t) \le J_d = \|r_1\|$ $\forall$ $t \in (0,\sigma(1)]_{\mathbb{S}}$. For $t_{i-1} \in (0,\sigma(1)]_{\mathbb{S}}$ and from $(\mathtt{C}_5)$,

$$\begin{aligned}
\int_0^{\sigma(1)} M(t_{j-1},t_j)\,\upsilon(t_j)\,h_j\big(r_1(t_j)\big)\,\nabla t_j &\le \int_0^{\sigma(1)} M(t_j,t_j)\,\upsilon(t_j)\,h_j\big(r_1(t_j)\big)\,\nabla t_j \\
&\le \frac{N_2 J_d}{2} \int_0^{\sigma(1)} M(t_j,t_j) \prod_{i=1}^m \upsilon_i(t_j)\,\nabla t_j \\
&\le \frac{N_2 J_d}{2} \|M\|_{L^\infty_\nabla} \left\| \prod_{i=1}^m \upsilon_i \right\|_{L^{p_i}_\nabla} \\
&\le \frac{N_2 J_d}{2} \|M\|_{L^\infty_\nabla} \prod_{i=1}^m \|\upsilon_i\|_{L^{p_i}_\nabla} \\
&\le \frac{J_d}{2} < J_d.
\end{aligned}$$

The same result can be obtained for t_{j-2} belonging to $(0,\sigma(1)]_{\mathbb{S}}$ by following analogous steps:

$$\begin{aligned}
&\int_0^{\sigma(1)} M(t_{j-2},t_{j-1})\,\upsilon(t_{j-1})\,h_{j-1}\left[\int_0^{\sigma(1)} M(t_{j-1},t_j)\,\upsilon(t_j)\,h_j\big(r_1(t_j)\big)\,\nabla t_j \right] \nabla t_{j-1} \\
&\le \int_0^{\sigma(1)} M(t_{j-2},t_{j-1})\,\upsilon(t_{j-1})\,h_{j-1}(J_d)\,\nabla t_{j-1} \\
&\le \int_0^{\sigma(1)} M(t_{j-1},t_{j-1})\,\upsilon(t_{j-1})\,h_{j-1}(J_d)\,\nabla t_{j-1} \\
&\le \frac{N_2 J_d}{2} \int_0^{\sigma(1)} M(t_{j-1},t_{j-1}) \prod_{i=1}^m \upsilon_i(t_{j-1})\,\nabla t_{j-1} \\
&\le \frac{N_2 J_d}{2} \|M\|_{L^\infty_\nabla} \prod_{i=1}^m \|\upsilon_i\|_{L^{p_i}_\nabla} \\
&\le \frac{J_d}{2} < J_d.
\end{aligned}$$

By further applying this iterative argument, we obtain

$$\int_0^{\sigma(1)} M(s,t_1)\upsilon(t_1)h_1\left[\int_0^{\sigma(1)} M(t_1,t_2)\upsilon(t_2)h_2\left[\int_0^{\sigma(1)} M(t_2,t_3)\cdots \right.\right.$$
$$\left.\left. \times h_{j-1}\left[\int_0^{\sigma(1)} M(t_{j-1},t_j)\upsilon(t_j)h_j(r_1(t_j))\nabla t_j \right] \cdots \nabla t_3 \right] \nabla t_2 \right] \nabla t_1 \le \frac{J_d}{2}.$$

Also, we note that

$$\begin{aligned}\frac{b}{a-b}\int_0^{\sigma(1)} M(\tau,t_1)\,\upsilon(t_1)\,h_1\Bigg[\int_0^{\sigma(1)} M(t_1,t_2)\,\upsilon(t_2)\cdots \\ h_{j-1}\Bigg[\int_0^{\sigma(1)} M(t_{j-1},t_j)\,\upsilon(t_j)\,h_j\big(r_1(t_j)\big)\,\nabla t_j\Bigg]\cdots\nabla t_2\Bigg]\nabla t_1 \\ \le \frac{b}{a-b}\int_0^{\sigma(1)} M(t_1,t_1)\,\upsilon(t_1)\,h_1(J_d)\,\nabla t_1 \\ \le \frac{b}{a-b}\cdot\frac{N_2 J_d}{2}\,\|M\|_{L^\infty_\nabla}\prod_{i=1}^m\|\upsilon_i\|_{L^{p_i}_\nabla} \\ \le \frac{J_d}{2}.\end{aligned}$$

Thus, $(\mho r_1)(s) \le \frac{J_d}{2}+\frac{J_d}{2}=J_d$. Since $J_d=\|r_1\|$ for $r_1\in \mathrm{H}_{\nu_d}\cap\partial q_{1,d}$, we get

$$\|\mho r_1\|\le\|r_1\|. \tag{8}$$

Now define $q_{2,d}=\{r_1\in\Upsilon:\|r_1\|<\mathrm{Q}_d\}$. Let $r_1\in\mathrm{H}_{\nu_d}\cap\partial q_{2,d}$ and $t\in[\nu_d,\,\sigma(1)-\nu_d]_{\mathbb{S}}$. Then, the argument leading to (8) can be done to the present case. □

Lastly, for the the case $\sum_{i=1}^m \frac{1}{p_i}>1$, we have the following result.

Theorem 6. *Suppose* $(\mathtt{C}_1)$–$(\mathtt{C}_2)$ *hold. Let* $\{\nu_d\}_{d=1}^\infty$ *be a sequence with* $s_{d+1}<\nu_d<s_d$. *Let* $\{J_d\}_{d=1}^\infty$ *and* $\{\mathrm{Q}_d\}_{d=1}^\infty$ *be such that*

$$J_{d+1}<\frac{\nu_d}{\sigma(1)}\mathrm{Q}_d<\mathrm{Q}_d<\theta\mathrm{Q}_d<J_d \quad \textit{and} \quad \frac{\nu_d}{\sigma(1)}<\frac{1}{2},\ d\in\mathbb{N}.$$

Also, assume that h_i *satisfies* $(\mathtt{C}_4)$ *and*

$(\mathtt{C}_6)$ $h_i(r)\le\frac{N_2J_d}{2}\ \forall\, s\in(0,\sigma(1)]_{\mathbb{S}},\ 0\le r\le J_d,$
where

$$N_2<\min\left\{\left[\|M\|_{L^\infty_\nabla}\prod_{i=1}^m\|\upsilon_i\|_{L^1_\nabla}\right]^{-1},\ \left[\frac{b}{a-b}\|M\|_{L^\infty_\nabla}\prod_{i=1}^m\|\upsilon_i\|_{L^1_\nabla}\right]^{-1}\right\}.$$

Then (1)-(2) *has countable positive solutions, denoted by* $\{(r_1^{[d]},r_2^{[d]},\dots,r_j^{[d]})\}_{d=1}^\infty$ *which satisfies* $r_i^{[d]}(s)\ge0$ *over* $(0,\sigma(1)]_{\mathbb{S}}$, $i=1,2,\dots,j$ *as well as* $d\in\mathbb{N}$.

Proof. The proof follows by adapting the arguments from the proof of Theorem 4. □

4 Applications

To illustrate the applicability of our main results, we consider the following BVP on the time scale $\mathbb{S}=[0,1]$.

Example 1. Consider

$$\begin{cases} r_i''(s)+\upsilon(s)h_i(r_{i+1}(s))=0, & s\in(0,\sigma(1)]_{\mathbb{S}},\ i=1,2,\\ r_3(s)=r_1(s), & s\in(0,\sigma(1)]_{\mathbb{S}}, \end{cases} \tag{9}$$

$$r_i'(0)=0,\ \ r_i(1)-\frac{1}{2}r_i\left(\frac{1}{3}\right)=0, \tag{10}$$

where we take $j=2, m=2$, $a=1$, $b=\frac{1}{2}$, $\tau_1=\frac{1}{3}$, and $\upsilon(s)=\upsilon_1(s)$ in which

$$\upsilon_1(s)=\frac{1}{\sqrt[4]{|s-\frac{1}{4}|}}.$$

For $i=1,2$, let

$$h_i(r)=\begin{cases} 0.59\times10^{-4}, & r\in(10^{-4},+\infty),\\ \dfrac{21\times10^{-(4d+3)}-0.59\times10^{-4d}}{10^{-(4d+3)}-10^{-4d}}(r-10^{-4d})+0.59\times10^{-8d}, & \\ & r\in\left[10^{-(4d+3)},10^{-4d}\right],\\ 21\times10^{-(4d+3)}, & r\in\left(0.59\times10^{-(4d+3)},10^{-(4d+3)}\right),\\ \dfrac{21\times10^{-(4d+3)}-0.59\times10^{-8d}}{0.59\times10^{-(4d+3)}-10^{-(4d+4)}}(r-10^{-(4d+4)})+0.59\times10^{-8d}, & \\ & r\in\left(10^{-(4d+4)},0.59\times10^{-(4d+3)}\right],\\ 0, & r=0, \end{cases}$$

for all $d\in\mathbb{N}$.
Also, let

$$s_d=\frac{2}{5}-\sum_{l=1}^{d}\frac{1}{8(l+1)^4}\ \text{ and }\ \nu_d=0.5(s_d+s_{d+1}),\ d\in\mathbb{N},$$

then $\nu_1=0.39$ and $s_{d+1}<\nu_d<s_d$, $\frac{1}{5}<\nu_d<\frac{1}{2}$. It is clear that

$$s_1=\frac{251}{640}<\frac{1}{2},\ \text{ and }\ s_d-s_{d+1}=\frac{1}{8(d+2)^4},\ d\in\mathbb{N}.$$

Since $\sum_{J=1}^{\infty}\frac{1}{J^4}=\frac{\pi^4}{90}$ and $\sum_{J=1}^{\infty}\frac{1}{J^2}=\frac{\pi^2}{6}$, it follows that

$$\begin{aligned} s^* &= \lim_{d\to\infty} s_d \\ &= \frac{2}{5}-\sum_{l=1}^{\infty}\frac{1}{8(l+1)^4} \\ &= \frac{21}{40}-\frac{\pi^4}{720} \\ &= 0.39. \end{aligned}$$

Also, we have $\eta_1=\frac{\sqrt[4]{108}}{3}$ and

$$\begin{aligned} \frac{\nu_1}{\sigma(1)}\prod_{i=1}^{m}\eta_i\int_{\nu_1}^{\sigma(1)-\nu_1}M(t,t)\Delta t &= 0.39\times\frac{\sqrt[4]{108}}{3}\int_{0.39}^{1-0.39}(1-t)dt \\ &\approx 0.04609905006, \end{aligned}$$

$$\begin{aligned} \frac{b}{a-b}\frac{\nu_1}{\sigma(1)}\prod_{i=1}^{m}\eta_i\int_{\nu_1}^{\sigma(1)-\nu_1}M(t,t)\nabla t &= 0.39\times\frac{\sqrt[4]{108}}{3}\int_{0.39}^{1-0.39}(1-t)dt \\ &\approx 0.04609905006. \end{aligned}$$

So,

$$\theta=\max\left\{\frac{1}{0.04609905006},\frac{1}{0.04609905006}\right\}=21.69242097.$$

Next, let $0<a<1$ be fixed. Then $\upsilon_1,\upsilon_2\in L^{1+a}[0,1]$.

$$\int_0^{\sigma(1)}\upsilon_1(s)ds=\frac{\sqrt{2}}{3}+\frac{\sqrt{2}}{3^{1/4}}.$$

So, let $\mathrm{H}_i=1$ for $i=1,2$. Then

$$\prod_{i=1}^{m}\|\upsilon_i\|_{L_\nabla^{p_i}}=\sqrt{1+\sqrt{3}}\approx 1.652891650,$$

and also $\|M\|_{L_\nabla^\infty}=1$. Therefore,

$$N_1<\left[\|M\|_\infty\prod_{i=1}^{m}\|\upsilon_i\|_{L_\nabla^{\mathrm{H}_i}}\right]^{-1}\approx 0.6050003338.$$

Take $N_1=\frac{3}{5}$. In addition, if we take

$$J_d=10^{-4d},\ \mathrm{Q}_d=10^{-(4d+3)},$$

then

$$\begin{aligned} J_{d+1}=10^{-(4d+4)}<\frac{1}{5}\times10^{-(4d+3)}<\frac{\nu_d}{\sigma(1)}\mathrm{Q}_d \\ <\mathrm{Q}_d=10^{-(4d+3)}<J_d=10^{-4d}, \end{aligned}$$

$\theta Q_d = 21.69242097 \times 10^{-(4d+3)} < \frac{3}{5} \times 10^{-4d} = N_1 J_d$, $d \in \mathbb{N}$ and $h_i(i = 1,2)$ meets the subsequent requirements:

$$h_i(r) \leq N_1 J_d = \frac{3}{5} \times 10^{-4d}, \quad r \in \left[0, 10^{-4d}\right],$$
$$h_i(r) \geq \theta Q_d = 21.69242097 \times 10^{-(4d+3)}, \quad r \in \left[0.59 \times 10^{-(4d+3)}, 10^{-(4d+3)}\right],$$

for $s \in \mathbb{N}$. All conditions of Theorem 4 are thus met. Consequently, by Theorem 4, the iterative BVP (9)-(10) has countable positive solutions $\{(r_1^{[d]}, r_2^{[d]})\}_{d=1}^{\infty}$ such that $r_i^{[d]}(s) \geq 0$ on $[0,1]$, $i = 1,2$ and $d \in \mathbb{N}$.

Example 2. Application to heat transfer in composite materials with discrete thermal properties: The BVP (9)-(10) can be applied to the study of heat transfer in composite materials, where each layer has different thermal properties. The system involves continuous heat conduction within each material layer, but experiences discrete changes at the interfaces due to differences in thermal conductivity. This kind of system is common in industries such as construction, aerospace, and energy, where thermal insulation is important. In this example

1. $r_i(s)$ represents the temperature profile in layer i at position s, which could denote the spatial variable across the material layers.

2. $\upsilon(s)h_i(r_{i+1}(s))$ models the heat exchange between layers, where $\upsilon(s)$ represents material properties (such as thermal conductivity), and $h_i(r)$ captures non-linear thermal interactions at the interfaces between layers.

3. The function $\upsilon_1(s) = \frac{1}{\sqrt[4]{|s-\frac{1}{4}|}}$ introduces a singularity, reflecting the possibility of abrupt changes in thermal conductivity or other material properties at certain points, such as $s = \frac{1}{4}$.

4. The relationship $r_i'(0) = 0$ represents no heat flux at one end, implying that the boundary is insulated.

5. The condition $r_3(s) = r_1(s)$ could represent a system where, after progressing through multiple stages (e.g., heat passing through different layers of a material), the conditions at the final stage mirror those of the initial stage. For example, in heat transfer, this might mean that after heat flows through three layers of a composite material, the temperature distribution in the third layer returns to the same as in the first, indicating a cyclical or repeating thermal pattern.

6. The condition $r_i(1) - \frac{1}{2} r_i\left(\frac{1}{3}\right) = 0$ reflects a situation where the temperature at the far boundary is related to the temperature at an intermediate point, possibly modeling thermal resistance at the boundary.

This system can be used to simulate and optimize thermal insulation in multi-layered structures, such as walls in high-temperature furnaces, where controlling heat flow is critical for efficiency and safety. The model captures both the continuous and discrete dynamics of heat transfer, offering valuable insights into the behavior of composite materials in real-world applications.

5 Conclusion

This study successfully establishes the existence of countable positive solutions for singular iterative systems in three-point BVPs on time scales, leveraging advanced mathematical tools such as Krasnoselskii's fixed point theorem and Hölder's inequality. The research contributes to the growing field of time scale calculus by unifying discrete and continuous dynamics, providing a versatile framework that can model a wide range of hybrid systems. The rigorous theoretical analysis and the example presented confirm the sufficiency of the derived conditions, emphasizing their applicability in solving complex boundary value problems with singularities. This work advances our understanding of dynamic systems, offering new insights into iterative processes with multiple singularities, which can be beneficial for various fields, including population dynamics, epidemiology, and engineering.

Future Work: Based on the main results presented, future research can be extended in several important directions. A concrete extension involves developing a numerical scheme for the operator $\mho$, such as an iterative fixed-point approximation method, and validating it specifically for the composite heat transfer model discussed in the applications section. Additionally, we plan to tackle specific challenges, such as ensuring the stability of numerical approximations on irregular time scales and analyzing the impact of singularities on convergence rates in these simulations. Further advancements could include exploring higher-order BVPs with nonlocal or impulsive conditions, integrating machine learning techniques for predicting solutions in data-driven scenarios like epidemiology, and investigating fractional-order dynamic equations on time scales to enhance modeling capabilities in biology and finance.

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