

Second order sensitivity analysis for shape optimization of continuum structures

M. Rezaiee-Pajand*, Y. Kadkhodaye Bahre

Department of Civil Engineering, Ferdowsi University of Mashhad, Mashhad, Iran

Received 11 February 2011; accepted 26 April 2011

Abstract

This study focuses on the optimization of the plane structure. Sequential quadratic programming (SQP) will be utilized, which is one of the most efficient methods for solving nonlinearly constrained optimization problems. A new formulation for the second order sensitivity analysis of the two-dimensional finite element will be developed. All the second order required derivatives will be calculated. These values will be used in SQP scheme for structural optimization. Both plane stress and plane strain problems are analyzed. Numerical examples show the success and effectiveness of the suggested formulation.

Keywords: Shape optimization; Finite element; Sensitivity analysis; SQP method; Second order derivatives; Plane problem.

1. Introduction

The optimization problem considered in this paper consists of minimizing an objective function subject to some constraints insuring the feasibility of the structural design. This kind of the problem can be written in the following general mathematical form:

$$\text{Minimize } f(x) \tag{1}$$

Subjects to the constraints:

$$c_j(x) \geq 0 \quad j = 1, m \tag{2}$$

In the classical weight minimization with sizing variables, the objective function (1) is very often a linear function of the design variables, x_i . For the sake of generalization, $f(x)$ is assumed to be a nonlinear function in the present paper. Consequently, the object function can represent any structural characteristic to be minimized, such as stress concentration, with sizing or shape design variables. The inequalities (2) are the behavior constraints that impose limitations on structural response quantities. For instance, the upper bounds on stresses and

* Corresponding author.

E-mail address: mrpajand@yahoo.com (M.Rezaee-Pajand)

Nomenclature

$[B]$	strain matrix of the initial finite element.
$[\hat{B}]$	strain matrix of the deformed finite element.
$[D_m]$	elasticity matrix.
$\{D\}$	nodal displacement.
f_i	design element shape function.
$[J]$	Jacobin matrix.
$[\hat{J}]$	Jacobin matrix of the deformed finite element.
$ J $	determinant of Jacobin matrix.
$ J _x, J _y$	derivatives of $ J $ with respect to master node coordinates.
N_i	finite element shape function.
r, s	local coordinates of the design element.
$[K]_e$	element stiffness matrix.
X, Y	global coordinates of design element nodes.
x, y	global coordinates of finite element nodes.
$\delta X_k, \delta Y_k$	design changes related to k th master node.
ξ, η	local coordinates of the finite element.

displacements under static loading cases are usually limited. These constraints are regularly non-linear functions, but in some situations, they might also include linear functions, as well.

For the sake of simplicity, no equality constraints are explicitly stated in the presented non-linear programming problem. However, it is important to mention that the proposed formulation in this paper can easily handle equality constraints. In fact, this kind of the constraints can be substituted by two opposed inequalities. For the manufacturing reasons, the design variables must frequently be bounded from below and above. To condense the formula, these side constraints are not written separately in the optimization problem statement. In other words, they are assumed to be included in the general constraints (2).

There are many practical strategies for solving this design optimization problem, such as: the gradient projection method, trust region technique and sequential quadratic programming (SQP) tactic. Among these approaches, SQP scheme is an important one. SQP method replaces the primary optimization problem with a sequence of quadratic sub-problems. In this tactic, each sub-problem is generated through second order Taylor's series' expansion of the Lagrangian function. The constraints are also written in terms of intermediate linear variables.

One of the main steps of the gradient-based optimization techniques is the sum of the first order derivatives (gradients) of structural response with respect to the design variables. Due to the implicit relations of the behavior and design variables, computation of such derivatives, which is called sensitivity analysis, is an expensive and time-consuming process in structural optimization. Finding this kind of derivative in structural optimization problems is widely achieved by using the well known finite difference techniques [1-3].

A disadvantage of the mentioned strategies is that a proper step size should be chosen for the design variables. Furthermore, for a problem with k design variables, finite difference calculations of the displacement derivatives, with respect to design variables, require analysis of $k+1$ different stiffness matrices. On the other hand, a large number of the analysis associated with finite difference estimations can be avoided by utilizing analytical computation of sensitivity derivatives. It is worth emphasizing that different shape optimization tactics for the varieties of structural systems are available in the literature [4-6]. Most of the optimization techniques utilize the finite element strategy and sensitivity analysis [7-8]. A key role is usually played by the procedure of calculating the sensitivity derivatives [9-11].

The purpose of this paper is to introduce an efficient second order sensitivity formulation for the shape optimization of continuum structures. This study has two different features. The first is the use of a limited number of master nodes to characterize the form of an isoparametric finite element, and the adoption of their coordinates as design variables for the shape optimization. The second is the derivation of analytical formulations for the gradients. Finally, the sensitivity analysis is incorporated into authors' finite element computer program. To show the effectiveness of the proposed formulation, some numerical examples are solved, and the related discussions are presented.

2. Sequential quadratic programming (SQP) method

The structural optimization tactic, which is used in this paper, is based on a well known strategy in mathematical programming. The sequential quadratic programming (SQP) procedure first was proposed by Wilson and referred to as the solver scheme. Following Fletcher, this technique is best interpreted by applying Newton's method to find the stationary point of the subsequent Lagrangian function:

$$L(x, r) = f(x) - \sum r_j c_j(x) \quad (3)$$

In this function, the r_j 's denote the Lagrangian multipliers associated with the constraints, $c(x)$. Hence, this approach is named 'Lagrange-Newton method'. It should be reminded; this process was initially intended to be a second order technique, i.e. a procedure using second derivatives of the objective and constraint functions. Because, second derivatives are usually quite cumbersome to evaluate and to store, the Lagrange-Newton method was later modified to replace second order information with updated formulas based on the first order derivatives. This quasi-Newton implementation, using the now well established DFP or BFGS update formulas, is widely used by the numerical optimization community. The method can be found in most general purposed non-linear programming packages under the appellation Sequential Quadratic Programming (SQP) approach. To set up the related formulation, analyst can transform the primary optimization problem (1,2) with a sequence of the following quadratic approximations:

$$\text{Minimize } \frac{1}{2}(x - x_0)^T A (x - x_0) + b^T (x - x_0) \quad (4)$$

$$\text{subject to } c(x_0) + C^T (x - x_0) \quad (5)$$

In this optimization problem, the vector b and the matrix C contain the first derivatives of the objective and constraint function, respectively. They can be evaluated at x_0 , as follows:

$$b_i = \left. \frac{df}{dx_i} \right|_{x_0} \quad (6)$$

$$c_{ij} = \left. \frac{dc_j}{dx_i} \right|_{x_0} \quad (7)$$

In the original Lagrange-Newton strategy, the symmetric matrix, A, represents the Hessian of the Lagrangian function. The next formula is used to find the required entries:

$$\begin{aligned} A_{ik} &= \left. \frac{d^2 f}{dx_i dx_k} \right|_{x_0, r_0} \\ &= \left. \frac{d^2 f}{dx_i dx_k} \right|_{x_0} - \sum r_j^0 \left. \frac{d^2 c_j}{dx_i dx_k} \right|_{x_0} \end{aligned} \quad (8)$$

In this relation, the r_j 's denote the current values of the Lagrangian multipliers associated with the constraints (r_j^0 is zero if the constraint c is inactive). As it was mentioned earlier, each quadratic programming (4,5) is generated from the primary problem (1,2) by replacing the constraints $c(x)$ with their first order Taylor series' expansions at the current design point. In addition, the objective function, $f(x)$, is replaced with a quadratic approximation (4). This is a second order Taylor series' expansion, with the addition of constraint curvature terms in the Hessian. The inclusion of the constraint curvature terms is very important, because it insures a second order rate of convergence, even if the constraints are non-linear. It is worth emphasizing, in the case of linear objective function, the process would degenerate into a sequential linear programming approach if the constraint curvature term was omitted. One example of linear objective function occurs in classical minimum weight design problems involving sizing variables.

The application of the sequential quadratic programming method needs some usual computations. A sensitivity analysis provides the first derivatives of the objective function and all the constraint functions that have been identified as potentially active. Furthermore, the A matrix needs evaluation of the second derivatives for the objective function and all the currently active constraints (i.e. the constraints associated with non-zero Lagrangian multipliers). In many cases, the required second order sensitivity analysis is not available, or it is computationally too expensive. For this reason, in the most commonly used SQP implementation, the A matrix is only an approximation to the Hessian of the Lagrangian function, gradually built up at each iteration from the first order derivatives.

3. Second order sensitivity analysis

Obviously, the SQP approach based upon quasi-Newton approximation will not, in general, converge as fast as a pure Newton scheme using the true second order derivatives. This paper develops an efficient formulation in order to generate the required second order sensitivity information with little computational time. The second order sensitivity analysis presented in this section is restricted to design optimization problems involving constraints on static structural responses. However, it is important to mention, for many other types of behavior constraints similar developments could probably be accomplished. For simplicity, only one applied static load case is assumed for the suggested equations. If there are several real load cases, the related equations should be repeated for each one.

The evaluation of sensitivity derivatives for static displacement and stress constraints is now well established and documented. This process starts by differentiating the following governing equilibrium equations of the finite element model:

$$[K] \{q\} = \{P\} \quad (9)$$

In this equation, $[k]$ is the global stiffness matrix of the structure, $\{q\}$ is the nodal displacements, and $\{p\}$ is the global vector of equivalent nodal forces. Both $[k]$ and $\{p\}$ are available by assembling all the element stiffness matrices, $[k]_e$, and the element nodal forces, $\{p\}_e$, respectively. The result of differentiation has the below form:

$$[K] \left\{ \frac{\partial q}{\partial X} \right\} + \left[\frac{\partial K}{\partial X} \right] \{q\} = \left\{ \frac{\partial P}{\partial X} \right\} \tag{10}$$

$$\{\tilde{p}_1\} = \left\{ \frac{\partial P}{\partial X} \right\} - \left[\frac{\partial K}{\partial X} \right] \{q\} \tag{11}$$

$$[K] \left\{ \frac{\partial q}{\partial X} \right\} = \{\tilde{p}_1\}$$

Differentiating once again gives the succeeding relation:

$$[K] \left\{ \frac{\partial^2 q}{\partial X_i \partial X_j} \right\} = \left\{ \frac{\partial^2 P}{\partial X_i \partial X_j} \right\} - \left[\frac{\partial^2 K}{\partial X_i \partial X_j} \right] \{q\} - \left[\frac{\partial K}{\partial X_i} \right] \left\{ \frac{\partial q}{\partial X_j} \right\} - \left[\frac{\partial K}{\partial X_j} \right] \left\{ \frac{\partial q}{\partial X_i} \right\} \tag{12}$$

Matrix $[\partial^2 q / \partial x_i \partial x_j]$ is the second order derivatives of the nodal displacements with respect to design variables. For practical sizing problems, where the number of design variables might be large, computing and storing the matrix $[\partial^2 k / \partial x_i \partial x_j]$ could become burdensome. On the other hand, for the shape optimal design applications, the stiffness matrix can no longer be expressed as a simple explicit form of the design variables. Moreover, its second derivatives can be difficult to evaluate. Hence, it is worth a lot to neglect the coupling between design variables, and also to restrict the matrix to its diagonal terms. Therefore, equation (12) will be reduced to the following form:

$$[K] \left\{ \frac{\partial^2 q}{\partial X_k^2} \right\} = \left\{ \frac{\partial^2 P}{\partial X_k^2} \right\} - \left[\frac{\partial^2 K}{\partial X_k^2} \right] \{q\} - 2 \left[\frac{\partial K}{\partial X_k} \right] \left\{ \frac{\partial q}{\partial X_k} \right\} \tag{13}$$

$$\{\hat{p}_2\} = \left\{ \frac{\partial^2 P}{\partial X_k^2} \right\} - \left[\frac{\partial^2 K}{\partial X_k^2} \right] \{q\} - 2 \left[\frac{\partial K}{\partial X_k} \right] \left\{ \frac{\partial q}{\partial X_k} \right\} \tag{14}$$

$$[K] \left\{ \frac{\partial^2 q}{\partial X_k^2} \right\} = \{\hat{p}_2\}$$

In these relations, $\{\tilde{p}_1\}$ and $\{\tilde{p}_2\}$ are referred to the vectors of pseudo-loads.

4. New formulation

To perform second order sensitive analysis, a new formulation for finding $[\partial^2 k / \partial x_k^2]$ is developed. In two-dimensional elasticity problem, the stiffness matrix of an isoparametric element can be formulated as follows:

$$[k]_e = \iint_A [B]^T [D] [B] t dx dy \tag{15}$$

$$= \int_{-1}^1 \int_{-1}^1 [B]^T [D] [B] t |J| d\xi d\eta$$

In this equation, $[B]$ is the strain matrix, which operates on the nodal displacements to produce element strains, $[D]$ is the elasticity matrix, which relates the element stresses and

strains, and $|J|$ is the determinant of the Jacobin matrix. It should be reminded, $[J]$ maps the area $dx dy$, in global coordinates, to $d\xi d\eta$, in curvilinear local coordinates or vice versa.

In most optimal design problems, the mechanical properties of material are prescribed and do not change during the optimization process. Based on this fact, the first and second order derivatives of the element stiffness matrix with respect to a shape variable X_k can be written in the usual manner as follows:

$$\begin{aligned} \left[\frac{\partial K}{\partial X_k} \right]_e &= \int_{-1}^1 \int_{-1}^1 \left(\frac{\partial [B]^T}{\partial X_k} [D_m] [B] |J| + [B]^T [D_m] \frac{\partial [B]}{\partial X_k} |J| + [B]^T [D_m] [B] \frac{\partial |J|}{\partial X_k} \right) t d\xi d\eta \\ \left[\frac{\partial^2 K}{\partial X_k^2} \right]_e &= \int_{-1}^1 \int_{-1}^1 \left(\frac{\partial^2 [B]^T}{\partial X_k^2} [D_m] [B] |J| + \frac{\partial [B]^T}{\partial X_k} [D_m] \frac{\partial [B]}{\partial X_k} |J| + \frac{\partial [B]^T}{\partial X_k} [D_m] [B] \frac{\partial |J|}{\partial X_k} \right. \\ &+ \frac{\partial [B]^T}{\partial X_k} [D_m] \frac{\partial [B]}{\partial X_k} |J| + [B]^T [D_m] \frac{\partial^2 [B]^T}{\partial X_k^2} |J| \\ &+ [B]^T [D_m] \frac{\partial [B]}{\partial X_k} \frac{\partial |J|}{\partial X_k} + \frac{\partial [B]^T}{\partial X_k} [D_m] [B] \frac{\partial |J|}{\partial X_k} \\ &\left. + [B]^T [D_m] \frac{\partial [B]}{\partial X_k} \frac{\partial |J|}{\partial X_k} + [B]^T [D_m] [B] \frac{\partial^2 |J|}{\partial X_k^2} \right) t d\xi d\eta \end{aligned} \tag{16}$$

It is well known that a typical coefficient in $[B]$ depends on local coordinates and has ξ, η polynomials in both numerator and denominator. Therefore, the parametric integration of the stiffness matrix and its derivatives are complex. However, this can be calculated numerically. It is obvious that the evaluation of equation (16) requires the derivatives of $|J|$ and $[B]$ with respect to shape variables. A new technique for finding such derivatives is introduced hereafter. In order to develop this formulation, an m -node isoparametric design element is selected with natural coordinates r and s . Figure 1 shows this kind of element, which is called the design element, and consists of several finite elements.

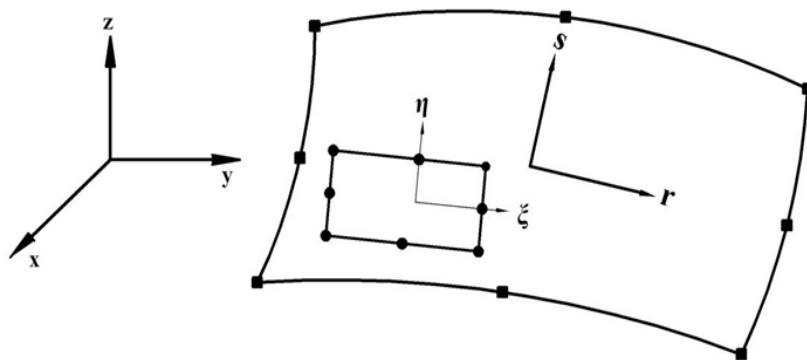


Figure 1. Design element and its associated finite element.
 ■ Design element nodes; ● Finite element nodes.

Once the nodal coordinates of the design element are determined, the coordinates of its internal nodes, such as finite elements nodal points, can be computed by using the technique of isoparametric mapping. It should be noted that a comprehensive description of this

approach for the shape representation of structures is available in the literature [12]. Using this technique, the coordinates of finite element nodal points are generated in the following form:

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \sum_{i=1}^m f_i(r, s) \begin{Bmatrix} X \\ Y \end{Bmatrix}_i \quad (17)$$

In this relation, X_i , Y_i and x , y are global coordinates of the design element and its associated finite element, respectively. Furthermore, f_i shows the isoparametric shape function corresponding to the i th node of the design element. By changing the coordinates X_k and Y_k , which are related to the k th master node of the design element, the shape of structure can be updated. Denoting the aforementioned changes by δX_k and δY_k , the nodal coordinates of the deformed finite element, which are shown by $\hat{\cdot}$, can be calculated by next equation:

$$\begin{Bmatrix} \hat{x} \\ \hat{y} \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix} + f_k(r, s) \begin{Bmatrix} \delta X_k \\ \delta Y_k \end{Bmatrix} \quad (18)$$

In this scheme, the components of an element stiffness matrix are computed in terms of the variation of a master node coordinates. At first, by using the equation (18), the Jacobin matrix and its determinant for an n -node deformed finite element are calculated as follows:

$$[\hat{J}] = \begin{bmatrix} \hat{x}_{i,\xi} & \hat{y}_{i,\xi} \\ \hat{x}_{i,\eta} & \hat{y}_{i,\eta} \end{bmatrix} = \sum_{i=1}^n \begin{bmatrix} \hat{x}_i N_{i,\xi} & \hat{y}_i N_{i,\xi} \\ \hat{x}_i N_{i,\eta} & \hat{y}_i N_{i,\eta} \end{bmatrix} \quad (19)$$

$$\begin{aligned} [\hat{J}] &= \sum_{i=1}^n \begin{bmatrix} (x_i + f_k \delta X_k) N_{i,\xi} & (y_i + f_k \delta Y_k) N_{i,\xi} \\ (x_i + f_k \delta X_k) N_{i,\eta} & (y_i + f_k \delta Y_k) N_{i,\eta} \end{bmatrix} \\ &= \sum_{i=1}^n \begin{bmatrix} x_i N_{i,\xi} & y_i N_{i,\xi} \\ x_i N_{i,\eta} & y_i N_{i,\eta} \end{bmatrix} + \sum_{i=1}^n f_k(r_i, s_i) \begin{bmatrix} N_{i,\xi} & 0 \\ N_{i,\eta} & 0 \end{bmatrix} \delta X_k + \sum_{i=1}^n f_k(r_i, s_i) \begin{bmatrix} 0 & N_{i,\xi} \\ 0 & N_{i,\eta} \end{bmatrix} \delta Y_k \end{aligned} \quad (20)$$

$$[J] = \sum_{i=1}^n \begin{bmatrix} x_i N_{i,\xi} & y_i N_{i,\xi} \\ x_i N_{i,\eta} & y_i N_{i,\eta} \end{bmatrix} \quad (21)$$

$$[J]_x = \sum_{i=1}^n f_k(r_i, s_i) \begin{bmatrix} N_{i,\xi} & 0 \\ N_{i,\eta} & 0 \end{bmatrix} \quad (22)$$

$$[J]_y = \sum_{i=1}^n f_k(r_i, s_i) \begin{bmatrix} 0 & N_{i,\xi} \\ 0 & N_{i,\eta} \end{bmatrix} \quad (23)$$

$$[\hat{J}] = [J] + [J]_x \delta X_k + [J]_y \delta Y_k \quad (24)$$

According to these formulations, the Jacobin matrix of the finite element has a linear form in terms of the design changes, δX_k and δY_k . It should be noted that $[J]$, $[J]_x$, and $[J]_y$ are the Jacobin matrix of the initial finite element and its derivatives with respect to the k th master node coordinates, X_k and Y_k . In the subsequent lines, the determinant of the Jacobin matrix is calculated by using equations (18) and (19):

$$[\hat{J}] = \sum_{i=1}^n \begin{bmatrix} (x_i + f_k \delta X_k) N_{i,\xi} & (y_i + f_k \delta Y_k) N_{i,\xi} \\ (x_i + f_k \delta X_k) N_{i,\eta} & (y_i + f_k \delta Y_k) N_{i,\eta} \end{bmatrix} \quad (25)$$

$$|J| = \sum_{i=1}^n \begin{vmatrix} x_i N_{i,\xi} & y_i N_{i,\xi} \\ x_i N_{i,\eta} & y_i N_{i,\eta} \end{vmatrix} \quad (26)$$

$$|J_x| = \sum_{i=1}^n \begin{vmatrix} f_k N_{i,\xi} & y_i N_{i,\xi} \\ f_k N_{i,\eta} & y_i N_{i,\eta} \end{vmatrix} \quad (27)$$

$$|J_y| = \sum_{i=1}^n \begin{vmatrix} x_i N_{i,\xi} & f_k N_{i,\xi} \\ x_i N_{i,\eta} & f_k N_{i,\eta} \end{vmatrix} \quad (28)$$

$$|\hat{J}| = |J| + |J_x| \delta X_k + |J_y| \delta Y_k \quad (29)$$

According to the last equation, the determinant of Jacobin matrix for the deformed finite element has a linear form in terms of the design changes, as well. It should be noted that $|J|_x$ and $|J|_y$ denote the derivatives of Jacobin determinant with respect to the design variables, which are calculated from equations (27) and (28), respectively. These are not equal to the determinants of matrix $[J]_x$ and $[J]_y$. In the next step, the derivatives of strain matrix $[B]$, with respect to the design variables, are calculated. The components of this matrix are the derivatives of finite element shape function with respect to the global coordinates x and y . The latest values are related to the problem type. When the n-node isoparametric element is used in the optimization process, its related strain matrix and the derivatives of shape functions are written as follows:

$$[B] = [B_1 \dots B_i \dots B_n] \quad (30)$$

$$[B_i] = \begin{bmatrix} N_{i,x} & 0 \\ 0 & N_{i,y} \\ N_{i,y} & N_{i,x} \end{bmatrix} \quad (31)$$

$$\begin{Bmatrix} N_{i,x} \\ N_{i,y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} N_{i,\xi} \\ N_{i,\eta} \end{Bmatrix} \quad (32)$$

The components of the strain matrix can be computed by using equation (32). It is obvious that for calculating the derivatives of shape function in global coordinates, the inverse of Jacobin matrix is required. In the same manner, to find these derivatives for the deformed finite element, the inverse of Jacobin matrix for this element must be calculated. Based on these points, the results have the subsequent appearance:

$$\begin{aligned}
 [\hat{J}]^{-1} &= \frac{[\hat{J}]^a}{|\hat{J}|} = \frac{1}{|\hat{J}|} \sum_{i=1}^n \begin{bmatrix} (y_i + f_k \delta Y_k) N_{i,\eta} & -(y_i + f_k \delta Y_k) N_{i,\xi} \\ -(x_i + f_k \delta X_k) N_{i,\eta} & (x_i + f_k \delta X_k) N_{i,\xi} \end{bmatrix} \\
 &= \frac{1}{|\hat{J}|} \left\{ \sum_{i=1}^n \begin{bmatrix} y_i N_{i,\eta} & -y_i N_{i,\xi} \\ -x_i N_{i,\eta} & x_i N_{i,\xi} \end{bmatrix} + \sum_{i=1}^n f_k(r_i, s_i) \begin{bmatrix} 0 & 0 \\ -N_{i,\eta} & N_{i,\xi} \end{bmatrix} \delta X_k \right. \\
 &\quad \left. + \sum_{i=1}^n f_k(r_i, s_i) \begin{bmatrix} N_{i,\eta} & -N_{i,\xi} \\ 0 & 0 \end{bmatrix} \delta Y_k \right\} \\
 &= \frac{1}{|\hat{J}|} \left\{ [J]^a + [J]_x^a \delta X_k + [J]_y^a \delta Y_k \right\}
 \end{aligned} \tag{33}$$

$$[J]^a = \sum_{i=1}^n \begin{bmatrix} y_i N_{i,\eta} & -y_i N_{i,\xi} \\ -x_i N_{i,\eta} & x_i N_{i,\xi} \end{bmatrix} = \begin{bmatrix} y_{,\eta} & -y_{,\xi} \\ -x_{,\eta} & x_{,\xi} \end{bmatrix} \tag{34}$$

$$[J]_x^a = \sum_{i=1}^n f_k(r_i, s_i) \begin{bmatrix} 0 & 0 \\ -N_{i,\eta} & N_{i,\xi} \end{bmatrix} \tag{35}$$

$$[J]_y^a = \sum_{i=1}^n f_k(r_i, s_i) \begin{bmatrix} N_{i,\eta} & -N_{i,\xi} \\ 0 & 0 \end{bmatrix} \tag{36}$$

$$[\hat{J}]^a = [J]^a + [J]_x^a \delta X_k + [J]_y^a \delta Y_k \tag{37}$$

In these formulations, $[J]^a$, $[J]_x^a$ and $[J]_y^a$ are the adjoint of the Jacobin matrix for the initial finite element, and its derivatives with respect to the coordinates of k th master node. According to equation (37), the adjoint of Jacobin matrix for the deformed finite element, has a linear form with respect to the design variables. On the other hand, the inverse of the Jacobin matrix for the deformed element is a nonlinear function of the design variables. This is because of the existing determinant of Jacobin matrix in the denominator of the fraction of equation (33). In the following lines, the derivatives of the shape functions and strain matrix of the deformed finite element are computed by using equations (18) and (19):

$$\begin{Bmatrix} \hat{N}_{i,x} \\ \hat{N}_{i,y} \end{Bmatrix} = [\hat{J}]^{-1} \begin{Bmatrix} N_{i,\xi} \\ N_{i,\eta} \end{Bmatrix} = \frac{[\hat{J}]^a}{|\hat{J}|} \begin{Bmatrix} N_{i,\xi} \\ N_{i,\eta} \end{Bmatrix} \tag{38}$$

$$|\hat{J}| \begin{Bmatrix} \hat{N}_{i,x} \\ \hat{N}_{i,y} \end{Bmatrix} = \begin{Bmatrix} \hat{N}_{i,x} \\ \hat{N}_{i,y} \end{Bmatrix}^a = [\hat{J}]^a \begin{Bmatrix} N_{i,\xi} \\ N_{i,\eta} \end{Bmatrix} \tag{39}$$

$$\begin{aligned} \begin{Bmatrix} \hat{N}_{i,x} \\ \hat{N}_{i,y} \end{Bmatrix}^a &= \begin{bmatrix} \sum (y_i + f_k \delta Y_k) N_{i,\eta} & -\sum (y_i + f_k \delta Y_k) N_{i,\xi} \\ -\sum (x_i + f_k \delta X_k) N_{i,\eta} & \sum (x_i + f_k \delta X_k) N_{i,\xi} \end{bmatrix} \begin{Bmatrix} N_{i,\xi} \\ N_{i,\eta} \end{Bmatrix} \\ &= \begin{Bmatrix} y_{,\eta} N_{i,\xi} - y_{,\xi} N_{i,\eta} \\ x_{,\xi} N_{i,\eta} - x_{,\eta} N_{i,\xi} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -N_{i,\xi} \left(\sum_{i=1}^n f_k N_{i,\eta} \right) + N_{i,\eta} \left(\sum_{i=1}^n f_k N_{i,\xi} \right) \end{Bmatrix} \delta X_k \\ &\quad + \begin{Bmatrix} N_{i,\xi} \left(\sum_{i=1}^n f_k N_{i,\eta} \right) - N_{i,\eta} \left(\sum_{i=1}^n f_k N_{i,\xi} \right) \\ 0 \end{Bmatrix} \delta Y_k \end{aligned} \tag{40}$$

$$\begin{Bmatrix} \hat{N}_{i,x} \\ \hat{N}_{i,y} \end{Bmatrix}^a = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ T_3 \end{Bmatrix} \delta X_k + \begin{Bmatrix} -T_3 \\ 0 \end{Bmatrix} \delta Y_k \tag{41}$$

$$T_1 = y_{,\eta} N_{i,\xi} - y_{,\xi} N_{i,\eta} \tag{42}$$

$$T_2 = x_{,\xi} N_{i,\eta} - x_{,\eta} N_{i,\xi} \tag{43}$$

$$T_3 = -N_{i,\xi} \left(\sum_{i=1}^n f_k (r_i, s_i) N_{i,\eta} \right) + N_{i,\eta} \left(\sum_{i=1}^n f_k (r_i, s_i) N_{i,\xi} \right) \tag{44}$$

$$\begin{bmatrix} \hat{B}_i \end{bmatrix}^a = \begin{bmatrix} \hat{N}_{i,x} & 0 \\ 0 & \hat{N}_{i,y} \\ \hat{N}_{i,y} & \hat{N}_{i,x} \end{bmatrix}^a = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \\ T_2 & T_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & T_3 \\ T_3 & 0 \end{bmatrix} \delta X + \begin{bmatrix} -T_3 & 0 \\ 0 & 0 \\ 0 & T_3 \end{bmatrix} \delta Y \tag{45}$$

$$[B_i]^a = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \\ T_2 & T_1 \end{bmatrix} \tag{46}$$

$$[B_i]_x^a = \begin{bmatrix} 0 & 0 \\ 0 & T_3 \\ T_3 & 0 \end{bmatrix} \tag{47}$$

$$[B_i]_y^a = \begin{bmatrix} -T_3 & 0 \\ 0 & 0 \\ 0 & T_3 \end{bmatrix} \tag{48}$$

$$[\hat{B}]^a = [B]^a + [B]_x^a \delta X_k + [B]_y^a \delta Y_k \tag{49}$$

$$[\hat{B}] = \frac{[\hat{B}]^a}{|\hat{J}|} = \frac{[B]^a + [B]_x^a \delta X_k + [B]_y^a \delta Y_k}{|J| + |J_x| \delta X_k + |J_y| \delta Y_k} \tag{50}$$

Up to this stage, the required components for calculating the derivatives of the element stiffness matrix are obtained. It should be added, another approach can be used to find these components, as well. In the following relation, the difference between the stiffness matrix of the initial and deformed finite element is computed and the general definition of the derivative is used to calculate the first-order derivative of the stiffness matrix:

$$\begin{aligned}
 [K]_e &= \int_{-1}^1 \int_{-1}^1 [B]^T [D_m] [B] |J| \, t d \xi d \eta \\
 &= \int_{-1}^1 \int_{-1}^1 \frac{([B]^a)^T [D_m] [B]^a}{|J|} t d \xi d \eta
 \end{aligned} \tag{51}$$

$$\begin{aligned}
 [\hat{K}]_e &= \int_{-1}^1 \int_{-1}^1 [\hat{B}]^T [D_m] [\hat{B}] |\hat{J}| \, t d \xi d \eta \\
 &= \int_{-1}^1 \int_{-1}^1 \frac{([\hat{B}]^a)^T [D_m] [\hat{B}]^a}{|\hat{J}|} t d \xi d \eta
 \end{aligned} \tag{52}$$

$$\left[\frac{\partial K}{\partial X_k} \right] = \lim_{\delta X_k \rightarrow 0} \frac{[\hat{K}]_e - [K]_e}{\delta X_k} \tag{53}$$

To simplify the presented formulation, the matrices $[E]$, $[F]$ and $[G]$ are used. In the succeeding lines, these matrices are defined. Afterward, equation (49) is used with the assumption of $\delta Y_k = 0$. Based on this fact, the components of the stiffness matrices for the initial and deformed finite element are computed.

$$[E] = ([B]_x^a)^T [D_m] [B]^a \tag{54}$$

$$[F] = ([B]^a)^T [D_m] [B]^a \tag{55}$$

$$[G] = ([B]_x^a)^T [D_m] [B]_x^a \tag{56}$$

By using presented formulation, the stiffness matrices of the initial and deformed finite elements are computed. As it is written below, by using equation (53), the derivative of the stiffness matrix with respect to the design variable X_k is obtained:

$$[K]_e = \int_{-1}^1 \int_{-1}^1 \frac{[F]}{|J|} t d \xi d \eta \tag{57}$$

$$[\hat{K}]_e = \int_{-1}^1 \int_{-1}^1 \frac{[G] \delta X_k^2 + ([E] + [E]^T) \delta X_k + [F]}{|J| + |J_x| \delta X_k} t d \xi d \eta \tag{58}$$

$$\begin{aligned}
 \left[\frac{\partial K}{\partial X_k} \right]_e &= \lim_{\delta X_k \rightarrow 0} \int_{-1}^1 \int_{-1}^1 \left(\frac{|J| [G] \delta X_k^2 + ([E] + [E]^T) \delta X_k + [F]}{|J| (|J| + |J_x| \delta X_k) \delta X_k} \right. \\
 &\quad \left. - \frac{(|J| + |J_x| \delta X_k) [F]}{|J| (|J| + |J_x| \delta X_k) \delta X_k} \right) t d \xi d \eta \\
 &= \int_{-1}^1 \int_{-1}^1 \left\{ \frac{([E] + [E]^T)}{|J|} - [F] \frac{|J|_x}{|J|^2} \right\} t d \xi d \eta
 \end{aligned} \tag{59}$$

By utilizing equation (59), the second order derivatives of the stiffness matrix can be formulated as follows:

$$\left[\frac{\partial^2 K}{\partial X_k^2} \right] = \lim_{\delta X_k \rightarrow 0} \frac{\left[\frac{\partial \hat{K}}{\partial X_k} \right] - \left[\frac{\partial K}{\partial X_k} \right]}{\delta X_k} \tag{60}$$

$$\begin{aligned} \left[\frac{\partial^2 K}{\partial X_k^2} \right] &= \int_{-1}^1 \int_{-1}^1 \left\{ \frac{|J|^2 [2[G] \delta X_k + ([E] + [E]^T)] (|J| + |J|_x \delta X_k)}{|J|^2 (|J| + |J|_x \delta X_k)^2 \delta X_k} \right. \\ &\quad - \frac{|J|^2 |J|_x ([G] \delta X_k^2 + ([E] + [E]^T) \delta X_k + [F])}{|J|^2 (|J| + |J|_x \delta X_k)^2 \delta X_k} \\ &\quad \left. - \frac{(|J| + |J|_x \delta X_k)^2 ([E] + [E]^T) + [F]}{|J|^2 (|J| + |J|_x \delta X_k)^2 \delta X_k} \right\} t d\xi d\eta \tag{61} \\ &= \int_{-1}^1 \int_{-1}^1 \left\{ \frac{2[G]}{|J|} - \frac{2|J|_x}{|J|^2} ([E] + [E]^T) + \frac{2|J|_x^2}{|J|^2} [F] \right\} t d\xi d\eta \end{aligned}$$

It should be noted that equations (59) and (61) are applicable to each of the nodal coordinates of the design element. Both $[\partial K / \partial X_k]$ and $[\partial^2 K / \partial X_k^2]$ are used to calculate the vector of the pseudo-loads. Subsequently, the pseudo-loads are utilized to compute the first and second order derivatives of the displacements.

In general, the global constraints involve stresses and displacements of the finite element model. To this end, the first and second order derivatives of the displacements were computed. In the next stage, a new formulation for finding $\partial^2 \{\sigma\} / \partial X_k^2$ is presented. The stresses of an isoparametric element can be found as below:

$$\{\sigma\} = [D] [B] \{q\}_e \tag{62}$$

The derivative of the stress with respect to shape variable can be written in the usual manner as follows:

$$\frac{\partial \{\sigma\}}{\partial X_k} = [D] [B] \left\{ \frac{\partial q}{\partial X_k} \right\}_e + [D] \left[\frac{\partial B}{\partial X_k} \right] \{q\}_e \tag{63}$$

Differentiating this equation once again will lead to following relation:

$$\frac{\partial^2 \{\sigma\}}{\partial X_k^2} = 2[D] \left[\frac{\partial B}{\partial X_k} \right] \left\{ \frac{\partial q}{\partial X_k} \right\}_e + [D] [B] \left[\frac{\partial^2 q}{\partial X_k^2} \right]_e + [D] \left[\frac{\partial^2 B}{\partial X_k^2} \right] \{q\}_e \tag{64}$$

It is clear that the calculation of the derivatives of the stresses requires finding the derivatives of the displacements and strain matrix. In the succeeding lines, the first and second derivatives of the strain matrix are computed:

$$\left[\frac{\partial B}{\partial X_k} \right] = \lim_{\delta X_k \rightarrow 0} \frac{[\hat{B}] - [B]}{\delta X_k} \tag{65}$$

$$\left[\frac{\partial B}{\partial X_k} \right] = \frac{[B_x]^a (|J| + |J|_x \delta X_k) - |J|_x ([B]^a + [B_x]^a \delta X_k)}{(|J| + |J|_x \delta X_k)^2} \quad (66)$$

$$\left[\frac{\partial B}{\partial X_k} \right] = \frac{[B_x]^a}{|J|} + \frac{|J|_x}{|J|^2} [B]^a \quad (67)$$

Differentiating the last relation once again gives the following results:

$$\left[\frac{\partial^2 B}{\partial X_k^2} \right] = \lim_{\delta X_k \rightarrow 0} \frac{\left[\frac{\partial \hat{B}}{\partial X_k} \right] - \left[\frac{\partial B}{\partial X_k} \right]}{\delta X_k} \quad (68)$$

$$\left[\frac{\partial^2 B}{\partial X_k^2} \right] = \frac{2|J|_x^2}{|J|^2} [B]^a - \frac{2|J|_x}{|J|^2} [B_x]^a \quad (69)$$

By using these formulations, the second order derivatives of stresses can be formulated as follows:

$$\begin{aligned} \frac{\partial^2 \{\sigma\}}{\partial X_k^2} &= \frac{2}{|J|} [D] [B_x]^a \left\{ \frac{\partial q}{\partial X_k} \right\}_e - \frac{2|J|_x}{|J|^2} [D] [B]^a \left\{ \frac{\partial q}{\partial X_k} \right\} \\ &+ \frac{1}{|J|} [D] [B]^a \left\{ \frac{\partial^2 q}{\partial X_k^2} \right\} + \frac{2|J|_x^2}{|J|^2} \{\sigma\}_e - \frac{2|J|_x}{|J|^2} [D] [B_x]^a \{q\}_e \end{aligned} \quad (70)$$

5. Numerical steps

To perform the second order sensitivity analysis, and solving the optimization problem, the first and second derivatives of the nodal displacements are required. In the following lines, the numerical steps for computing these values are given:

1. Computing the Jacobin matrix.
2. Calculating the determinant of the Jacobin matrix.
3. Picking up the adjoin entries of the Jacobin matrix.
4. Computing the adjoin derivatives of the shape functions as follows:

$$\begin{Bmatrix} N_{i,x} \\ N_{i,y} \end{Bmatrix}^a = [J]^a \begin{Bmatrix} N_{i,\xi} \\ N_{i,\eta} \end{Bmatrix} \quad (71)$$

5. Forming the adjoin entries of the strain matrix.
6. Computing the following matrices:

$$[E] = ([B]_x^a)^T [D_m] [B]^a \quad (72)$$

$$[F] = ([B]^a)^T [D_m] [B]^a \quad (73)$$

$$[G] = ([B]_x^a)^T [D_m] [B]_x^a \tag{74}$$

7. Computing the first and second derivatives of the finite element stiffness matrix with respect to the design variables:

$$\left[\frac{\partial K}{\partial X_k} \right]_e = \int_{-1}^1 \int_{-1}^1 \left\{ \frac{([E] + [E]^T)}{|J|} - [F] \frac{|J|_x}{|J|^2} \right\} t d\xi d\eta \tag{75}$$

$$\left[\frac{\partial^2 K}{\partial X_k^2} \right] = \int_{-1}^1 \int_{-1}^1 \left\{ \frac{2[G]}{|J|} - \frac{2|J|_x}{|J|^2} ([E] + [E]^T) + \frac{2|J|_x^2}{|J|^2} [F] \right\} t d\xi d\eta \tag{76}$$

8. Calculating the vector of pseudo-load related to each finite element as follows:

$$\{\tilde{p}_1\} = - \left[\frac{\partial K}{\partial X} \right] \{q\} \tag{77}$$

9. Repeating steps 1 through 8 for all finite elements and assembling the structural pseudo-load.
10. Finding the derivatives of the nodal displacements with respect to design variables by solving the following system of equations:

$$[K] \left\{ \frac{\partial q}{\partial X} \right\} = \{\tilde{p}_1\} \tag{78}$$

11. Forming the vector of pseudo-load related to each finite element as follows:

$$\{\hat{p}_2\} = - \left[\frac{\partial^2 K}{\partial X_k^2} \right] \{q\} - 2 \left[\frac{\partial K}{\partial X_k} \right] \left\{ \frac{\partial q}{\partial X_k} \right\} \tag{79}$$

12. Computing the second order derivatives of the nodal displacements with respect to design variables by solving the following system of equations:

$$[K] \left\{ \frac{\partial^2 q}{\partial X_k^2} \right\} = \{\hat{p}_2\} \tag{80}$$

6. Numerical examples

6.1. Plane stress problems

Most of the three-dimensional design problems can be converted to the two-dimensional ones using some simplification assumptions. In the cases where one dimension of the structure is too small compared to the other dimensions, the stress components in the small direction can be neglected and a two-dimensional state of stress can be considered. This is the case of the plane stress problems, which is considered here. In this paper, the optimum solution is obtained by writing a computer program. This program consists of several subroutines, which can be run on microcomputers. The correct performances of the writers' program are checked by solving some benchmark problems.

6.1.1. Cantilever beam

For the first example, a design of a two-dimensional cantilever beam is considered here. The structure is under a concentrated moment of 518 kg cm at its free end. The initial shape of the beam is shown in Figure 2. The properties of the structure are: length $L = 25.4$ cm, thickness $t = 2.54$ cm, modulus of elasticity $E = 700000$ kg/cm², and Poisson’s ratio $\nu = 0.3$. In this problem, the y -coordinate of the master nodes i, j and k are selected as design variables. Two behavior constraints are considered for the design of this beam. The first behavior constraint is the vertical displacement of the free end of the structure, which is limited to 1.27 cm. Furthermore, the second behavior constraint is the bending stress at the beam cross section, which is limited to 2100 kg/cm².

As it is shown in Figure 2, the solution of this problem has been found after eight design iterations. It is evident that the initial volume of the structure (i.e. 163.87 cm³) is reduced to 54.19 cm³. The optimum shape of the beam is presented in Figure 3 and in Table 1, the results obtained by the proposed formulation are presented along with the pervious findings by Rezaiee and Salary [1], Prasad and Ding [12], Haftka [13], and Braibant and Fleury [14]. Based on the presented values, it is observed that all of the results are comparable to each other.

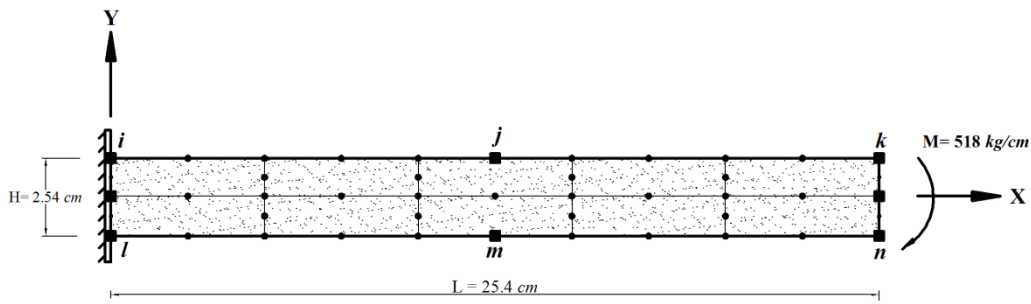


Figure 2. Initial shape of the beam.

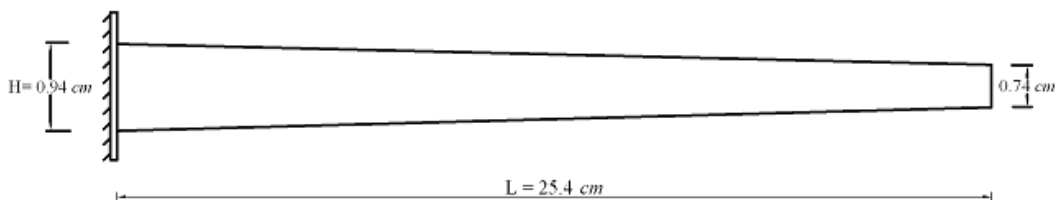


Figure 3. Final design of the beam.

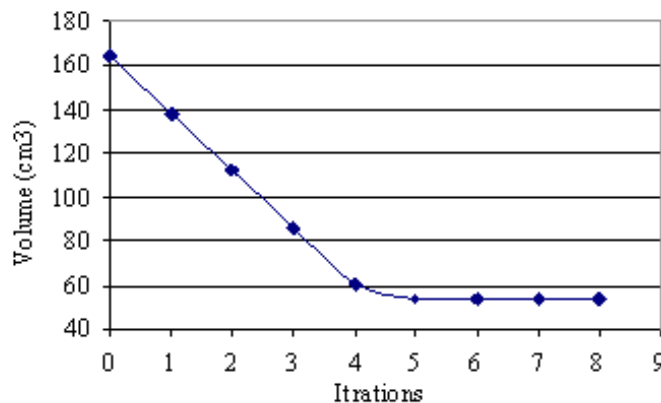


Figure 4. Volume reduction history of the beam.

Table1. Comparison of the result.

Height of the beam (cm)					
$x(cm)$	This paper	Ref. [1]	Ref. [12]	Ref. [13]	Ref. [14]
0.00	94.00	1.152	1.121	1.076	1.178
2.54	0.92	1.094	1.064	1.036	1.137
5.08	0.90	1.040	1.017	0.996	1.095
7.62	0.88	0.990	0.977	0.956	1.053
10.16	0.86	0.944	0.940	0.918	1.011
12.70	0.84	0.922	0.905	0.882	0.969
15.24	0.82	0.881	0.868	0.846	0.928
17.78	0.80	0.826	0.833	0.813	0.886
20.32	0.78	0.794	0.803	0.783	0.845
22.86	0.76	0.764	0.782	0.765	0.803
25.40	0.74	0.740	0.775	0.763	0.762
Volume (cm ³)	54.19	59.09	59.48	58.04	63.61

6.1.2. Cantilever beam under a uniformly distributed load

The initial shape of this structure is shown in Figure 5. In the second numerical example, the cantilever beam is under a uniformly distributed load. The density of the distributed load is equal to 100 kg/m. The properties of the structure are: length $L = 8$ m, thickness $t = .8$ m, modulus of elasticity $E = 210000$ kg/cm², and Poisson’s ratio $\nu = 0.2$. To optimize the shape of this beam, the y -coordinate of master nodes i and j are selected as design variables. Moreover, the behavior constraint is the stress at the beam cross section, which is limited to 25 and 250 kg/cm², for the tension and compression, respectively.

After performing six design iterations, the solution of this problem has been archived. In the optimization process, the initial volume of the structure (i.e. 38.4 cm³) is reduced to 23.47 cm³. The structural volume variation is shown in Figure 6.

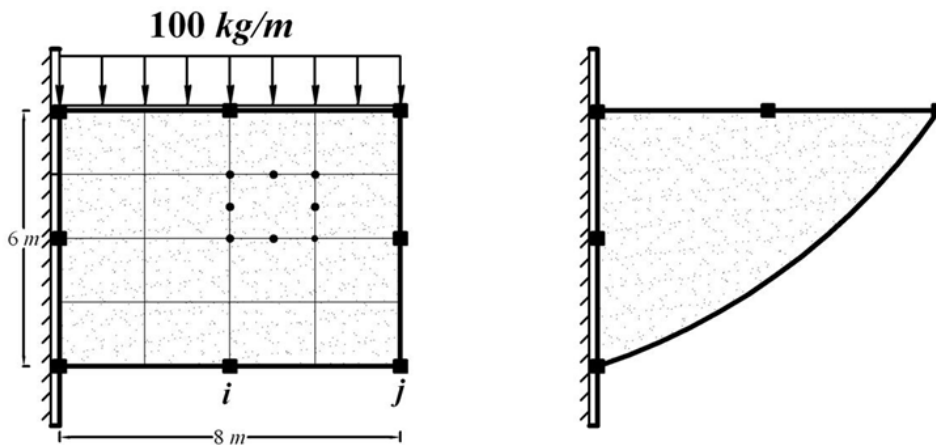


Figure 5. Initial and final design of the beam.



Figure 6. Volume reduction history of the beam.

6.1.3. Concrete pole

An optimized design of a concrete pole is considered, as a third example for plane stress problem. The initial shape of the pole is shown in Figure 7. A uniform load of 1000 kg/m is applied throughout the pole. The allowable strength of the concrete is 25 and 250 kg/cm², for the tension and compression, respectively. Furthermore, the modulus of elasticity and Poisson’s ratio have the values of $E = 210000$ kg/cm² and $\nu = 0.2$. The thickness of the pole is 80 cm, and it is considered fixed during the design process. The design variables contain the x -coordinates of the master nodes i , j , k and l .

As it is shown in Figure 7, the optimum design is obtained after 9 design iterations. The structural volume is reduced from the initial value of 76.8 m³ to the optimum one of 23.36 m³. The variation of the pole volume is shown in Figure 8.

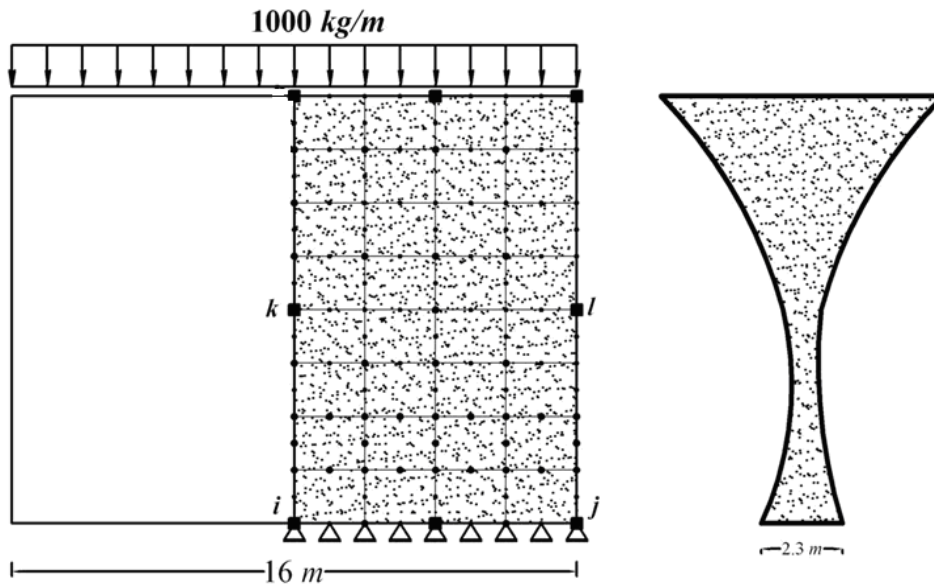


Figure 7. Initial and final shape of the concrete pole.

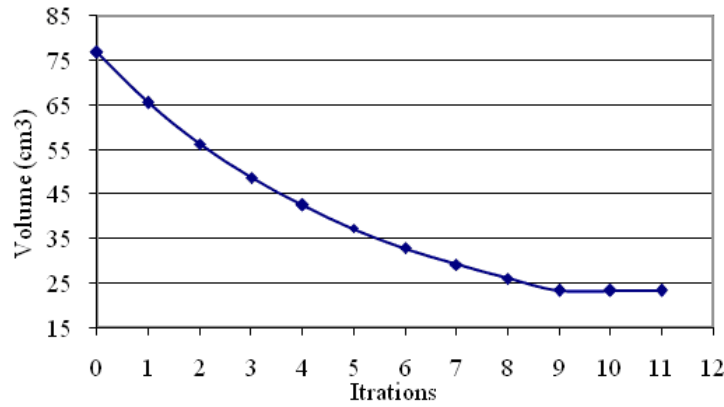


Figure 8. Volume reduction history of the concrete pole.

6.2. Plane strain problems

For all of the plane problems, depending on the geometry of the structure and the applied loads, only the in-plane components of the strains may be considered. In fact, the other components of strains can be neglected. The finite element solutions of a few such continua, which are referred to as plane strain problems, are considered in the following lines.

6.2.1. Gravity dam

The first plane strain problem, an optimum design of the down-stream side of a gravity dam, is considered. The initial shape of the dam is shown in Figure 9. The dam is built of the concrete materials with the following properties: modulus of elasticity $E = 210000 \text{ kg/cm}^2$, Poisson's ratio $\nu = 0.2$, tensile allowable stress 10 kg/cm^2 , compressive allowable stress 100 kg/cm^2 . This structure is designed only against water pressure loads, and principal stresses at the Gauss points are limited to the maximum allowable values. In this problem, the width of the dam at its head has a limited value of 6.0 m.

To optimize this structure, the shape of the downstream wall is controlled by using three design variables: X_i , X_j and X_k . After performing 11 design iterations, the volume of the dam is reduced from 3000 to 1351 m³. The variation of the structural volume is shown in Figure 10. To demonstrate the finding, the final shape of the dam is presented in Figure 9. It is worth emphasizing that this problem was solved previously by Rezaiee and Salary [1], and the optimized volume of the dam was reported 1665 m³.

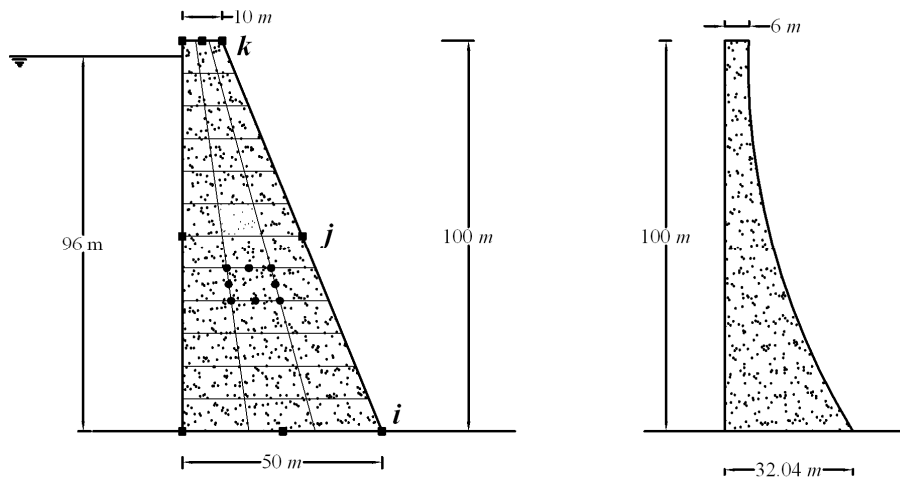


Figure 9. Initial and final shape of the dam.

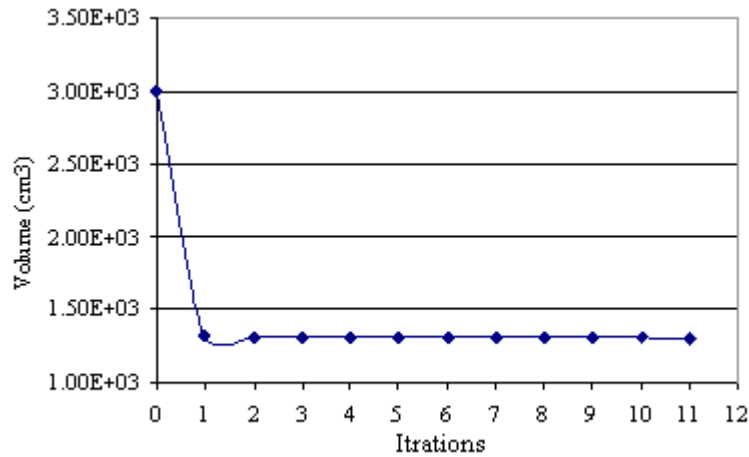


Figure 10. Volume reduction history of the dam.

6.2.2. Tunnel

In this example problem, an optimum design of the internal face of a tunnel is considered. This structure is shown in Figure 11. The initial shape of the tunnel is a semi-ellipse with long and short diameters of 13 and 9 m, respectively. This tunnel is running through unconsolidated clay with a saturated density of $\gamma = 2 \text{ t/m}^3$. The earth pressure is considered equal to the weight of the earth column of height z (i.e. γz), which is applied as radial forces at any point of the tunnel face. This plane strain structure is built of un-reinforced concrete with the tensile and compressive strengths of 20 and 200 kg/cm^2 , respectively. The principal stresses at the tunnel are considered as the stress constraints. To perform the optimization process, the nodal coordinates: X_i, X_j, Y_j and Y_k are selected as the design variables. The solution of the problem is found after 5 design iterations. Figure 12 reveals the optimum shape of the structure. It is evident that the volume of the tunnel is reduced from the initial value of 9.4 m^3 to the optimized one of 6.59 m^3 . This problem was solved previously by Rezaiee-Pajand and Salary [1]. According to the reported results, the final volume of the tunnel was found to be 6.0 m^3 , after 13 iterations.

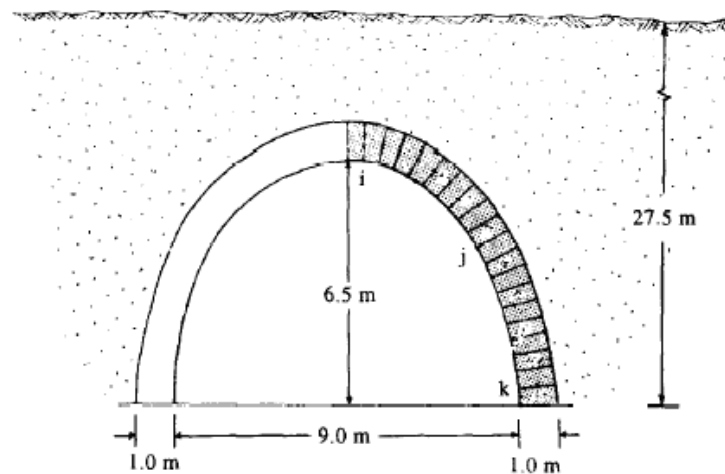


Figure 11. Initial shape of the tunnel.

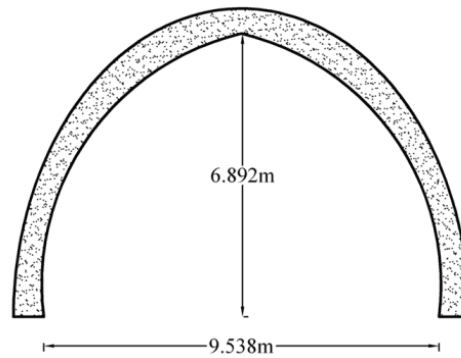


Figure12. Final shape of the tunnel.

7. Conclusion

This study deals with the second order sensitivity analysis of the plane structural problem. The goal is to find the optimized continuum shaped. To perform the optimization process, the finite element method and the design element technique were utilized. In this article, the required formulations for calculating the sensitivity derivatives of a general finite element were presented. Afterward, the developed formulations are used to compute the second order derivatives of the stiffness matrix of an eight-node isoparametric element. According to the numerical results, it is proven that the proposed second order sensitivity analysis has the effective capability for the shape optimization of the continuum structures, when used along with SQP method.

References

- [1] M. Rezaiee Pajand, M.R. Salary, Two-dimensional sensitivity analysis, *Computers & Structures*, Vol. 47, 3 (1996) 563-571.
- [2] M. Rezaiee Pajand, M.R. Salary, Three-dimensional sensitivity analysis using a factoring technique, *Computers & Structures*, Vol. 49, 1 (1993) 157-165.
- [3] Y. Ding, Shape optimization of structures: a literature survey, *Computers & Structures*, Vol. 24, 6 (1986) 985-1004.
- [4] H. Uysal, R. Gül, Ü. Uzman, Optimum shape design of shell structures, *Eng. Struct*, 29 (2007) 80-87.
- [5] D.C. Lee, J.I. Lee, An integrated design for double-layered structures, *Finite Elements in Analysis and Design*, Vol. 41, 2 (2004) 133-146.
- [6] J.J. Lee, G.J. Park, Shape optimization of the initial blank in the sheet metal forming process using equivalent static loads, *International Journal for Numerical Methods in Engineering*, Vol. 85, 2 (2011) 247-268.
- [7] S.J. Lee, E. Hinton, Dangers inherited in shells optimized with linear assumptions, *Computers & Structures*, 78 (2000) 473- 486.
- [8] X. Duan, T. Sheppard, Shape optimization using FEA software: A V-shaped anvil as an example, *J. Mater. Proc. Technol*, 120 (2002) 426- 431.
- [9] H. Uysal, N. Asci Demiröz, U. Uzman, Optimum design of planar beams based on sensitivity analysis, *Scientific Research and Essays*, Vol. 17, 5 (2010) 2413-2423.
- [10] R.T. Haftka, Z. Mroz, First and second order sensitivity analysis of linear and non linear structures, *AIAA J*, 24 (1986) 1187-1192.
- [11] J. Akbari, N.H. Kim, M.T. Ahmadi, Shape sensitivity analysis with design-dependent loadings-equivalence between continuum and discrete derivatives, *Struct Multidisc Optim*, 40 (2010) 353–364.
- [12] Y. Ding, Shape optimization of two-dimensional elastic structure with optimal thicknesses for fixed parts, *Computers & Structures*, 27 (1987) 729-743.

- [13] B. Prasad, R.T. Haftka, Optimal structural design with plate finite element, *J. Struct. Div, ASCE*, 105 (1979) 2367-2382.
- [14] V. Braibant, C. Fleury, Shape optimal design using B-splines, *Comput. Meth. appl. Mech. Engng*, 44 (1984) 247-267.