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Improved Fuzzy Bayesian Reliability Analysis of Coherent Systems via the α -Pessimistic Method

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ABSTRACT

In real-world reliability analysis, the underlying data and prior knowledge are often imprecise, posing significant challenges to classical probabilistic models. This study presents a novel fuzzy Bayesian approach for analyzing the reliability of coherent systems under imprecise prior information, where system lifetimes follow a Pascal distribution. We construct uncertain Bayes estimators using both squared error and precautionary loss functions by modelling the system reliability as a fuzzy random variable with a prior fuzzy distribution. A key innovation of the proposed approach is the application of the α-pessimistic method, which allows for the estimation process to be carried out without relying on complex non-linear programming, a common limitation in existing literature. Instead, this technique simplifies the computational procedure while enhancing interpretability and analytical tractability. The framework is applied to coherent systems, including parallel, series, and k-out-of-m structures, using Mellin transform techniques to derive the estimators. A numerical example is provided to demonstrate the practical applicability and effectiveness of the proposed method.

1. Introduction

For some experiments, the costs of testing can depend more on the number of failures than on the number, which may be more influenced by the number of trials n. This is especially true if failures have a far higher economic cost than survivors. In this case, it will be more advantageous to fix the number of failures (m) and treat (N) as a random variable instead of a fixed number of trials. The

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Pascal distribution is an appropriate tool fr modelling in such situations [20]. For a given r, the conditional probability function that N trials are necessary to obtain (m) failures is provided by

$$f\ (n\,|\,r)=\binom{n-1}{m} r^{n-m}(1-r)^{m},\ \ 0<r<1,\ m=1,2,...,\quad n=m,m+1,....$$

When the sample size or (the number) of statistical data is insufficient for more advanced (complex) statistical analyses, we must use another source of information. So, we employ Bayesian approaches. In a Bayesian analysis, there are data (a statistical model), the prior distributions for parameters, and the loss functions. Unlike the Bayesian approach, where the parameter is a random variable with a distribution function called the prior distribution, classical statistics treats the parameter as an unknown constant. After observing the sample, the extra information about the parameter is combined prior to obtaining the posterior distribution. A Bayesian estimator is obtained by minimizing the posterior loss function's expected value [19].

In classical reliability theory it is assumed that the component lifetimes are formulated by exact numbers. However, due to uncertainty and inaccuracy in data, it is sometimes complicated to determine the exact values of these parameters in real-world systems. In fact, in analyzing system reliability, the uncertainty is an important aspect which needs to be considered. After introducing fuzzy set theory by Zadeh [28], many researchers have applied this theory in order to quantify the uncertainty, especially in the field of probability and statistics [23, 29].

To the best of our knowledge, application of fuzzy probability to fault-tree analysis was first studied by Tanaka et al. [21] and by Furuta and Shiraishi [3]. After that, numerous researchers have made significant contributions to imprecise system reliability assessment [12, 16]. In the following, we review the primary studies on Bayesian fuzzy reliability. Wu [25, 26] studied a Bayesian approach to system reliability based on fuzzy random variables. He assumes the fuzzy parameters as fuzzy random variables with prior fuzzy distributions. To determine the membership function of the estimates of the reliability function of multi-parameter lifetime distributions, Huang et al. [9] applied an artificial neural network and a genetic algorithm. The fuzzy Bayes point estimators of system reliability based on the Exponential distribution for a coherent system were investigated by Wu [27]. After that, some classical and Bayesian procedures for reliability estimation were provided by Viertl [24] based on fuzzy data. Liu et al. [15] propose a new method for determining the membership functions of parameter estimates and, the reliability functions of multi-parameter lifetime distributions and formulate a preventive maintenance policy using a fuzzy reliability framework. Taheri and Zarei [22] developed a Bayesian approach to coherent system reliability analysis based on the vague set (intuitionistic fuzzy set) theory. Görkemli and Ulusoy [8] investigate a novel approach to compute the reliability and availability of a production system. Based on Wu's approach [27], the fuzzy Bayes estimation of system reliability was extended by Gholizadeh et al. [6, 7] based on prior two-parameter exponential and Pascal distributions. Zarei et al. [30] provide the Bayes estimator of system aging (failure rate and mean time to failure) based on vague lifetime data in the case of complete and censored data sets. Based on Bayesian inference with fuzzy probabilities, Bamrungsetthapong and Pongpullponsak [1] developed the posterior fuzzy system reliability of a non-repairable multi-state series–parallel system. They consider the fuzzy failure rate function as an exponential fuzzy number. Hryniewicz [10] presents some results related to fuzzy Bayes methodology of imprecise reliability analysis and obtained some useful approximations according to shadowed sets. Gholizadeh et al. [5] using E-Bayesian estimation approach, investigate a modified Bayesian estimator for system reliability and apply it to present a methodology for discussing uncertainty in the reliability assessment of the production system. Jegatheesan and Gundala [11] investigate the fuzzy Bayesian reliability assessment of the linear (circular) consecutive k-out-of-n: F system based on the squared error loss function. They consider the parameters as fuzzy random variables in the process of obtaining fuzzy Bayesian reliability. Recently, a hybrid reliability analysis framework was provided by Fang et al. [4]. For this purpose, they decomposed the structural parameters and external loads having fuzziness and extended them to fuzzy sets.

In the present work, the Bayesian approach to system reliability estimation is incorporated into the fuzzy set theory to deduce the so-called imprecise Bayes estimator. First, in section 2, we briefly recall some notions of fuzzy random variables. Then, the concept of the Mellin transformation was recalled and the Bayesian approach to the modelling system reliability was presented. In Section 3, we explore the derivation of the Bayes estimator for structural system reliability using the α-pessimistic method. A numerical example illustrating the proposed approach is presented in Section 4. Finally, concluding remarks are given in Section 5.

2. Problem Formulation

In this section, we suggest the Bayesian point estimator of system reliability in imprecise environments.

2.1. Fuzzy Numbers

A fuzzy set $\widetilde{M}$ of $\mathcal{R}$ is called a fuzzy number

(1) If $\forall\alpha \in [0,1]$, the set $\widetilde{M}[\alpha]$ will be represented by $[\widetilde{M}^L[\alpha], \tilde{A}^U[\alpha]]$, as a non-empty compact interval. Where $\widetilde{M}^L[\alpha] = inf\{x \in R | \widetilde{M} \geq \alpha\}$ and $\widetilde{M}^U[\alpha] = sup\{x \in \mathcal{R} | \widetilde{M}(x) \geq \alpha\}$ respectively.

(2) There is a single real number $x^* = x^*_{\widetilde{M}} \in \mathcal{R}$ such that $\widetilde{M}(x^*) = 1$, i.e. $\widetilde{M}[1]$ is a singleton set and this real number is unique.

$\mathcal{F}(\mathcal{R})$ represents the set of all fuzzy numbers of $\mathcal{R}$.

One of the most popular fuzzy numbers is the LR-fuzzy number. $\widetilde{M} = (m; \delta, \gamma)_{LR}$ stands for the LR fuzzy number $\widetilde{M}$ with center value $m \in \mathcal{R}$, right and left spread $\gamma \in \mathcal{R}^+, \delta \in \mathcal{R}^+$. The following membership function is present.

$$\widetilde{M}(x) = \begin{cases} L(\frac{m-x}{\delta}) & if \quad x \leq m, \\ R(\frac{x-m}{\gamma}) & if \quad x \geqslant m, \end{cases}$$

$R: \mathcal{R}^+ \rightarrow [0,1]$ and $L: \mathcal{R}^+ \rightarrow [0,1]$, with $L(0) = R(0) = 1$, are decreasing right- and left-shaped functions.

The following formula makes it simple to compute the α-cut of $\widetilde{M}$

$$\widetilde{M}[\alpha] = [m - L^{-1}(\alpha)\delta, m + R^{-1}(\alpha)\gamma]. \alpha \in [0,1].$$

Remark 2.1 *The α-pessimistic of $\widetilde{M} \in \mathcal{F}(\mathcal{R})$ is a mapping $\widetilde{M}_\alpha: [0,1] \to \mathcal{R}$, for any $\widetilde{M} \in \mathcal{F}(\mathcal{R})$, and it is defined by:*

$$\widetilde{M}_\alpha = \begin{cases} \widetilde{M}^L[2\alpha] & \alpha \in [0,0.5], \\ \widetilde{M}^U[2(1-\alpha)] & \alpha \in (0.5,1]. \end{cases}$$

where $\widetilde{M}^L[\alpha]$ and $\widetilde{M}^U[\alpha]$ represent the lower and the upper bounds of α-cuts of $\widetilde{M}$, respectively. It is clear that:

$$\widetilde{M}[\alpha] = [\widetilde{M}^L[\alpha], \widetilde{M}^U[\alpha]] = [\widetilde{M}_{\frac{\alpha}{2}}, \widetilde{M}_{1-\frac{\alpha}{2}}].$$

For an LR-fuzzy number $\widetilde{M} = (n; \delta, \gamma)_{LR}$, the α-pessimistic are determined as follows

$$\widetilde{M}_\alpha = \begin{cases} m - \delta L^{-1}(2\alpha) & \alpha \in [0,0.5], \\ m + \gamma R^{-1}(2(1-\alpha)) & \alpha \in (0.5,1]. \end{cases}$$

In the case of triangular fuzzy number $\widetilde{M} = (m; \delta, \gamma)_T$, we have that

$$\widetilde{M}_\alpha = \begin{cases} (m-\delta) + 2\delta\alpha & \alpha \in [0,0.5], \\ m + \gamma - 2\gamma(1-\alpha) & \alpha \in (0.5,1]. \end{cases}$$

2.2. Fuzzy Random Variables

A well-stated and well-supported model for the random mechanisms generating fuzzy data in the probabilistic setting is random fuzzy numbers, or more generally, random fuzzy sets. They combine randomness and fuzziness so that the former impacts the creation of experimental data and the latter affects the nature of experimental data that are presumed to be intrinsically ambiguous. In some ways, the concept of the random fuzzy set may be formalized.

Definition 2.2 *A probability space $(\Omega, \mathcal{A}, \mathcal{P})$ describes a random experiment. If $\forall\alpha \in [0,1]$, the real-valued mapping $\widetilde{W}_\alpha: \Omega \to \mathcal{R}$ is a random variable with real-valued on $(\Omega, \mathcal{A}, \mathcal{P})$, the fuzzy-valued mapping $\widetilde{W}: \Omega \to \mathcal{F}(\mathcal{R})$ is called an frv. All random variables in this study are supposed to have the same space of probability $(\Omega, \mathcal{A}, \mathcal{P})$.*

The concept of $frvs$ was first suggested by Kwakernaak [14] and it was later developed clearly by Meyer and Kruse [13]. If the two real-valued mappings $\widetilde{W}_\alpha^L: \Omega \to \mathcal{R}$ and $\widetilde{W}_\alpha^U: \Omega \to \mathcal{R}$ are both real-valued random variables $\forall\alpha \in [0,1]$, a mapping $\widetilde{W}: \Omega \to \mathcal{F}(\mathcal{R})$ is said to be an frv in a probability space $(\Omega, \mathcal{A}, \mathcal{P})$.The following relationships between the definition of frv proposed in this study and Kwakernaak and Kruse's definition are easily demonstrated.

$$\widetilde{W}_\alpha = \begin{cases} \widetilde{W}^L_{2\alpha} & \alpha \in [0,0.5], \\ \widetilde{W}^U_{2(1-\alpha)} & \alpha \in (0.5,1]. \end{cases}$$

$$\widetilde{W}[\alpha] = [\widetilde{W}_{\frac{\alpha}{2}}, \widetilde{W}_{1-\frac{\alpha}{2}}].$$

The first equation shows that the information contained in the two-dimensional variable $(\widetilde{W}^L_\alpha, \widetilde{W}^U_\alpha)$ is summarized in the one-dimensional variable $\widetilde{W}_\alpha$ making the computational procedures in the problem easier.

2.3. Imprecise Bayes Estimation

To construct the Bayesian point estimator (BPE) of system reliability, each component of the underlying system was given certain prior distributions. As a result, we can use Bayes' theorem to obtain the posterior distribution of each component's reliability. The posterior distribution of the reliability of each component is then used to compute the posterior distribution of the system's reliability.

Suppose that the prior pdf of the i-th reliability component, R_i, is $\pi_i(r_i)$. Then, the posterior distribution of R_i, denoted by $\pi_i(r_i|m_i)$, is obtained as follows:

$$\pi_i(r_i|m_i) = \frac{r_i^{n_i-m_i}(1-r_i)^{m_i}\pi_i(r_i)}{\int_0^1 t^{n_i-m_i}(1-t)^{m_i}\pi_i(t)dt}, \quad 0 < r_i < 1, \tag{1}$$

where m_i is the total number of failures n_i trials that were detected.

It should be noted that the system reliability can be expressed by multiplying independent random variables that correspond to component unreliability in parallel systems or component reliability in series systems. The main problem is to deduce the *pdf* of such random variables. In other words, under the square error loss function, the BPE of the system reliability is the mean of the posterior distribution, under the precautionary loss function, it is the square root of the second moment of the posterior distribution.

The Mellin transform [2], which is described as follows, is an appropriate technique for answering this question.

Definition 2.3 *For a non-negative random variable X with pdf $f(x)$. The Mellin transform of f, concerning the complex parameter u, is defined by:*

$$M(f;u) = \int_0^\infty x^{u-1}f(x)dx = E(X^{u-1}). \tag{2}$$

Theorem 2.4 *Suppose that X_i are independent random variables with pdf f_i for $i = 1,\dots,k$ and $Y = \Pi_{i=1}^k X_i$ has a pdf denoted by $g_k(y)$. Then:*

$$M(g_k;u) = \Pi_{i=1}^k M(f_i;u). \tag{3}$$

Theorem 2.5 *Let $\hat{\theta}$ be a Bayesian estimator of the parameter θ. Then,*

(I) [19] Under the squared error loss function $L(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2$, denoted by SEL, the Bayesian point of estimation yields $\hat{\theta}_s = E(\theta|X)$.

(II) *[7] Under the precautionary loss function* $L(\hat{\theta},\theta)=\frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}$*, denoted by PL, the Bayesian point of estimation yields* $\hat{\theta}_p^2=E(\theta^2|X)$.

3. Fuzzy Bayesian Estimator of System Reliability

In this section, we will derive a fuzzy BPE of system reliability based on Pascal distribution under the squared error loss function and the precautionary loss function for parallel, series, and k-out-of-m systems.

3.1. Series Systems

A suitable prior distribution for R_i with the following pdf is the Beta distribution, $Beta(m_{i0},n_{i0})$.

$$\pi_i(r_i)=\frac{\Gamma(n_{i0})}{\Gamma(m_{i0})\Gamma(n_{i0}-m_{i0})}r_i^{m_{i0}-1}(1-r_i)^{n_{i0}-m_{i0}-1}, 0\le r_i\le 1,\quad m_{i0},n_{i0}>0. \tag{4}$$

The posterior distribution of the i-th component of reliability R_i is computed using ***Eqs. (1)*** and ***(4)*** as shown below:

$$\pi_i(r_i|m_i)=\frac{\Gamma(n_i+n_{i0})}{\Gamma(n_i-m_i+m_{i0})\Gamma(n_{i0}-m_{i0}+m_i)}r_i^{n_i-m_i+m_{i0}-1}(1-r_i)^{n_{i0}-m_{i0}+m_i-1}. \tag{5}$$

$Beta(n_i-m_i+m_{i0},n_i+n_{i0})$ is the posterior distribution as a result.

The Mellin transform of $\pi_i(r_i|m_i)$ derived from ***Eqs. (2)*** and ***(5)***, is:

$$M\{\pi_i(r_i|m_i);u\}=E(R_i^{u-1}|m_i)=\frac{\Gamma(n_i+n_{i0})}{\Gamma(n_i-m_i+m_{i0})}\times\frac{\Gamma(n_i-m_i+m_{i0}+u-1)}{\Gamma(n_i+n_{i0}+u-1)}$$

$$=\frac{(n_i+n_{i0}-1)!}{(n_i-m_i+m_{i0}-1)!}\times\frac{1}{(n_i-m_i+m_{i0}+u-1)(n_i-m_i+m_{i0}+u)\ldots(n_i+n_{i0}+u-2)}, \tag{6}$$

where $Re(u)>-(n_i-m_i+m_{i0}-1)$.

Consider a series system with k independent components. $R=\Pi_{i=1}^{k}R_i$ is the reliability of the system. Obtaining a fuzzy BPE of the reliability of a fuzzy system is our goal here. From Theorem 2.4, Theorem 2.5 and ***Eq. (6)***, the Mellin transform $\pi(r|s;n)$ of system reliability R is thus given by

$$M\{\pi(r|m);u\}=\Pi_{i=1}^{k}E(R_i^{u-1}\big|m_i)$$

$$=\Pi_{i=1}^{k}\left[\frac{(n_i+n_{i0}-1)!}{(n_i-m_i+m_{i0}-1)!}.\frac{1}{(n_i-m_i+m_{i0}+u-1)(n_i-m_i+m_{i0}+u)\ldots(n_i+n_{i0}+u-2)}\right], \tag{7}$$

In this case $Re(u)>-(n_i-m_i+m_{i0}-1),\quad \forall i=1,2,3,\ldots,k$.

Now, we are going to construct the uncertain BPE of system reliability based on two types of loss functions.

Under the SEL function, the BPE of system reliability is the mean of the posterior distribution. As a result, with $u=2$, we can apply ***Eq. (7)*** to get the BPE of system reliability

$$\hat{r}_s=E(R|m)=M\{\pi(R|m);u=2\}=\Pi_{i=1}^{k}\left(\frac{n_i-m_i+m_{i0}}{n_{i0}+n_i}\right). \tag{8}$$

In the case of a series system, the BPE of the system reliability R is the second root of the second moment of the posterior distribution. We put $u = 3$ in ***Eq. (7)*** and we have

$$\hat{r}_p = \Pi_{i=1}^{k}[E(R_i^2|m_i)]^{\frac{1}{2}}$$

$$= \Pi_{i=1}^{k}[M\{\pi_i(r_i|m_i); u = 3\}]^{\frac{1}{2}}$$

$$= \Pi_{i=1}^{k}[\frac{(n_i+n_{i0}-1)!}{(n_i-m_i+m_{i0}-1)!}.\frac{1}{(n_i-s_i+s_{i0}+2)(n_i-s_i+s_{i0}+3)\cdots(n_i+n_{i0}+1)}]^{\frac{1}{2}}$$

$$= \Pi_{i=1}^{k}[\frac{(n_i-m_i+m_{i0})}{(n_i+n_{i0})}.\frac{(n_i-m_i+m_{i0}+1)}{(n_i+n_{i0}+1)}]^{\frac{1}{2}}. \quad \textbf{(9)}$$

Now suppose that the component reliabilities are not exact values that can be measured, but rather are thought of as fuzzy random variables $\tilde{R}_i$ using the assumption that n_{i0} is a known integer (representing the pseudo number of items for the i-th component) and $\tilde{s}_{i0}$ is a known fuzzy number (representing our imprecise knowledge of the pseudo number of failures). In actuality, $\widetilde{m}_{i0}$ is taken as a fuzzy number because it's possible that the exact number of failures wasn't accurately recorded. In such a situation, the uncertain BPE of $(\tilde{r}_s)_\alpha$ and $(\tilde{r}_p)_\alpha$ are obtained from ***Eqs. (8)*** and ***(9)*** as

$$(\hat{\tilde{r}}_s)_\alpha = \Pi_{i=1}^{k}(\frac{n_i-m_i+(\widetilde{m}_{i0})_\alpha}{n_{i0}+n_i}), \quad \forall\alpha \in [0,1]. \quad \textbf{(10)}$$

$$(\hat{\tilde{r}}_p)_\alpha = \pi_{i=1}^{k}[\frac{(n_i-m_i+(\widetilde{m}_{i0})_\alpha)}{(n_i+n_{i0})}.\frac{(n_i-m_i+(\widetilde{m}_{i0})_\alpha+1)}{(n_i+n_{i0}+1)}]^{\frac{1}{2}}. \quad \textbf{(11)}$$

3.2. Parallel Systems

We will consider a parallel system with k-independent components. The system reliability is obtained as $R = 1 - \Pi_{i=1}^{k}(1 - R_i)$, where R_i is the component reliability of the system's $i - th$ component. We employ the unreliability of the system, $Q = 1 - R = \Pi_{i=1}^{k}Q_i$ and $Q_i = 1 - R_i$.

Applying ***Eq. (5)*** and with a variable change, the posterior density function of $Q_i = 1 - R_i$ is as follows

$$\pi_i(q_i|m_i) = \frac{\Gamma(n_i+n_{i0})}{\Gamma(n_i+m_{i0}-m_i)\Gamma(n_{i0}+m_i-m_{i0})}(1-q_i)^{n_i+m_{i0}-m_i-1}q_i^{n_{i0}+m_i-m_{i0}-1}.$$

Consequently,

$$Q_i|m_i = (1 - R_i)|m_i \sim Beta(n_{i0} - m_{i0} + m_i, n_i + n_{i0}).$$

In the same way as the preceding subsection (series systems) and the Mellin transform, we have

$$E(Q_i|m_i) = \frac{n_{i0}-m_{i0}+m_i}{n_i+n_{i0}}, \quad \mathrm{i} = 1,2,\dots,k.$$

So, the following is how we can obtain the BPE of the system's unreliability for a SEL function:

$$E(Q|m) = \Pi_{i=1}^{k}E(Q_i|m_i) = \Pi_{i=1}^{k}(\frac{n_{i0}-m_{i0}+m_i}{n_i+n_{i0}}).$$

The BPE of system reliability R, for the SEL function, would be

$$\hat{r}_s = E(R|m) = 1 - \Pi_{i=1}^{k}E(Q_i|m_i)$$

$$= 1 - \Pi_{i=1}^{k}[1 - \frac{n_i - m_i + m_{i0}}{n_i + n_{i0}}]. \quad (12)$$

The BPE of the system reliability R for a parallel system and the PL function is

$$\hat{r}_p = [E(R^2|m)]^{\frac{1}{2}} = [E(1-Q)^2|m]^{\frac{1}{2}} = [E(1 - 2Q + Q^2)|m]^{\frac{1}{2}}$$

$$= [1 - 2\Pi_{i=1}^{k} \frac{n_{i0} - m_{i0} + m_i}{n_i + n_{i0}} + \Pi_{i=1}^{k} \frac{n_{i0} - m_{i0} + m_i}{n_i + n_{i0}} . \frac{n_{i0} - m_{i0} + m_i + 1}{n_i + n_{i0} + 1}]^{\frac{1}{2}}. \quad (13)$$

With ***Eqs. (12)*** and ***(13)***, and under uncertain assumptions, we arrive to

$$(\hat{\tilde{r}}_s)_\alpha = 1 - \Pi_{i=1}^{k}[1 - \frac{n_i - m_i + (\tilde{m}_{i0})_\alpha}{n_i + n_{i0}}]. \quad (14)$$

$$(\hat{\tilde{r}}_p)_\alpha = [1 - 2\Pi_{i=1}^{k} \frac{n_{i0} - (\tilde{m}_{i0})_\alpha + m_i}{n_i + n_{i0}} + \Pi_{i=1}^{k} \frac{n_{i0} - (\tilde{m}_{i0})_\alpha + m_i}{n_i + n_{i0}} . \frac{n_{i0} - (\tilde{m}_{i0})_\alpha + m_i + 1}{n_i + n_{i0} + 1}]^{\frac{1}{2}}. \quad (15)$$

3.3. k-out-of-l System

The system reliability for a k-out-of-l system made up of l independent and identical components is given by the formula

$$r_{kl} = \Sigma_{j=k}^{l} \binom{l}{j} r^j (1-r)^{l-j}, \quad (16)$$

where r represents the reliability of each system component. For this issue, we can apply the Bayesian technique in a manner similar to earlier ones. It is assumed that the reliability of component r is a random variable R with a prior Beta distribution $Beta(m_0, n_0)$.

The posterior distribution of R is a $Beta(m + m_0, n + n_0)$ distribution with the following pdf, according to ***Eq. (5)***.

$$\pi_R(r|m, m_0) = \frac{\Gamma(n+n_0)}{\Gamma(n-m+m_0)\Gamma(n_0+m-m_0)} r^{n-m+m_0-1}(1-r)^{n_0+m-m_0-1}. \quad (17)$$

The BPE of system reliability R_{kl} under a SEL function would be:

$$\hat{r}_{kl_s} = E(R_{kl_s}|m, m_0) = \Sigma_{j=k}^{l} \binom{l}{j} \int_0^1 r^j (1-r)^{l-j} \pi_R(r|m, m_0) dr$$

$$= \Sigma_{j=k}^{l} \binom{l}{j} \int_0^1 \frac{\Gamma(n+n_0)}{\Gamma(n-m+m_0)\Gamma(n_0+m-m_0)} \times r^{n-m+m_0+j-1}(1-r)^{n_0+l+m-m_0-j-1} dr$$

$$= \frac{\Gamma(n+n_0)}{\Gamma(n-m+m_0)\Gamma(n_0+m-m_0)} \times [\Sigma_{j=k}^{l} \binom{l}{j} \frac{\Gamma(n-m+m_0+j)\Gamma(n_0+l+m-m_0-j)}{\Gamma(l+n+n_0)}]. \quad (18)$$

Under uncertain assumptions and using ***Eq. (18)***, the BPE of $(\tilde{r}_{kls})_\alpha$ is:

$$(\hat{\tilde{r}}_{kls})_\alpha = \frac{\Gamma(n+n_0)}{\Gamma(n-m+(\tilde{m}_0)_\alpha)\Gamma(n_0+m-(\tilde{m}_0)_\alpha)}$$

$$\times \left[\Sigma_{j=k}^{l} \binom{l}{j} \frac{\Gamma(n-m+(\tilde{m}_0)_\alpha+j)\Gamma(n_0+l+m-(\tilde{m}_0)_\alpha-j)}{\Gamma(l+n+n_0)}\right]. \quad (19)$$

The BPE of system reliability under a PL function would be

$$\hat{r}_{klp} = [E(R_{klp}^2|m,m_0)]^{\frac{1}{2}} = [\Sigma_{j=k}^{l}\Sigma_{i=k}^{l}\binom{l}{j}\binom{l}{i}\int_0^1 r^{j+i+2}(1-r)^{2l-j-i}\pi_R(r|m,m_0)dr]^{\frac{1}{2}}$$

$$= [\Sigma_{j=k}^{l}\Sigma_{i=k}^{l}\binom{l}{j}\binom{l}{i}\int_0^1 \frac{\Gamma(n+n_0)}{\Gamma(n-m+m_0)\Gamma(n_0+m-m_0)}$$

$$\times r^{n-m+m_0+j+i+1}(1-r)^{n_0+2l+m-m_0-j-i-1}dr]^{\frac{1}{2}}$$

$$= [\frac{\Gamma(n+n_0)}{\Gamma(n-m+m_0)\Gamma(n_0+m-m_0)}.(\Sigma_{j=k}^{l}\Sigma_{i=k}^{l}\binom{l}{j}\binom{l}{i}$$

$$.\frac{\Gamma(n-m+m_0+j+i+2l)\Gamma(n_0+2l+m-m_0-j-i)}{\Gamma(2l+n+n_0)})]^{\frac{1}{2}}. \quad \textbf{(20)}$$

Eq. (20) and uncertain conditions give us

$$(\hat{\tilde{r}}_{klp})_\alpha = [\frac{\Gamma(n+n_0)}{\Gamma(n-m+(\tilde{m}_0)_\alpha)\Gamma(n_0+m-(\tilde{m}_0)_\alpha)}.(\Sigma_{j=k}^{l}\Sigma_{i=k}^{l} l_j \; l_i$$

$$.\frac{\Gamma(n-m+(\tilde{m}_0)_\alpha+j+i+2l)\Gamma(n_0+2l+m-(\tilde{m}_0)_\alpha-j-i)}{\Gamma(2l+n+n_0)})]^{\frac{1}{2}}. \quad \textbf{(21)}$$

4. Numerical Example

Consider a series system that contains four independent components. We'll presume the following information was gathered during a test. There are $n_1 = 13$ tested items in the first component, of which $m_1 = 2$ are failures. There are $n_2 = 11$ and $m_2 = 2$, $n_3 = 9$ and $m_3 = 1$, and $n_4 = 6$ and $m_4 = 1$ in the second, third, and fourth components, respectively.

The percentage of system failures is "about 20%" for the first component, "about 20%" for the second component, "about 15%"for the third component, "about 20%" for the fourth component based on previous information and experiences. As a result, we can infer that $\tilde{m}_{10} = \tilde{m}_{20} = \tilde{3}$, $\tilde{m}_{30} = \tilde{m}_{40} = \tilde{2}$, $n_{10} = n_{20} = 12$, $n_{30} = 11$, and $n_{40} = 9$, where $\tilde{3} = (3,4,5)$, and $\tilde{2} = (1,2,3)$ are triangular fuzzy real numbers. The α-pesemesics of $\tilde{2}$, and $\tilde{3}$, are, respectively, $\tilde{2}_\alpha = 1 + 2\alpha$, and $\tilde{3}_\alpha = 2 + 2\alpha$, $0 \leq \alpha \leq 1$.

Now, using ***Eq. (10)***, the α-level of uncertain BPE $(\tilde{r}_s)_\alpha$ under SEL function is obtained as follows:

$$(\hat{\tilde{r}}_s)_\alpha = \Pi_{i=1}^{k}(\frac{n_i-m_i+(\tilde{m}_{i0})_\alpha}{n_{i0}+n_i}) = \frac{(13+2\alpha)(11+2\alpha)(9+2\alpha)(6+2\alpha)}{172500}, \quad 0 \leq \alpha \leq 1.$$

In a PL function, for $(\tilde{r}_p)_\alpha$ the uncertain BPE using ***Eq. (11)*** is:

$$(\hat{\tilde{r}}_p)_\alpha = \pi_{i=1}^{k}[\frac{(n_i-m_i+(\tilde{m}_{i0})_\alpha)}{(n_i+n_{i0})}.\frac{(n_i-m_i+(\tilde{m}_{i0})_\alpha+1)}{(n_i+n_{i0}+1)}]^{\frac{1}{2}}$$

$$= [\frac{(13+2\alpha)(14+2\alpha)}{650}.\frac{(11+2\alpha)(12+2\alpha)}{552}.\frac{(9+2\alpha)(10+2\alpha)}{420} \times \frac{(6+2\alpha)(7+2\alpha)}{240}]^{\frac{1}{2}}.$$

Finally, one can obtain the degree of membership of system reliability. The resulting system reliability functions, expressed in terms of α-levels, provide a clear view of how uncertainty affects reliability estimates. As shown in ***Figure 1***, both SEL- and PL-based reliability estimates exhibit a slight decline as α increases, reflecting the growing pessimism in prior knowledge. Notably, the PL-based reliability is consistently lower, highlighting its conservative nature and suitability for safety-critical applications. The graphical comparison confirms that the α-pessimistic technique not only captures uncertainty effectively but also eliminates the need for complex non-linear programming. This significantly simplifies computation while offering interpretable, robust, and flexible reliability assessments across varying degrees of imprecision in prior information.

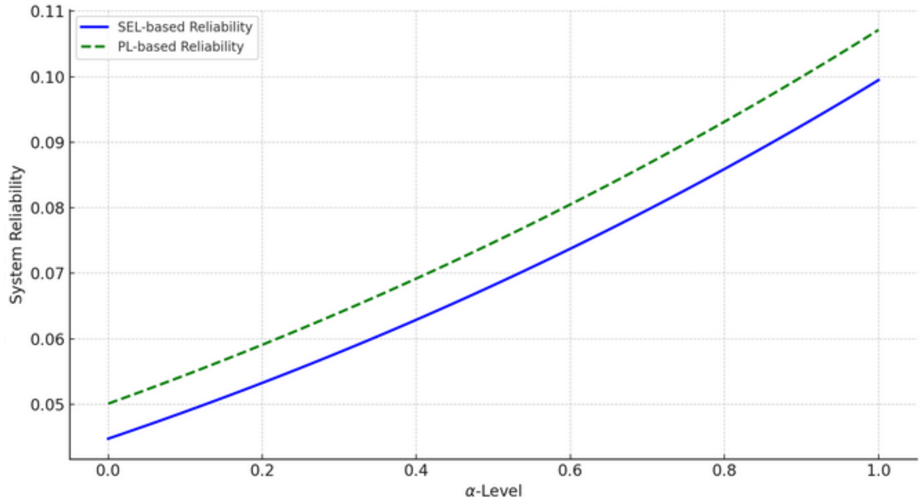

Figure 1: Fuzzy Bayes estimation of series system reliability (SEL vs PL)

To further demonstrate the flexibility of the proposed α-pessimistic fuzzy Bayesian approach, a parallel system composed of four independent components was analyzed using the same prior fuzzy information. In this case, the uncertain BPE using ***Eqs. (14)*** and ***(15)*** for $(\tilde{r}_p)_\alpha$ under the SEL function and PL function as follows

$$(\hat{\tilde{r}}_s)_\alpha = 1 - \Pi_{i=1}^{k}[1 - \frac{n_i - m_i + (\tilde{m}_{i0})_\alpha}{n_i + n_{i0}}].$$

$$= 1 - \frac{(12-2\alpha)(12-2\alpha)(11-2\alpha)(9-2\alpha)}{172500}, \quad 0 \le \alpha \le 1.$$

$$(\hat{\tilde{r}}_p)_\alpha = [1 - 2\Pi_{i=1}^{k}\frac{n_{i0} - (\tilde{m}_{i0})_\alpha + m_i}{n_i + n_{i0}} + \Pi_{i=1}^{k}\frac{n_{i0} - (\tilde{m}_{i0})_\alpha + m_i}{n_i + n_{i0}} . \frac{n_{i0} - (\tilde{m}_{i0})_\alpha + m_i + 1}{n_i + n_{i0} + 1}]^{\frac{1}{2}}.$$

$$= [1 - \frac{(12-2\alpha)(12-2\alpha)(11-2\alpha)(9-2\alpha)}{86250}$$

$$+ \frac{(12-2\alpha)(13-2\alpha)}{650} . \frac{(12-2\alpha)(13-2\alpha)}{552} . \frac{(11-2\alpha)(12-2\alpha)}{420} \times \frac{(9-2\alpha)(10-2\alpha)}{240}]^{\frac{1}{2}}.$$

In the case of the parallel system, the uncertain Bayes point estimates under both the SEL and PL functions were computed using α-pessimistic fuzzy priors. ***Figure 2*** displays the behaviour of system reliability across varying levels of pessimism ($\alpha \in [0,1]$). As expected, the reliability values are substantially higher than those in the series configuration, reflecting the inherent robustness of parallel systems in tolerating individual component failures. The reliability estimates gradually decrease as α increases, indicating that greater uncertainty in prior information results in more conservative reliability predictions. The PL-based reliability function again yields lower estimates than the SEL-based function, reinforcing its suitability for safety-critical assessments where conservative judgment is preferred. Notably, the smooth and interpretable decline of both curves demonstrates the advantage of the α-pessimistic technique, which allows nuanced control over uncertainty without requiring non-linear optimization [25, 26, 27, 30]. This computational simplicity, combined with the flexibility of fuzzy Bayesian inference, makes the method particularly useful in practical engineering scenarios where prior information is imprecise but essential for decision-making.

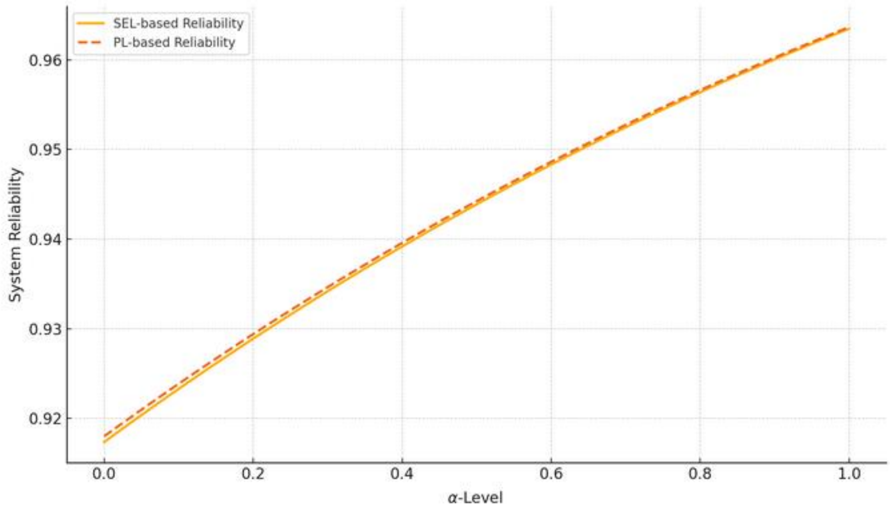

Figure 2: Fuzzy Bayes estimation of parallel system reliability (SEL vs PL)

In the following, we consider a 2-out-of-4 system as having four independent and identical components. When ten items are included in a test, let's assume that there are three failures, or $n = 12$ and $m = 3$. The failure rate, according to previous testing and experiences, is "around 20%". Consequently, the fuzzy prior Beta distribution $Beta(\tilde{2}, 10)$ is employed and $\tilde{2} = (1,2,3)$ is a triangular fuzzy real number.

Eqs. (19) and ***(21)*** can be used to obtain the uncertain BPE for $(\tilde{r}_p)_\alpha$ under the SEL function and PL function as follows

$$(\hat{\tilde{r}}_{kls})_\alpha = \frac{\Gamma(n+n_0)}{\Gamma(n-m+(\tilde{m}_0)_\alpha)\Gamma(n_0+m-(\tilde{m}_0)_\alpha)}$$

$$\times \left[\Sigma_{j=k}^{l} \binom{l}{j} \frac{\Gamma(n-m+(\tilde{m}_0)_\alpha+j)\Gamma(n_0+l+m-(\tilde{m}_0)_\alpha-j)}{\Gamma(l+n+n_0)}\right]$$

$$= \frac{\Gamma(22)}{\Gamma(10+2\alpha)\Gamma(12-2\alpha)} \times \left[\Sigma_{j=2}^{4} \binom{4}{j} \frac{\Gamma(10+2\alpha+j)\Gamma(16-2\alpha-j)}{\Gamma(26)}\right]$$

$$(\hat{\tilde{r}}_{klp})_\alpha = \left[\frac{\Gamma(n+n_0)}{\Gamma(n-m+(\tilde{m}_0)_\alpha)\Gamma(n_0+m-(\tilde{m}_0)_\alpha)} . \left(\Sigma_{j=k}^{l}\Sigma_{i=k}^{l} \binom{l}{j}\binom{l}{i}\right.\right.$$

$$\left.\left. . \frac{\Gamma(n-m+(m_0)_\alpha+j+i+2l)\Gamma(n_0+2l+m-(m_0)_\alpha-j-i)}{\Gamma(2l+n+n_0)}\right)\right]^{\frac{1}{2}}$$

$$= \left[\frac{\Gamma(22)}{\Gamma(10+2\alpha)\Gamma(12-2\alpha)} . \left(\Sigma_{j=2}^{4}\Sigma_{i=2}^{4} \binom{4}{j}\binom{4}{j} . \frac{\Gamma(18+2\alpha+j+i)\Gamma(20-2\alpha-j-i)}{\Gamma(30)}\right)\right]^{\frac{1}{2}}$$

The reliability estimates were computed using both the SEL and PL loss functions. As depicted in ***Figure 3***, the estimated system reliability under both loss functions consistently decreases with increasing α, reflecting higher levels of pessimism in prior belief. Notably, the PL-based estimator provides slightly more conservative (lower) estimates compared to the SEL-based estimator, particularly as approaches 1. This behaviour underscores the effect of precautionary modelling in uncertainty propagation. The figure also visually confirms the robustness and interpretability of the proposed fuzzy Bayes approach using the α-pessimistic method for k-out-of-l systems.

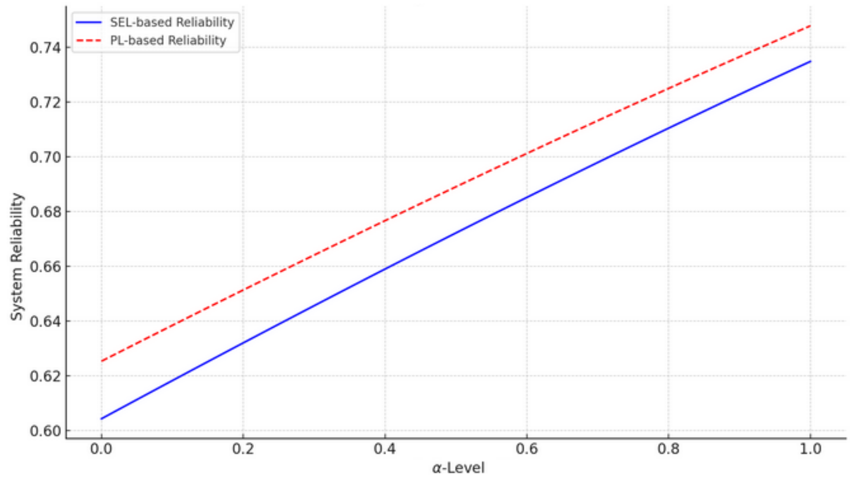

Figure 3: Fuzzy Bayes estimation of 2-out-of-4 system reliability (SEL vs PL)

5. Summary and Conclusion

This paper introduced a novel fuzzy Bayesian framework for system reliability analysis that effectively handles imprecise prior information by modelling parameters as fuzzy random variables with prior fuzzy distributions. This represents a significant advancement over traditional methods, which often lack a theoretical foundation in fuzzy random variables and fail to treat uncertainty at the parameter level naturally and realistically. We constructed fuzzy Bayes point estimators of system reliability based on the Pascal distribution for coherent systems, including parallel, series, and k-out-of-m configurations. These estimators were derived using the concept of uncertain Bayes estimation under squared error and precautionary loss functions, facilitated by the Mellin transform.

A key innovation of our approach lies in the use of the α-pessimistic technique, which plays a central role in circumventing the complex non-linear programming procedures commonly required in related works. This technique significantly simplifies the computational process while enhancing the interpretability and applicability of the results. The proposed method offers three significant advantages: It leverages prior system knowledge through fuzzy distributions, provides a deeper and more precise analysis of system behaviour via uncertainty theory and eliminates dependency on non-linear programming through the α-pessimistic approach, enabling efficient and tractable reliability analysis.

In conclusion, our methodology presents a flexible, theoretically grounded, and computationally efficient framework for reliability analysis in uncertain environments. Future research directions include extending the model to dynamic or multi-state systems and applying the framework to real-world case studies in industrial reliability engineering.

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