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Modeling and optimization of the number of graduates in a multi-specialization study program

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Abstract. Consider a study program which offers a number of specializations, and requires all students to be enrolled in exactly one specialization at any given time. We construct a continuous mathematical model governing the time evolution of the number of students enrolled in each of the program's specializations. Using the model, we further construct an optimization problem describing the search of an intervention strategy which maximizes the program's total number of graduates, along with a framework for sensitivity analysis. We discretize the constructs accordingly, and employ a coordinate-descent method to solve the optimization problem numerically in two simulated scenarios involving two and four specializations, respectively, describing the results' practical implications.

Keywords: Specialization, optimization, sensitivity analysis, discretization, coordinate descent.
AMS Subject Classification 2010: 34C60, 92D25.

1 Introduction

Studies have acknowledged the benefits of offering specialization choices in higher education [13, 21, 22, 32]. On the students' side, the decision of choosing a specialization is known to be influenced by a range of factors, including expected future earnings [4–6, 24], gender and family background [7, 15, 25, 41, 42], and most notably, peer influence [2, 14, 29, 30, 34]. The unfortunate prevalence of the latter could be associated with a broad academic concern: low graduation rates [10, 12, 26, 36–38]. Accordingly, on the institutions' side, a problem of interest is to determine what forms of intervention could be implemented in multi-specialization study programs in order to secure the largest possible number of graduates. In this paper, we aim to address this problem from a mathematical perspective. While many existing studies implemented linear programming approaches for such an intention [3, 11, 35], we implement a novel

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approach. Specifically, we draw insight from mathematical epidemiology and construct a model governing the time evolution of the number of students enrolled in each of the specializations offered in a study program, resembling those that describe disease spread. Using the model, we formulate and solve a suitable optimization problem, and analyze the sensitivity of the optimal solution with respect to the model's parameters, with the aim of identifying the appropriate forms of intervention.

We organize our work as follows. In Section 2, we begin by assuming that a study program offers a number of specializations, and requires all students to be enrolled in exactly one specialization at any given time during their studies. Assuming that each specialization has a constant registration rate, a linear drop-out rate, and a linear graduation rate, and that transitions from one specialization to another occur at bilinear rates (i.e., rates proportional to the number of students registered in both the source and target specializations), we construct a system of differential equations governing the time evolution of the number of students enrolled in each specialization, and establish the non-negativity and boundedness of the model's solutions (Theorem 1). Subsequently, specializing to the case of zero registration rates and finitely many equilibria, we show that the origin is the model's unique equilibrium (Lemmas 1–2 and Theorem 2) and establish its local asymptotic stability (Theorem 3). At the end of the section, we accordingly describe our desired optimization problem (Problem 1) and framework for sensitivity analysis.

In Section 3, to establish a basis for numerical analysis, we formulate a discretized version of the constructed model, using Euler's method [16, sec. 22.3] with an arbitrary step size. We show that, in the case of zero registration rates and finitely many equilibria, the origin is also the unique equilibrium of the discretized model (Theorem 4), which is also locally asymptotically stable provided a sufficiently small step size (Theorem 5). Using numerical methods for differentiation and integration [16, Chap. 20–21], we state a discrete analogue of our optimization problem (Problem 2) and explain how our sensitivity analysis will be carried out in a discrete setting in the subsequent section.

In Section 4, we use our discretized model to study numerically our main problem —determining the appropriate forms of intervention— with simulated values of parameters, in the cases of two and four specializations. In each case, fixing the values of all other parameters, we use a coordinate-descent method [20, sec. 8.9] to determine the values of the coefficients of the graduation rates of the existing specializations for which the total number of graduates achieves a maximum. At the maximum state, we visualize the time evolution of the number of students in each specialization, assess the sensitivity of the total number of graduates with respect to small value changes of each of the specializations' graduation rate coefficients, and discuss the practical implications accordingly. In the final Section 5, we present our conclusions and outline avenues for future research.

2 Model construction and analysis

In this section, we first construct a system of differential equations which model the time evolution of the number of students enrolled in each of the specializations offered in a study program. We then analyze the model from the viewpoint of dynamical systems theory (see [33, 39] for background), before describing our optimization problem and framework for sensitivity analysis.

2.1 Model construction

To begin our model's construction, let n be a natural number. Consider a study program which offers n different specializations, say 1, ..., n, and requires all students to be registered in exactly one specialization at any given time during their studies. For every $i \in \{1,\dots,n\}$, let $S_i = S_i(t) \geqslant 0$ be the number of students enrolled in specialization i at time $t \geqslant 0$. For every $i \in \{1,\dots,n\}$, let $\overline{S}_i = \overline{S}_i(t) \geqslant 0$ be the number of students already graduated with specialization i at time $t \geqslant 0$. We assume that the following hold at any given time:

(i) each specialization i has a constant registration rate of $\lambda_i \geqslant 0$ students per time unit;

(ii) each specialization i has a linear drop-out rate of $\mu_i S_i$ students per time unit, where $\mu_i > 0$;

(iii) each specialization i has a linear graduation rate of $\gamma_i S_i$ students per time unit, where $\gamma_i > 0$;

(iv) students change specialization, say from specialization i to specialization j, at the bilinear rate of $\beta_j S_i S_j$, where $\beta_j \geqslant 0$.

The symbols λ_i and μ_i are borrowed from standard symbols in mathematical epidemiology denoting, respectively, the birth and death rates of an indexed subpopulation i. The final assumption models the idealized situation where the students' changes of specialization are driven only by peer influence.

Accordingly, we construct as our model a system of $2n$ differential equations, consisting of the n equations

$$\frac{\mathrm{d}S_i}{\mathrm{d}t} = \lambda_i - (\mu_i + \gamma_i)S_i + S_i \sum_{j=1}^{n} (\beta_i - \beta_j) S_j, \qquad i \in \{1,\dots,n\}, \tag{1}$$

accompanied by the n equations

$$\frac{\mathrm{d}\overline{S}_i}{\mathrm{d}t} = \gamma_i S_i, \qquad i \in \{1,\dots,n\}. \tag{2}$$

2.2 Non-negativity and boundedness of solutions

Let us now establish the non-negativity and boundedness of the solutions of model (1). Consider a solution $(S_1,\dots,S_n) = (S_1(t),\dots,S_n(t))$ associated to an initial condition $(S_1(0),\dots,S_n(0)) \in \mathbb{R}_+^n$, where $\mathbb{R}_+ := [0,\infty)$. For every $i \in \{1,\dots,n\}$, we have that at every time $t \geqslant 0$ satisfying $S_i(t) = 0$,

$$\frac{\mathrm{d}S_i}{\mathrm{d}t} = \lambda_i - (\mu_i + \gamma_i)0 + 0 \sum_{j=1}^{n} (\beta_i - \beta_j) S_j = \lambda_i \geqslant 0,$$

which means that the function S_i is non-decreasing at t. Since $S_i(0) \geqslant 0$, this implies that $S_i(t) \geqslant 0$ at every time $t \geqslant 0$.

Next, letting

$$N = N(t) = \sum_{i=1}^{n} S_i(t)$$

and adding (1) for $i \in \{1,\dots,n\}$, one obtains that

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n} (\mu_i + \gamma_i) S_i \leqslant \lambda - \sum_{i=1}^{n} \kappa S_i = \lambda - \kappa N,$$

where

$$\lambda := \lambda_1 + \cdots + \lambda_n \geqslant 0 \qquad \text{and} \qquad \kappa := \min\{\mu_1+\gamma_1,\dots,\mu_n+\gamma_n\} > 0.$$

Therefore, we have

$$\frac{\mathrm{d}N}{\mathrm{d}t} + \kappa N \leqslant \lambda,$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{\kappa t} N(t)\right] \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\lambda}{\kappa}\mathrm{e}^{\kappa t} - \frac{\lambda}{\kappa} + N(0)\right].$$

This means that, at any given time $t \geqslant 0$, the slope of the function $\mathrm{e}^{\kappa t}N(t)$ is bounded above by that of the function $(\lambda/\kappa)\mathrm{e}^{\kappa t} - \lambda/\kappa + N(0)$. Since both functions evaluate to $N(0)$ at $t = 0$, it follows that at any given time $t \geqslant 0$ we have

$$\mathrm{e}^{\kappa t} N(t) \leqslant \frac{\lambda}{\kappa}\mathrm{e}^{\kappa t} - \frac{\lambda}{\kappa} + N(0),$$

i.e.,

$$N(t) \leqslant \frac{\lambda}{\kappa} + \left[N(0) - \frac{\lambda}{\kappa}\right]\mathrm{e}^{-\kappa t} \xrightarrow{t\to\infty} \frac{\lambda}{\kappa}.$$

Consequently, if $N(0) \leqslant \lambda/\kappa$, then at any given time $t \geqslant 0$ we have that $N(t) \leqslant \lambda/\kappa$. Therefore, we have proved the following theorem.

Theorem 1. *The sets* $\mathbb{R}_+^n$ *and*

$$\left\{(S_1,\dots,S_n) \in \mathbb{R}_+^n : S_1 + \cdots + S_n \leqslant \frac{\lambda_1 + \cdots + \lambda_n}{\min\{\mu_1+\gamma_1,\dots,\mu_n+\gamma_n\}}\right\}$$

are both positively invariant under model (1)*. Moreover, every solution of model* (1) *associated to a non-negative initial condition is bounded.*

2.3 Dynamical analysis

Let us now analyze model (1) dynamically. For simplicity, we shall only work on the case where every specialization has a zero registration rate, i.e., $\lambda_i = 0$ for every $i \in \{1,\dots,n\}$. In this case, we can rewrite (1) as

$$\frac{\mathrm{d}S_i}{\mathrm{d}t} = S_i\left[-\mu_i - \gamma_i + \sum_{j=1}^{n} (\beta_i - \beta_j) S_j\right], \qquad i \in \{1,\dots,n\}. \tag{3}$$

We shall study this zero-registration model in the positively invariant domain $\mathbb{R}_+^n$.

First, we shall determine the equilibria of model (3). For this purpose, we use the following two lemmas.

Lemma 1. *Consider the system of n equations*

$$-\mu_i-\gamma_i+\sum_{j=1}^{n}(\beta_i-\beta_j)\,S_j=0,\qquad i\in\{1,\dots,n\} \tag{4}$$

in $S_1,\dots,S_n\in\mathbb{R}_+$. *For* $n=2$, *the system has a unique solution if and only if* $\beta_1\neq\beta_2$. *For* $n\geqslant 3$, *the system has either no solution or infinitely many solutions.*

Proof. The system's $n\times n$ coefficient matrix

$$\begin{pmatrix} 0 & \beta_1-\beta_2 & \beta_1-\beta_3 & \cdots & \beta_1-\beta_n \\ \beta_2-\beta_1 & 0 & \beta_2-\beta_3 & \cdots & \beta_2-\beta_n \\ \beta_3-\beta_1 & \beta_3-\beta_2 & 0 & \cdots & \beta_3-\beta_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_n-\beta_1 & \beta_n-\beta_2 & \beta_n-\beta_3 & \cdots & 0 \end{pmatrix}$$

has determinant $(\beta_1-\beta_2)^2$ if $n=2$, and 0 if $n\geqslant 3$. The lemma follows. □

Lemma 2. *Let* $i,j\in\{1,\dots,n\}$, *where* $i\neq j$. *The system of two equations*

$$\begin{cases} 0 = S_i\big[-\mu_i-\gamma_i+(\beta_i-\beta_j)\,S_j\big], \\ 0 = S_j\big[-\mu_j-\gamma_j+(\beta_j-\beta_i)\,S_i\big] \end{cases} \tag{5}$$

in $S_i,S_j\in\mathbb{R}_+$ *has only the trivial solution* $(0,0)$.

Proof. Clearly, $(0,0)$ is a solution of the system. Suppose by contradiction that $(S_i,S_j)\neq(0,0)$ is a solution. If $S_i=0$ and $S_j>0$, then the system's second equation implies

$$0=(\beta_j-\beta_i)\,S_i=\mu_j+\gamma_j>0,$$

a contradiction. Similarly, if $S_i>0$ and $S_j=0$, then the system's second equation implies

$$0=(\beta_i-\beta_j)\,S_j=\mu_i+\gamma_i>0,$$

another contradiction. Finally, if $S_i>0$ and $S_j>0$, then the system's first equation implies $(\beta_i-\beta_j)\,S_j=\mu_i+\gamma_i>0$, which forces $\beta_i>\beta_j$, while the system's second equation implies $(\beta_j-\beta_i)\,S_i=\mu_j+\gamma_j>0$, which forces $\beta_j>\beta_i$, contradicting each other. The lemma is proved. □

Applying the above lemmas, let us now derive a result on the equilibria of model (3).

Theorem 2. *Suppose that model* (3) *in* $\mathbb{R}_+^n$ *has only finitely many equilibria. Then the only equilibrium of the model is* $(\underbrace{0,\dots,0}_{n})$.

Proof. If the case of $n=2$, the model's equilibria $(S_1,S_2)\in\mathbb{R}_+^2$ are the solutions of a system of the form (5). By Lemma 2, $(0,0)$ is the model's only equilibrium.

Now consider the case $n=3$. The model's equilibria $(S_1,S_2,S_3)\in\mathbb{R}_+^3$ are the solutions of the system

$$\begin{cases} 0 &= S_1\left[-\mu_1-\gamma_1+(\beta_1-\beta_2)S_2+(\beta_1-\beta_3)S_3\right],\\ 0 &= S_2\left[-\mu_2-\gamma_2+(\beta_2-\beta_1)S_1+(\beta_2-\beta_3)S_3\right],\\ 0 &= S_3\left[-\mu_3-\gamma_3+(\beta_3-\beta_1)S_1+(\beta_3-\beta_2)S_2\right].\end{cases}$$

If none of S_1, S_2, and S_3 are zero, then (S_1,S_2,S_3) satisfies a system of the form (4) which, by Lemma 1, has either no solution or infinitely many solutions, a contradiction. If exactly one of S_1, S_2, and S_3 is zero, then the non-zero variables satisfy a system of the form (5), contradicting Lemma 2. If exactly two of S_1, S_2, and S_3 are zero, then one of the equations in the above system is not satisfied, a contradiction. If S_1, S_2, and S_3 are all zero, we obtain the equilibrium $(0,0,0)$.

Finally, consider the case $n\geqslant 4$. Assume that $(S_1,\ldots,S_n)\in\mathbb{R}_+^n$ is an equilibrium. Suppose that exactly k of S_1, …, S_n are zero. If $k\leqslant n-3$, then the non-zero S_is satisfy an $(n-k)\times(n-k)$ system of the form (4), where $n-k\geqslant 3$, which has either no solution or infinitely many solutions by Lemma 1, a contradiction. If $k=n-2$, then the two non-zero S_is satisfy a system of the form (5), contradicting Lemma 2. If $k=n-1$, then one of the equilibrium equations is not satisfied, a contradiction. Finally, if $k=n$, we obtain the equilibrium $(\underbrace{0,\ldots,0}_{n})$. □

Let us next show that the zero equilibrium of model (3) stated in Theorem 2 is locally asymptotically stable, using [33, Thm. 4.6(a)].

Theorem 3. *The equilibrium* $(\underbrace{0,\ldots,0}_{n})$ *of model* (3) *is locally asymptotically stable.*

Proof. The (i,j)-entry of the Jacobian of model (3) is given by

$$\frac{\partial}{\partial S_j}\left[S_i\left[-\mu_i-\gamma_i+\sum_{j=1}^{n}(\beta_i-\beta_j)S_j\right]\right]=\begin{cases}-\mu_i-\gamma_i+\sum\limits_{j=1}^{n}(\beta_i-\beta_j)S_j, & \text{if } i=j;\\ (\beta_i-\beta_j)S_i, & \text{if } i\neq j,\end{cases}$$

which, at the equilibrium $(\underbrace{0,\ldots,0}_{n})$, evaluates to $-\mu_i-\gamma_i$ if $i=j$, and to 0 if $i\neq j$. Therefore, the eigenvalues of the Jacobian of model (3) at the equilibrium $(\underbrace{0,\ldots,0}_{n})$ are $-\mu_1-\gamma_1$, …, $-\mu_n-\gamma_n$, which are all negative since $\mu_1,\ldots,\mu_n,\gamma_1,\ldots,\gamma_n>0$. It follows that the equilibrium $(\underbrace{0,\ldots,0}_{n})$ is locally asymptotically stable. □

The main problem of this paper concerns the number of students graduated with each specialization. In the notation of our model, we are interested in the limit

$$\lim_{t\to\infty}\overline{S}_i(t)$$

for every $i \in \{1,\ldots,n\}$. For every $i \in \{1,\ldots,n\}$, equation (2) gives

$$\mathrm{d}\overline{S} = \gamma_i S_i \,\mathrm{d}t, \qquad \text{and so} \qquad \overline{S}_i(t) = \gamma_i \int_0^t S_i(\tau)\,\mathrm{d}\tau.$$

Therefore, assuming that $\gamma_1, \ldots, \gamma_n$ are the only parameters amenable to intervention, we are interested in the limit

$$\mathscr{G}_i(\gamma_1,\ldots,\gamma_n) := \lim_{t\to\infty} \overline{S}_i(t) = \gamma_i \int_0^\infty S_i(\tau)\,\mathrm{d}\tau \tag{6}$$

for every $i \in \{1,\ldots,n\}$. Our problem of maximizing the total number of graduates can thus be formulated as the following optimization problem.

Problem 1. *Fix a set of values for the parameters $\lambda_1,\ldots,\lambda_n \in \mathbb{R}_+$ and $\mu_1,\ldots,\mu_n,\beta_1,\ldots,\beta_n \in (0,\infty)$, and fix an initial condition $(S_1(0),\ldots,S_n(0)) \in \mathbb{R}^n_+$. Determine $(\gamma_1,\ldots,\gamma_n) \in \mathbb{R}^n_+$ such that the solution of model (1), i.e.,*

$$\frac{dS_i}{dt} = \lambda_i - (\mu_i+\gamma_i)S_i + S_i \sum_{j=1}^n (\beta_i - \beta_j)S_j, \qquad i \in \{1,\ldots,n\},$$

associated to the fixed initial condition $(S_1(0),\ldots,S_n(0))$, maximizes the value of

$$\mathscr{G}(\gamma_1,\ldots,\gamma_n) := \sum_{i=1}^n \mathscr{G}_i(\gamma_1,\ldots,\gamma_n) = \sum_{i=1}^n \left[\gamma_i \int_0^\infty S_i(\tau)\,d\tau\right].$$

Suppose that the function $\mathscr{G}$ is found to achieve a maximum at $(\gamma_1^*,\ldots,\gamma_n^*) \in \mathbb{R}^n_+$. At this maximum state, the sensitivity index of $\mathscr{G}$ with respect to each γ_i is given by

$$\Upsilon^{\mathscr{G}}_{\gamma_i} := \frac{\partial \mathscr{G}}{\partial \gamma_i} \cdot \frac{\gamma_i}{\mathscr{G}} \bigg|_{(\gamma_1,\ldots,\gamma_n)=(\gamma_1^*,\ldots,\gamma_n^*)}, \tag{7}$$

i.e., the ratio of the relative change in $\mathscr{G}$ to the relative change in γ_i [8]. In the upcoming section we shall introduce a discretized version of this index, which will be used in our numerical analysis in Section 4.

3 Discretization

To prepare a setting for our numerical study in Section 4, let us now discretize our model given by (1) and (2) using Euler's method [16, sec. 22.3] with an arbitrary step size. We shall show that, in the case of zero registration rate and finitely many equilibria, the discretized model also has a unique equilibrium at the origin, which is also locally asymptotically stable provided a sufficiently small time step. Subsequently, we describe how our optimization problem will be solved numerically, and how our sensitivity analysis will be conducted numerically in the upcoming section.

Fix an arbitrary step size $\Delta t > 0$. Euler's method [16, sec. 22.3] discretizes our model (1) into the system of difference equations

$$S_{i,k+1} = S_{i,k} + \left[\lambda_i - (\mu_i+\gamma_i)S_{i,k} + S_{i,k}\sum_{j=1}^n (\beta_i-\beta_j)S_{j,k}\right]\Delta t, \qquad i \in \{1,\ldots,n\}, \tag{8}$$

where $S_{i,0} = S_i(0)$ for every $i \in \{1,\dots,n\}$. In the zero-registration case $\lambda_i = 0$ for every $i \in \{1,\dots,n\}$, model (8) reduces to

$$S_{i,k+1} = S_{i,k} + S_{i,k}\left[-\mu_i - \gamma_i + \sum_{j=1}^{n}(\beta_i - \beta_j)\,S_{j,k}\right]\Delta t, \qquad i \in \{1,\dots,n\}. \tag{9}$$

Any $(S_1,\dots,S_n) \in \mathbb{R}_+^n$ is an equilibrium of model (9) if and only if for every $i \in \{1,\dots,n\}$ we have

$$S_i\left[-\mu_i - \gamma_i + \sum_{j=1}^{n}(\beta_i - \beta_j)\,S_j\right] = 0,$$

i.e., if and only if it is an equilibrium of model (3). This implies the following theorem.

Theorem 4. *Suppose that model (9) in $\mathbb{R}_+^n$ has only finitely many equilibria. Then the only equilibrium of the model is $(\underbrace{0,\dots,0}_{n})$.*

Furthermore, the (i,j)-entry of the Jacobian of model (9),

$$\frac{\partial}{\partial S_j}\left[S_i + S_i\left[-\mu_i - \gamma_i + \sum_{j=1}^{n}(\beta_i - \beta_j)\,S_j\right]\Delta t\right],$$

in the case of $i = j$ and $i \neq j$ reads, respectively,

$$1 + \left[-\mu_i - \gamma_i + \sum_{j=1}^{n}(\beta_i - \beta_j)\,S_j\right]\Delta t \qquad \text{and} \qquad (\beta_i - \beta_j)\,S_i\,\Delta t,$$

which, at the equilibrium $(\underbrace{0,\dots,0}_{n})$, evaluate to, respectively, $1 + (-\mu_i - \gamma_i)\,\Delta t$ and 0. The Jacobian's eigenvalues are thus $1 + (-\mu_1 - \gamma_1)\,\Delta t,\ \dots,\ 1 + (-\mu_n - \gamma_n)\,\Delta t$. By [33, Thm. 12.3(a)], we have the following theorem.

Theorem 5. *The equilibrium $(\underbrace{0,\dots,0}_{n})$ of model (9) is locally asymptotically stable if*

$$\Delta t < \min\left\{\frac{2}{\mu_1 + \gamma_1},\dots,\frac{2}{\mu_n + \gamma_n}\right\}.$$

Next, using the trapezoidal rule [16, Sec. 21.3] to discretize the integral appearing in (6), we define the following discrete analogue for the estimated number of students graduated with specialization i:

$$\tilde{\mathscr{G}}_i := \frac{1}{2}\gamma_i\,\Delta t \sum_{k=0}^{\infty}(S_{i,k} + S_{i,k+1})$$

for every $i \in \{1,\dots,n\}$, and thus reformulate our optimization problem in a discrete setting as follows.

Problem 2. *Fix a set of values for the parameters $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ and $\mu_1, \dots, \mu_n, \beta_1, \dots, \beta_n \in (0, \infty)$, and fix an initial condition $(S_{1,0}, \dots, S_{n,0}) \in \mathbb{R}_+^n$. Determine $(\gamma_1, \dots, \gamma_n) \in \mathbb{R}_+^n$ such that the solution of model (8), i.e.,*

$$S_{i,k+1} = S_{i,k} + \left[\lambda_i - (\mu_i + \gamma_i) S_{i,k} + S_{i,k} \sum_{j=1}^{n} (\beta_i - \beta_j) S_{j,k} \right] \Delta t, \qquad i \in \{1, \dots, n\},$$

associated to the fixed initial condition $(S_{1,0}, \dots, S_{n,0})$, maximizes the value of

$$\tilde{\mathscr{G}}(\gamma_1, \dots, \gamma_n) := \sum_{i=1}^{n} \tilde{\mathscr{G}}_i(\gamma_1, \dots, \gamma_n) = \frac{\Delta t}{2} \sum_{i=1}^{n} \left[\gamma_i \sum_{k=0}^{\infty} (S_{i,k} + S_{i,k+1}) \right].$$

This optimization problem can be solved numerically —as we shall see in the upcoming section— using a coordinate-descent method [20, Sec. 8.9] which generates recursively a sequence $\left(\gamma^{(0)}, \gamma^{(1)}, \dots\right)$ of vectors in $\mathbb{R}_+^n$ as follows. First, choose an initial guess $\gamma^{(0)} \in \mathbb{R}_+^n$, and a step size $\Delta\gamma > 0$. We employ the following recursion for every $\ell \in \mathbb{N}$:

$$\gamma^{(\ell)} = \begin{cases} \gamma^{(\ell-1)}, & \text{if } \tilde{\mathscr{G}}\left(\gamma_{\max}^{(\ell-1)}\right) \leqslant \tilde{\mathscr{G}}\left(\gamma^{(\ell-1)}\right); \\ \gamma_{\max}^{(\ell-1)}, & \text{otherwise,} \end{cases}$$

where

$$\gamma_{\max}^{(\ell-1)} \in \left\{ \check{\gamma}_1^{(\ell-1)}, \dots, \check{\gamma}_n^{(\ell-1)}, \hat{\gamma}_1^{(\ell-1)}, \dots, \hat{\gamma}_n^{(\ell-1)} \right\}$$

is chosen such that $\tilde{\mathscr{G}}\left(\gamma_{\max}^{(\ell-1)}\right)$ is maximum among $\tilde{\mathscr{G}}\left(\check{\gamma}_1^{(\ell-1)}\right)$, $\dots$, $\tilde{\mathscr{G}}\left(\check{\gamma}_n^{(\ell-1)}\right)$, $\tilde{\mathscr{G}}\left(\hat{\gamma}_1^{(\ell-1)}\right)$, $\dots$, $\tilde{\mathscr{G}}\left(\hat{\gamma}_n^{(\ell-1)}\right)$, the notation $\hat{\gamma}_i^{(\ell-1)}$ denoting the vector obtained from $\gamma^{(\ell-1)}$ by replacing its i-th entry $\gamma_i^{(\ell-1)}$ with $\gamma_i^{(\ell-1)} + \Delta\gamma$, while $\check{\gamma}_i^{(\ell-1)}$ denoting the vector obtained from $\gamma^{(\ell-1)}$ by replacing its i-th entry $\gamma_i^{(\ell-1)}$ with $\max\left\{\gamma_i^{(\ell-1)} - \Delta\gamma, 0\right\}$. The resulting sequence $\left(\gamma^{(0)}, \gamma^{(1)}, \dots\right)$ is thus expected to stabilize at a reasonable estimate for a point $\gamma \in \mathbb{R}_+^n$ at which $\tilde{\mathscr{G}}$ achieves a maximum.

Finally, using the forward difference estimation [20, Sec. 20.2] for the derivative $\partial\mathscr{G}/\partial\gamma_i$ in (7), we define the discretized sensitivity index of $\tilde{\mathscr{G}}$ with respect to each γ_i at the maximum state $\gamma^* = (\gamma_1^*, \dots, \gamma_n^*) \in \mathbb{R}_+^n$ as

$$\hat{\Upsilon}_{\gamma_i}^{\tilde{\mathscr{G}}} := \frac{\tilde{\mathscr{G}}(\hat{\gamma}_i) - \tilde{\mathscr{G}}(\gamma)}{\Delta\gamma} \cdot \frac{\gamma_i}{\tilde{\mathscr{G}}(\gamma)} \Bigg|_{\gamma = (\gamma_1^*, \dots, \gamma_n^*)},$$

where, as above, $\hat{\gamma}_i$ denotes the vector obtained from γ by replacing its i-th entry γ_i with $\gamma_i + \Delta\gamma$. In our numerical simulations, by computing this index numerically for each γ_i, we shall identify the parameter γ_i upon which $\tilde{\mathscr{G}}$ depends most sensitively, and interpret the results in connection to the optimal intervention strategy.

4 Numerical simulations

In this section, we shall use our zero-registration discrete model (9) to address our main problem —determining an optimal intervention strategy— in two simulated cases involving, respectively, two and four specializations. In each case, using simulated values of drop-out rate coefficients and specialization-change rate coefficients, we determine the graduation rate coefficients γ_i which maximize the total number of graduates, analyze the sensitivity of the total number of graduates at its maximum with respect to each γ_i, and interpret the results in connection to the suggested strategy. All numerical simulations are performed using Maple 2016 on a personal computer equipped with an Intel Core i7 processor.

4.1 A two-specialization case

For our first simulation, consider a study program offering $n = 2$ specializations 1 and 2, with drop-out rate coefficients $\mu_1 = 0.001$ and $\mu_2 = 0.002$, respectively, and specialization-change rate coefficients $\beta_1 = 0.25$ and $\beta_2 = 0.3$, respectively. Suppose that the numbers of students enrolled in these specializations are initially $S_1(0) = 150$ and $S_2(0) = 100$, respectively. Let us set $\Delta t = 0.01$, and —for the estimation of the total number of graduates— carry out $N = 500$ iterations:

$$\tilde{\mathscr{G}}(\gamma_1, \gamma_2) \approx \frac{\Delta t}{2}\left[\gamma_1 \sum_{k=0}^{N}(S_{1,k} + S_{1,k+1}) + \gamma_2 \sum_{k=0}^{N}(S_{2,k} + S_{2,k+1})\right]. \tag{10}$$

One way to obtain numerically a pair (γ_1, γ_2) maximizing the value of $\tilde{\mathscr{G}}$ is to assume that such a pair is to be sought within a specific square, say, $[0,9]^2$, and to discretize the square into a lattice of a specified step size, e.g., $\Delta\gamma = 0.01$:

$$\begin{aligned}\mathscr{L} = \{ &(0.00, 0.00) \quad (0.01, 0.00) \quad \cdots \quad (9.00, 0.00),\\ &(0.00, 0.01) \quad (0.01, 0.01) \quad \cdots \quad (9.00, 0.01),\\ &\quad\vdots \qquad\qquad\quad \vdots \qquad\quad \ddots \qquad\quad \vdots\\ &(0.00, 9.00) \quad (0.01, 9.00) \quad \cdots \quad (9.00, 9.00) \}.\end{aligned}$$

Using (10) to estimate the values of $\tilde{\mathscr{G}}(\gamma_1, \gamma_2)$ for all $(\gamma_1, \gamma_2) \in \mathscr{L}$, one finds that the maximum value of $\tilde{\mathscr{G}}$ over $\mathscr{L}$ is approximately 240.462298, which is achieved at $(\gamma_1, \gamma_2) = (1.04, 7.46)$. For a significant reduction of computational effort, the same result could instead be obtained by using the coordinate-descent method described in the previous section, again with $\Delta\gamma = 0.01$. See Figure 1. Notice that the method necessitates not a domain over which the maximum point is to be sought, but an initial guess of the point, for which one could choose, e.g., $\left(\gamma_1^{(0)}, \gamma_2^{(0)}\right) = (0.1, 0.1) \in \mathbb{R}_+^2$. For $(\gamma_1, \gamma_2) = (1.04, 7.46)$, the time evolution of the number of students enrolled in each specialization is visualized in Figure 2. Notice the expected monotonic convergence towards the zero equilibrium.

The fact that the maximum number of graduates is achieved at the state where $\gamma_1 = 1.04$ and $\gamma_2 = 7.46$ implies that, in order to achieve a maximum total number of graduates from the two specializations, interventions must be directed towards raising the graduation rate coefficient of specialization 2 while keeping the graduation rate coefficient of specialization 1 relatively low. This is not surprising since our parameter values reflect that specialization 2 is more attractive to students while also having a higher drop-out rate coefficient. Furthermore, at the maximum state, the discretized sensitivity indices of $\tilde{\mathscr{G}}$ with respect to γ_1 and γ_2 are given by

$$\hat{\Upsilon}_{\gamma_1}^{\tilde{\mathscr{G}}} = \frac{\tilde{\mathscr{G}}(1.05, 7.46) - \tilde{\mathscr{G}}(1.04, 7.46)}{0.01} \cdot \frac{1.04}{\tilde{\mathscr{G}}(1.04, 7.46)} \approx -0.000085$$

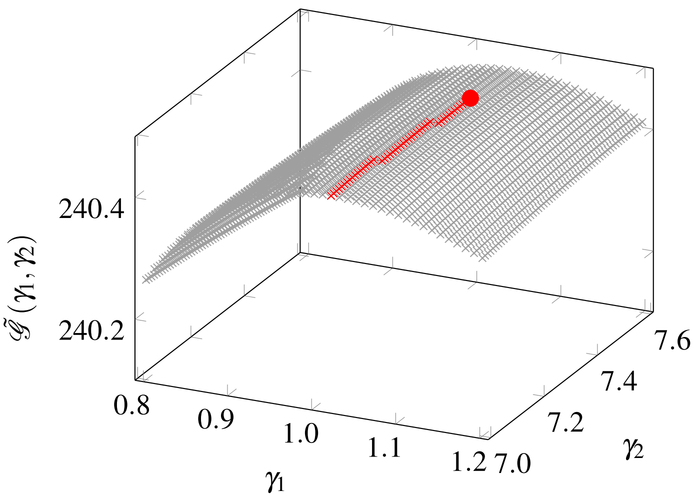

Figure 1: The points $(\gamma_1, \gamma_2, \tilde{\mathscr{G}}(\gamma_1, \gamma_2))$ for all $(\gamma_1, \gamma_2) \in \mathscr{L} \cap [0.8, 1.2] \times [7.0, 7.6]$. The red point $(1.04, 7.46, 240.462298)$ is the maximum point of $\tilde{\mathscr{G}}$ over $\mathscr{L}$. The coordinate-descent method generates the red trajectory, which emanates from the initial point $(0.1, 0.1, 93.196405)$ to the maximum point.

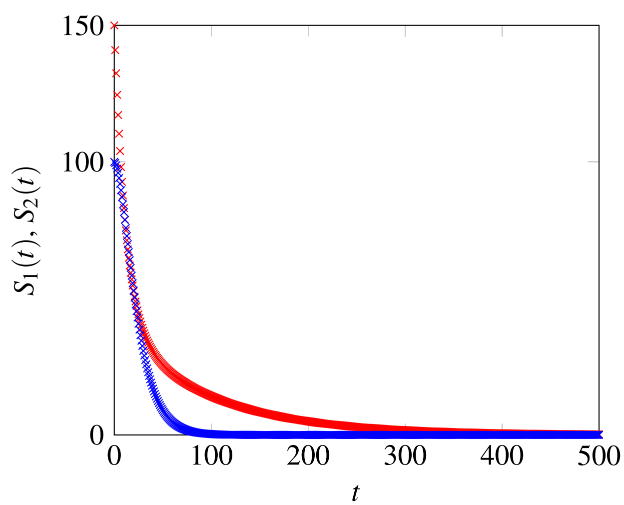

Figure 2: The time evolution of the number of students enrolled in specializations 1 (red) and 2 (blue) in our two-specialization case.

and

$$\widehat{\Upsilon}_{\gamma_2}^{\tilde{\mathscr{G}}} = \frac{\tilde{\mathscr{G}}(1.04, 7.47) - \tilde{\mathscr{G}}(1.04, 7.46)}{0.01} \cdot \frac{7.46}{\tilde{\mathscr{G}}(1.04, 7.46)} \approx -0.000035,$$

respectively, which means that the total number of graduates, at its maximum, depends with negative correlation on both γ_1 and γ_2, and is more sensitive to changes on γ_1 than to those on γ_2: a 10% increase in γ_1 results in a 0.00085% decrease in the total number of graduates, while a 10% increase in γ_2 results in a 0.00035% decrease in the total number of graduates.

4.2 A four-specialization case

Let us now turn our attention to a scenario involving $n = 4$ specializations 1, 2, 3, and 4, with drop-out rate coefficients $\mu_1 = 0.001$, $\mu_2 = 0.002$, $\mu_3 = 0.004$, and $\mu_4 = 0.003$, respectively, and specialization-change rate coefficients $\beta_1 = 0.25$, $\beta_2 = 0.3$, $\beta_3 = 0.1$, and $\beta_4 = 0.2$, respectively. Consider the initial condition given by $S_1(0) = 150$, $S_2(0) = 100$, $S_3(0) = 120$, and $S_4(0) = 140$. Again using $\Delta t = 0.01$ and performing $N = 500$ iterations, one makes the estimation

$$\tilde{\mathscr{G}}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \approx \frac{\Delta t}{4} \left[\sum_{i=1}^{4} \gamma_i \sum_{k=0}^{N} (S_{i,k} + S_{i,k+1}) \right].$$

Using the initial guess $\left(\gamma_1^{(0)}, \gamma_2^{(0)}, \gamma_3^{(0)}, \gamma_4^{(0)}\right) = (0.1, 0.1, 0.1, 0.1) \in \mathbb{R}_+^4$, the coordinate-descent method reveals that $\tilde{\mathscr{G}}$ achieves a maximum of 477.213418 at $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (15.96, 58.93, 0, 1.14)$. Thus, in the case of the specializations possessing such different levels of popularity, interventions must be directed only towards raising the graduation rate coefficients of the popular specializations. At this maximum state, the evolution of the number of students enrolled in each specialization over time is plotted in Figure 3, and the discretized sensitivity indices of $\tilde{\mathscr{G}}$ with respect to the γ_is are given by

$$\widehat{\Upsilon}_{\gamma_1}^{\tilde{\mathscr{G}}} \approx -0.000002, \qquad \widehat{\Upsilon}_{\gamma_2}^{\tilde{\mathscr{G}}} \approx -0.000005, \qquad \widehat{\Upsilon}_{\gamma_3}^{\tilde{\mathscr{G}}} \approx 0, \qquad \text{and} \qquad \widehat{\Upsilon}_{\gamma_4}^{\tilde{\mathscr{G}}} \approx -0.000112.$$

Thus, at the maximum state, the total number of graduates depends most sensitively on γ_4, with a 10% increase in γ_4 leading to 0.00112% decrease in the total number of graduates.

5 Conclusions and future research

Adopting the perspective of a mathematical epidemiologist, we have constructed a continuous model governing the time evolution of the number of students enrolled in each specialization offered by a multi-specialization study program, under the assumption that students change specializations solely due to peer influence. We have established the non-negativity and boundedness of the model's solutions. In the case of each specialization having a zero registration rate and the model having only finitely many equilibria, we have proved that the origin is the model's only equilibrium, which is locally asymptotically stable. We have also formulated an optimization problem aimed at identifying an intervention strategy which maximizes the program's total number of graduates, and a sensitivity index as a key quantity for sensitivity analysis. Finally, we have constructed discretized versions of the model, the optimization problem, and the sensitivity index, and exploited them to carry out numerical experiments in two simulated cases involving, respectively, two and four specializations, solving the optimization problem using a coordinate-descent method.

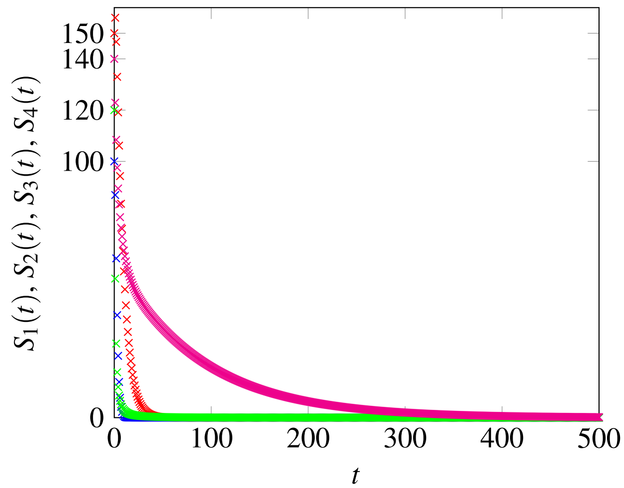

Figure 3: The time evolution of the number of students enrolled in specializations 1 (red), 2 (blue), 3 (green), and 4 (magenta) in our four-specialization case.

There are various avenues for extending our work. First, for both our continuous and discretized models with zero registration rates and finitely many equilibria, one could investigate whether it is possible to prove that the unique equilibrium is globally asymptotically stable, using Lyapunov functions [17,33,40]. On the other hand, the case of infinitely many equilibria may also be worth addressing. If an empirical dataset of the time-dependent number of students at each specialization from a certain study program is available, one could conduct a parameter estimation using machine-learning methods such as neural networks [1,31] or Bayesian inference [9,19,28]. Furthermore, one could employ multiple discretization methods —not only Euler's method but also the fourth-order Runge-Kutta method or even non-standard finite-difference methods— and investigate how the spectral radius of the Jacobian associated with each method depends on the discretization step size [18] while also comparing their computational efficiency. Finally, one could attempt accelerating the convergence of the coordinate-descent method employed to solve our optimization problem by using a relatively large step size during early iterations, to be gradually reduced as the iterations proceed. Finally, one could attempt

Our model itself is open to various modifications. One could investigate, for instance, whether it is appropriate to replace the bilinear form of our specialization-change rates —which assumes that specialization changes are driven purely by peer influence— with more sophisticated forms such as the Holling forms or the Beddington-DeAngelis form [27]. Additionally, in the zero-registration case, the solutions of our present model approach the origin only asymptotically. For an improvement, it may be of interest to develop a model according to which the number of students enrolled in each specialization actually becomes zero at some finite time. To construct such a model, one could adopt the idea of Mickens [23] who has realized the same improvement to the Kermack-McKendrick SIR-type epidemic model.

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