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On a generalization of regular rings with central nilpotents

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Abstract. A ring R is called π-regular if, for every $x \in R$, there exists $y \in R$ such that $x^n = x^n y x^n$ for some positive integer n. Here, we shall give some characterizations of π-regular rings in which nilpotent elements lie in the center. It is shown that these rings can be formulated in a way motivated by recent works of P. Danchev, leading to new insights into π-regular rings and providing partial answers to a question posed by him. In the end, we aim to classify this class of rings, up to an isomorphism.

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1 Introduction

Throughout this paper, all rings are assumed to be associative with a nonzero identity, and R is always a ring. In addition, $J(R)$ denotes the Jacobson radical of R, $\mathrm{Nil}(R)$ the set of nilpotent elements of R, $\mathrm{C}(R)$ the center of R, and $\mathbb{Z}_n$ the ring of integers modulo n.

A ring R is said to be *(von Neumann) regular* when each $x \in R$ satisfies $x \in xRx$. Although regular rings have been introduced and studied for a long time, their variations, generalizations, and closely related notions continue to attract significant attention; see, for instance, [1–4, 6, 8]. A well-studied generalization, called *π-regular rings*, are rings in which each $x \in R$ satisfies $x^n \in x^n R x^n$ for some positive integer n.

Reduced π-regular rings are known to be regular (see, for instance, [13, Proposition 24.5]). However, what happens if in π-regular rings we have nontrivial nilpotents but they are all central, and where does this class of rings lie? To investigate this, we turn our attention to another well-known class of rings. A ring R is called *strongly π-regular* if, for every $x \in R$, there

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is a positive integer n such that $x^n \in x^{n+1}R$ (equivalently, for every $x \in R$, there is a positive integer n such that $x^n \in Rx^{n+1}$). Examples of this class include Artinian rings and perfect rings. It is well-known that if a ring R is strongly π-regular, then there exists $y \in R$ that commutes with x and $x^n = x^{n+1}y$ for some positive integer n (see [13, Theorem 23.7]). Consequently, it follows that strongly π-regular rings are also π-regular. For more examples, some background or a complete study of this subject area, see related sections in [13].

Some recent works of Danchev concerns variations of rings defined above (see [4, 5]). As an inspiration, in the next section, we first introduce a new class of rings which are defined to be exactly π-regular rings whose nilpotent elements are central. By providing several examples, we aim to explore their connections with classical rings. Following this, we investigate their properties, symmetries, uniqueness, and the rings constructed from them (particularly, polynomial rings). Subsequently, we present some characterizations of such rings. Finally, our objective is to classify these rings when they are finite, Artinian or semilocal.

2 The Main Results

Firstly, we introduce centrally π-regular rings, which we shall observe to be a special case of π-regular rings.

Definition 1. *Let R be a ring. An element $x \in R$ is called* centrally π-regular *in R if there exists an idempotent $e \in xR$ such that $x(1-e) \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$. A ring whose elements are centrally π-regular is called centrally π-regular.*

It is not hard to see that units, idempotents and central nilpotent elements in a ring are centrally π-regular. Therefore, division rings and Boolean rings are examples of centrally π-regular rings. In addition, we have the following example.

Example 1. Finite commutative rings are centrally π-regular. To see this, note that for each element x of a finite ring R, we have $x^{2n} = x^n$ for some positive integer n. Hence, $\big(x(1-x^n)\big)^n = 0$. Therefore, $x(1-e)$ is a nilpotent element, where $e = x^n$. The combination of this fact with commutativity implies the desired result.

For a more nontrivial example, we recall that a ring R is said to be *strongly regular* if every element x in R satisfies $x \in x^2R$.

Proposition 1. *A ring is strongly regular if and only if it is reduced and centrally π-regular.*

Proof. ($\Rightarrow$) Suppose that R is a strongly regular ring. Letting $x \in R$, we can find $y \in R$ such that $x = x^2y$. Consequently, $x = x^2y = x^3y^2 = \cdots = x^ny^{n-1}$, which implies that R is reduced. Moreover, it can be seen that $(x - xyx)^2 = 0$. Since R is reduced, it can be concluded that $x = xyx$. Hence, $x(1-e) = 0 \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$, where $e = xy$ is an idempotent, concluding the result.

($\Leftarrow$) Assume that R is a reduced centrally π-regular ring. To show that R is strongly regular, let $x \in R$. Then, there exists $y \in R$ such that $x(1-xy) \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$. Since R is reduced, we conclude that $x(1-xy) = 0$. Therefore, $x = x^2y$, as required. □

Recall that a ring is strongly regular exactly when it is both regular and reduced. Therefore, in reduced rings, the notions of regularity and being centrally π-regular coincide. However, there exists a centrally π-regular ring that is not regular. For instance, consider $\mathbb{Z}_4$, which is a finite commutative ring, and by Example 1, it is centrally π-regular, but not regular (more generally, take $\mathbb{Z}_n$ where n is not a square-free integer). Furthermore, there exist regular rings which are not centrally π-regular. To show this, we require a crucial lemma, which plays the main role in all our subsequent works.

Lemma 1. *A nilpotent element in a ring is centrally π-regular if and only if it is central.*

Proof. ($\Rightarrow$) Suppose that in a ring R, x is a nilpotent element which is also centrally π-regular. There exists $y \in R$ such that $(xy)^2 = xy$ and $x(1-xy) \in \mathrm{C}(R)$. Hence, $x(1-xy)x = xx(1-xy)$, and thereby $x^2yx = x^3y$, implying $x^2yxy = x^3y^2$. Since xy is an idempotent, we obtain that $x^2y = x^3y^2$. Therefore, $x^2y = x^3y^2 = x^4y^3 = \cdots = x^ny^{n-1}$. As x is a nilpotent element, it can be concluded that $x^2y = 0$, and consequently, $x = x(1-xy) \in \mathrm{C}(R)$, as desired.

($\Leftarrow$) Let $x \in \mathrm{C}(R)$ and $x^n = 0$ for some positive integer n. Then, $x = x(1-x^n) = 0 \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$. □

Example 2. There exists a regular ring that is not centrally π-regular. Consider $M_n(F)$ for some field F and positive integer n greater than 1. Note that $M_n(F)$ is always a regular ring (see, for example [12, Corollary 4.24]). On the other hand, since $n > 1$, it is not hard to see that $M_n(F)$ contains non-central nilpotents. Therefore, by Lemma 1, the result is obtained.

Therefore, regular rings and centrally π-regular rings are not generally related to each other. Indeed, we show that centrally π-regular rings are strongly π-regular. First, we state a proposition that presents some necessary conditions for a ring to be centrally π-regular. Recall that a ring is said to be *Abelian* when its idempotents are central.

Proposition 2. *If R is a centrally π-regular ring, then R is Abelian and $J(R) = \mathrm{Nil}(R) \subseteq \mathrm{C}(R)$.*

Proof. Since by Lemma 1, all nilpotent elements of R lie in the center, it is not difficult to see that R is an Abelian ring (note that $eR(1-e), (1-e)Re \subseteq \mathrm{Nil}(R)$ for any idempotent $e \in R$). Additionally, $J(R)$ contains all central nilpotents, implying that $\mathrm{Nil}(R) \subseteq J(R)$. Now, consider $x \in J(R)$. There exists an idempotent $e \in xR$ such that $x(1-e) \in \mathrm{Nil}(R)$. Since $J(R)$ contains no nonzero idempotents, we deduce that $e = 0$. Consequently, $x = x(1-e) \in \mathrm{Nil}(R)$, from which the assertion follows. □

Now, we present one of our main theorems that provides some characterizations of centrally π-regular rings. Following [4, p. 703], a ring R is said to be *Utumi* when, for every $x \in R$, there exists $y \in R$ depending on x such that $x - x^2y \in \mathrm{Nil}(R)$. Moreover, if $xy = yx$, the ring R is called *strongly Utumi*, and if $x - x^2y \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$, the ring R is said to be *centrally Utumi.*

Theorem 1. *The following statements hold true.*

1. *R is a centrally π-regular ring;*

2. *R is a π-regular ring whose nilpotents are central;*

3. R is a strongly π-regular ring whose nilpotents are central;

4. R is a centrally Utumi ring.

Proof. (1) $\Rightarrow$ (2) Considering Lemma 1, it follows that $\mathrm{Nil}(R) \subseteq \mathrm{C}(R)$. Additionally, for any $x \in R$, we can find $y \in R$ such that $(xy)^2 = xy$ and $x(1-xy) \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$. Consequently, $\big(x(1-xy)\big)^n = 0$ for some positive integer n, implying $x^n(1-xy) = 0$ (note that from $ab \in \mathrm{C}(R)$ we can deduce that $(ab)^m = a^m b^m$ for any positive integer m). As a result, $x^n = x^n(xy)$. According to Proposition 2, R is Abelian. Thus, xy is in the center, so that $xy = (xy)^n = x^n y^n$. Therefore, $x^n = x^n(xy) = (xy)x^n = x^n y^n x^n$, as required.

(2) $\Rightarrow$ (3) As nilpotent elements of R are central, R is Abelian. Moreover, it is clear that Abelian π-regular rings are strongly π-regular. Hence, R is strongly π-regular, as claimed.

(3) $\Rightarrow$ (1) and (4) Letting $x \in R$, there exists $y \in R$ that commutes with x and satisfies $x^n = x^{n+1}y$ for some positive integer n. It follows that $x^n = x^{n+1}y = x^{n+2}y^2 = \cdots = x^n y^n x^n$. Consequently, $e = x^n y^n \in xR$ is an idempotent and $\big(x(1-e)\big)^n = 0$. Since nilpotent elements of R are central, we obtain $x(1-e) \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$, completing the proof.

(4) $\Rightarrow$ (3) According to [14, p. 1], nilpotent elements of a centrally Utumi ring lie in the center. Considering $x \in R$, we can find $y \in R$ such that $x - x^2y \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$. Consequently, $\big(x(1-xy)\big)^n = x^n(1-xy)^n = 0$ for some positive integer n. Using binomial expansion, it is evident that $(1-xy)^n = 1 - xz$ for some $z \in R$. Therefore, $x^n(1-xz) = 0$, and it follows that R is strongly π-regular. □

As an immediate consequence of Theorem 1, we have the following corollary.

Corollary 1. *The center of a (centrally) π-regular ring is centrally π-regular.*

Proof. According to [9, Theorem 1], the center of a π-regular ring is π-regular. Utilizing Theorem 1, the assertion follows. □

Recall that a ring R is called *exchange* if its idempotents lift modulo every left (right) ideal of R (see [13, Theorem 29.2]). In [4, Problem 3.3], Danchev posed the question of whether (centrally) Utumi rings possess the exchange property. In the following, we establish a positive answer to this question. First we show that strongly Utumi rings are exactly strongly π-regular ones.

Proposition 3. *A ring is strongly Utumi if and only if it is strongly π-regular.*

Proof. ($\Rightarrow$) Suppose that R is a strongly Utumi ring. For $x \in R$, there exists $y \in R$ that commutes with x and satisfies $x - x^2y \in \mathrm{Nil}(R)$. Hence, similarly to the last part of the proof in Theorem 1, the result follows.

($\Leftarrow$) Assume that R is strongly π-regular. For $x \in R$, there exists $y \in R$ such that $x^n = x^{n+1}y$ and $xy = yx$ for some positive integer n. Consequently, $x^n(1-xy) = 0$, and thereby $\big(x(1-xy)\big)^n = 0$, from which the assertion follows. □

Corollary 2. *Every strongly Utumi ring is exchange.*

Proof. Since strongly π-regular rings are exchange (see, for instance [11, Theorem 1]), the result follows. □

In view of Theorem 1, we see that centrally Utumi rings are strongly π-regular. Hence, using Proposition 3, we obtain the following corollary.

Corollary 3. *Every centrally Utumi ring is strongly Utumi. In particular, centrally Utumi rings are exchange.*

As mentioned before, for Abelian rings, the notions of strongly π-regular and π-regular coincide. The natural question that arises here is whether Abelian π-regular rings are centrally π-regular. In the following example, we see that this is not the case.

Example 3. There exist π-regular rings which are Abelian but not centrally π-regular. To observe this, let F be a finite field with characteristic 2, and G a finite non-Abelian 2-group with the identity element e. Consider the group algebra $F[G]$. Since $F[G]$ is finite, it is π-regular. Moreover, by [12, Proposition 19.11], $F[G]$ is a local ring and, hence, Abelian. However, as G is non-Abelian, we can choose a non-central element g in G. Since $|G|$ is a power of 2, it follows that $(e+g)^{|G|} = e^{|G|} + g^{|G|} = 2e = 0$. On the other hand, as g is not in the center of G, it is clear that $e+g$ is a nilpotent element that is not central. Therefore, $F[G]$ is not centrally π-regular. (For further illustration, consider the case where $F = \mathbb{Z}_2$, $G = Q_8$, the quaternion group, $e = 1$, and $g = i$.)

Now, we aim to identify symmetries within centrally π-regular rings.

Proposition 4. *A ring R is centrally π-regular if and only if for every $x \in R$, there exists an idempotent $e \in Rx$ such that $(1-e)x \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$.*

Proof. $(\Rightarrow)$ Let R be a centrally π-regular ring. Then, for $x \in R$, we can find $y \in R$ such that $e = xy$ is an idempotent, and $x(1-e) \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$. Since R is Abelian by Proposition 2, it follows that $(1-e)x \in \mathrm{C}(R)$. Consequently, $y(1-e)x = (1-e)xy$. However, $(1-e)xy = (1-e)e = 0$, so $y(1-e)x = 0$. Hence, $yx = yex = (yx)^2$. Therefore, yx is an idempotent, making it central. Now, it can be seen that

$$e = xy = (xy)^2 = x(yx)y = (yx)xy = yx(xy) = y(xy)x = (yx)^2 = yx.$$

Therefore, $(1-e)x \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$, where $e = yx = xy$ is an idempotent.

$(\Leftarrow)$ Given $x \in R$, there exists $y \in R$ such that $e = yx$ is an idempotent and $(1-e)x \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$. This implies $x(1-e)x = (1-e)xx$, leading to $xex = ex^2$. Therefore, $xyx^2 = yx^3$, and hence $yxyx^2 = y^2x^3$. Since $(yx)^2 = yx$, it can be concluded that $yx^2 = y^2x^3$. Consequently, $yx^2 = y^2x^3 = \cdots = y^{n-1}x^n$. Assuming that x is a nilpotent, we obtain that $ex = yx^2 = 0$, implying $x = (1-e)x \in \mathrm{C}(R)$. Therefore, nilpotent elements of R lie in the center, making R Abelian. Now, in a similar manner to the proof of the implication $(\Rightarrow)$, it can be seen that $e = yx = xy$, from which the assertion follows. $\square$

According to the proof of Proposition 4, in centrally π-regular rings, for any element x, the associated idempotent e is of the form $e = xy$, where $xy = yx$ for some element y. Next, we establish the uniqueness of this idempotent. Before that, it is worth noting that if R is a centrally π-regular ring, then for every $x \in R$, there exists an idempotent $e \in Rx$ such that $x(1-e) \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$. However, the converse is not true in general. To see this, observe that regular rings have the mentioned property, but by Example 2, there exist regular rings which are not centrally π-regular. Therefore, this new class of rings includes regular rings and centrally π-regular rings. Furthermore, in the Abelian case, it coincides with centrally π-regular rings. One could study and explore the properties of this newfound class.

Proposition 5. *A ring R is centrally π-regular if and only if for every $x \in R$, there exists a unique idempotent $e \in xR$ such that $x(1-e) \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$.*

Proof. We only need to prove the implication ($\Rightarrow$). Suppose that $x \in R$ and there are idempotents $e = xy$ and $f = xz$ such that $x(1-e)$ and $x(1-f)$ are central nilpotents. Therefore, $x(1-e)z$ and $x(1-f)y$ are nilpotent elements. Since R is Abelian by Proposition 2, $x(1-e)z = xz(1-e) = f(1-e)$. Similarly, $x(1-f)y = xy(1-f) = e(1-f)$. Now, $f(1-e)$ and $e(1-f)$ are both nilpotent elements that are also idempotents. Consequently, $f(1-e) = e(1-f) = 0$, implying $e = f$, as desired. □

In what follows, we aim to study rings constructed from centrally π-regular rings. Initially, we begin with a characterization theorem. Note that it is evident that the homomorphic images of centrally π-regular rings retain the same property.

Theorem 2. *Let I be a nil ideal of a ring R. Then, R is centrally π-regular if and only if R/I is centrally π-regular and $\mathrm{Nil}(R) \subseteq \mathrm{C}(R)$.*

Proof. ($\Rightarrow$) Since every homomorphic image of a centrally π-regular ring is centrally π-regular, together with Proposition 2, the implication follows.

($\Leftarrow$) Let $x \in R$ and denote the image of an element $r \in R$ in R/I by $\bar{r}$. As R/I is centrally π-regular, there exists $\bar{y} \in R/I$ such that $\bar{x}(\bar{1} - \bar{x}\bar{y}) \in \mathrm{Nil}(R/I)$. Consequently, $x(1-xy) \in \mathrm{Nil}(R)$, since I is a nil ideal. By the assumption, $\mathrm{Nil}(R) \subseteq \mathrm{C}(R)$, implying that R is centrally Utumi. Therefore, by Theorem 1, the proof is complete. □

Note that for a ring R with $J(R) = 0$, the concepts of centrally π-regular and strongly regular coincide. Also, if $R/J(R)$ is strongly regular, then $\mathrm{Nil}(R) \subseteq J(R)$. Since in centrally π-regular rings, the Jacobson radical is always nil, we obtain the following.

Corollary 4. *A ring R is centrally π-regular if and only if $R/J(R)$ is centrally π-regular (strongly regular) and $J(R)$ is nil and central.*

Remark 1. *In Theorem 2, the result is not guaranteed when we consider I as a non-nil ideal. For example, take $R = \mathbb{Z}$, which is not centrally π-regular and $I = 2\mathbb{Z}$ (observe that an integral domain is centrally π-regular precisely whenever it is a field).*

Next, we study the direct product of centrally π-regular rings.

Proposition 6. *Let R_i be rings for all $i \in I$. Then, $\prod_{i \in I} R_i$ is centrally π-regular if and only if each R_i is centrally π-regular and $\prod_{i \in I} J(R_i)$ is nil.*

Proof. Reminding Proposition 2, we only need to prove ($\Leftarrow$). Let $(x_i) \in \prod_{i \in I} R_i$. Then, there exist idempotents $e_i \in x_i R_i$ such that $x_i(1-e_i) \in \mathrm{Nil}(R_i) \cap \mathrm{C}(R_i)$. By Proposition 2, we have $\mathrm{Nil}(R_i) \cap \mathrm{C}(R_i) = J(R_i)$. Therefore, one infers that

$$(x_i)\big((1) - (e_i)\big) \in \prod_{i \in I} \big(\mathrm{Nil}(R_i) \cap \mathrm{C}(R_i)\big) = \prod_{i \in I} J(R_i)$$

$$\subseteq \mathrm{Nil}(\prod_{i\in I} R_i) \cap \prod_{i\in I} \mathrm{C}(R_i) = \mathrm{Nil}(\prod_{i\in I} R_i) \cap \mathrm{C}(\prod_{i\in I} R_i).$$

From this, the result follows. $\square$

Remark 2. *The condition "$\prod_{i\in I} J(R_i)$ is nil" in Proposition 6 cannot be omitted. To observe this, by Example 1, we know that for a positive integer n, the ring $\mathbb{Z}_{2^n}$ is centrally π-regular. However, $\prod_{n=1}^{\infty} \mathbb{Z}_{2^n}$ is not centrally π-regular by Proposition 2, because $J(\prod_{n=1}^{\infty} \mathbb{Z}_{2^n}) = \prod_{n=1}^{\infty} 2\mathbb{Z}_{2^n}$.*

Corollary 5. *Every finite direct product of rings is centrally π-regular if and only if so is each of them.*

As a result, we observe that centrally π-regular rings are closed under the operation of taking corners.

Corollary 6. *Suppose that e is an idempotent of a ring R. Then, R is centrally π-regular if and only if the corner rings eRe and $(1-e)R(1-e)$ are centrally π-regular and e is central.*

Proof. If e is central, then it can be concluded that R has a ring decomposition $R = Re \times R(1-e)$, where $Re = eRe$ and $R(1-e) = (1-e)R(1-e)$. Considering Proposition 2 and Corollary 5, the rest is straightforward. $\square$

Note. It is noteworthy that if we omit the centrality condition for e in Corollary 6, the converse is not necessarily true in general. For instance, consider $M_2(F)$, where F is a field. Let E_{ij} be the matrix whose (i,j)-th entry is 1, and all other entries are 0. It is routine to see that $E_{11}M_2(F)E_{11}$ and $(I - E_{11})M_2(F)(I - E_{11})$ are both isomorphic to F, which is centrally π-regular. However, $M_2(F)$ is not centrally π-regular, since E_{12} is a nilpotent element that is not central.

It is worth noting that polynomial rings are never centrally π-regular. For if $R[x]$ is a centrally π-regular ring, then there exists $e \in xR[x]$ such that $x^n(1-e) = 0$ for some positive integer n. However, this implies $e = 1$, making x a unit, which is a contradiction. However, we shall show that some elements in the rings of polynomials are so. First, we need two lemmas.

Lemma 2. *Let x be a central element of a ring R. If $x(1-e) \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$ for some idempotent $e \in xR$, then e is central. Moreover, x is centrally π-regular in $\mathrm{C}(R)$.*

Proof. Let $e = xy$ for some $y \in R$. As x and $x - xe$ lie in the center of R, it can be asserted that $x^2y = xe \in \mathrm{C}(R)$. Consequently, for any $r \in R$, it can be seen that

$$\begin{aligned} re &= re^2 = rxyxy = r(x^2y)y = x^2yry \\ &= yr(x^2y) = x^2yyr = (xy)^2r = e^2r = er. \end{aligned}$$

Therefore, $e = xy \in \mathrm{C}(R)$. Moreover, one could infer that

$$r(yxy) = (xy)ry = yrxy = y(xy)r = (yxy)r.$$

Hence, $yxy \in \mathrm{C}(R)$. Consequently, $e = xy = x(yxy) \in x\mathrm{C}(R)$, from which the result follows. $\square$

Lemma 3. *Let R be a ring and $x, q \in \mathrm{C}(R)$. If x is centrally π-regular and q is nilpotent, then $x+q$ is centrally π-regular in R.*

Proof. Since x is centrally π-regular, there exists $y \in R$ such that $e = xy$ is an idempotent and $x(1-e) \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$. Consequently, $x^n = x^n e$ for some positive integer n. Now, as q is nilpotent, $q^m = 0$ for some positive integer m. Set $z = x + q$ and $t = n + m - 1$. One could infer that

$$z^t = (x+q)^t = x^t + c_1 x^{t-1} q + \cdots + c_{m-1} x^n q^{m-1}$$

for some $c_1, \ldots, c_{m-1} \in \mathrm{C}(R)$. Thus, $z^t = x^n w$ for some $w \in \mathrm{C}(R)$. From $x^n = x^n e$, we obtain that $z^t = z^t e$, implying $\big(z(1-e)\big)^t = 0$, since z is central. Now, with the help of Lemma 2, we have $e \in \mathrm{C}(R)$, so that $z(1-e) \in \mathrm{Nil}(R) \cap \mathrm{C}(R)$. Furthermore, it can be seen that

$$\begin{aligned} e &= xy = x^m y^m = (z-q)^m y^m \\ &= \big(z^m + c'_1 z^{m-1}(-q) + \cdots + c'_{m-1} z(-q)^{m-1}\big) y^m \end{aligned}$$

for some $c'_1, \ldots, c'_{m-1} \in R$. Hence, $e \in zR$ and the result follows. □

Before continuing, it is worth mentioning that Lemma 3 generalizes the well-known fact that in commutative rings, the sum of two nilpotents is nilpotent, and the sum of a nilpotent and a unit is a unit. However, the sum of an idempotent and a nilpotent is not necessarily idempotent or nilpotent (consider the sum of the idempotent 4 and the nilpotent 6 in $\mathbb{Z}_{12}$), but we can always say that this sum is centrally π-regular. It is also known that the sum of two orthogonal idempotents is idempotent. We generalize this in the following remark.

Remark 3. *Let R be a ring and $x, y \in R$ be centrally π-regular elements such that $xy = 0$. If either x or y is central, then $x + y$ is centrally π-regular. To demonstrate this, suppose that $e = xz$ and $f = yw$ are idempotents such that $x(1-e)$ and $y(1-f)$ are central nilpotents. Since $xy = yx = 0$, it can be seen that $(x+y)\big(1-(e+f)\big) = x(1-e) + y(1-f)$ is a central nilpotent element. Assume that x is central, so $e + f$ is idempotent. Moreover, $e + f = (xz)^2 + (yw)^2 = (x+y)(zxz + wyw) \in (x+y)R$, as desired.*

Now, we continue our work in polynomial rings.

Proposition 7. *Let R be a ring and let $n > 1$ be a positive integer. Then, $R[x]/\langle x^n \rangle$ is centrally π-regular if and only if R is a commutative (centrally) π-regular ring.*

Proof. ($\Rightarrow$) Assume that $R[x]/\langle x^n \rangle$ is centrally π-regular. Since R is a quotient of $R[x]/\langle x^n \rangle$, it follows that R is centrally π-regular, hence π-regular. To establish its commutativity, consider $r, s \in R$. Note that $\bar{r}\bar{x}$ is nilpotent, implying its centrality by Theorem 1. Thus, $\bar{s}(\bar{r}\bar{x}) = (\bar{r}\bar{x})\bar{s}$. Consequently, $(sr - rs)x \in \langle x^n \rangle$, from which it follows that $sr = rs$, as required.

($\Leftarrow$) If R is a commutative π-regular ring, it is not difficult to observe that $f(x) = \bar{a}_0 + \bar{a}_1 \bar{x} + \cdots + \bar{a}_{n-1} \bar{x}^{n-1} = \bar{a_0} + \bar{q} \in R[x]/\langle x^n \rangle$, where $\bar{a_0}$ is centrally π-regular, and $\bar{q}$ is a nilpotent element. Now, with the aid of Lemma 3, the assertion follows. □

It is well-known that $R[x]/\langle x^n \rangle$ is isomorphic to a special subring of the ring of triangular matrices $T_n(R)$ (see, for instance, [10]). So, although $M_n(R)$ and $T_n(R)$ are never centrally π-regular for $n > 1$, they may possess subrings that are.

Proposition 8. *Let R be a commutative ring. Then, $f = a_0 + a_1x + \cdots + a_nx^n \in R[x]$ is (centrally) π-regular in $R[x]$ if and only if a_0 is (centrally) π-regular in R and $a_1, \ldots, a_n$ are nilpotent elements of R.*

Proof. First note that in a commutative ring, an element is centrally π-regular precisely whenever it is π-regular.

($\Rightarrow$) Since f is centrally π-regular in $R[x]$, there exists $h \in R[x]$ such that $f^n = f^{n+1}h$ for some positive integer n. Assume that P is a prime ideal of R. By taking images in the integral domain $(R/P)[x]$, we obtain that $\deg(\bar{f}^n) = \deg(\bar{f}^{n+1}) + \deg(\bar{h})$. This implies that $n\deg(\bar{f}) = (n+1)\deg(\bar{f}) + \deg(\bar{h})$, thus $\deg(\bar{f}) = 0$. Therefore, $\bar{f}$ is a constant polynomial, thereby $a_1, \ldots, a_n \in P$. Consequently, $a_1, \ldots, a_n \in \mathrm{Nil}(R)$. On the other hand, considering Lemma 3, it is clear that a_0 is centrally π-regular in R since $a_0 = f - (a_1x + \cdots + a_nx^n)$, concluding the result.

($\Leftarrow$) According to Lemma 3, the result follows. □

Corollary 7. *Let R be a ring. Then, a central polynomial $f = a_0 + a_1x + \cdots + a_nx^n \in R[x]$ is centrally π-regular in $R[x]$ if and only if a_0 is centrally π-regular in R and $a_1, \ldots, a_n$ are nilpotent elements of R.*

Proof. ($\Rightarrow$) Since f is centrally π-regular in $R[x]$ and $f \in \mathrm{C}(R[x])$, by Lemma 2 it follows that f is centrally π-regular in $\mathrm{C}(R[x])$, which is a commutative ring. Therefore, Proposition 8 concludes the result.

($\Leftarrow$) According to Lemma 3, the assertion follows. □

Note. If we let f to be a non-central polynomial in Corollary 7, then the assertion may not be true. For instance, consider $R = M_2(F)$ for some field F. To see why ($\Leftarrow$) may not be true, observe that $f = E_{12}x$ is a non-central nilpotent in $R[x]$, which implies that it is not centrally π-regular. Also, for ($\Rightarrow$), consider $g = (E_{12} + E_{21}) + E_{11}x$ and $h = (E_{12} + E_{21}) - E_{22}x$. It is easy to see that $gh = hg = I_2$, implying that g is a unit, making it centrally π-regular. However, the coefficient of x in g, that is E_{11}, is not a nilpotent element.

Now, we begin classifying centrally π-regular rings, up to an isomorphism. Recall that a ring R for which $R/J(R)$ is semisimple is called *semilocal*.

Theorem 3. *Let R be a semilocal ring. Then, R is centrally π-regular if and only if it is isomorphic to a finite direct product of local centrally π-regular rings.*

Proof. ($\Rightarrow$) Suppose that R is centrally π-regular. According to Corollary 4, $R/J(R)$ is centrally π-regular. Since R is semilocal, let $R/J(R) \simeq M_{n_1}(D_1) \times \cdots \times M_{n_t}(D_t)$ for some division rings D_i and positive integers n_i. If for some i we have $n_i > 1$, then $R/J(R)$ contains a non-central nilpotent element, leading to a contradiction. Therefore, all the n_i's are equal to 1. Consequently, $R/J(R) \simeq D_1 \times \cdots \times D_t$. Let e_i denote the t-tuple whose only nonzero entry 1 is in the i-th place. Then, $e_1, \ldots, e_t$ form a complete set of orthogonal idempotents for $R/J(R)$. Since R is centrally π-regular, by Proposition 2, it is Abelian and $J(R)$ is nil. Utilizing [12, Proposition 21.25], it is not hard to see that by lifting e_i's, a complete set of orthogonal idempotents for R is obtained. Consequently, R is isomorphic to $R_1 \times \cdots \times R_t$ for some rings R_i, where $R_i/J(R_i) \simeq D_i$. So, each R_i is a local ring, and by Corollary 5 it is also centrally π-regular, from which the assertion follows.

($\Leftarrow$) The result follows by Corollary 5. □

In light of Theorem 3, the significance of local centrally π-regular rings becomes evident. The following proposition highlights some properties of these rings.

Proposition 9. *For a local centrally π-regular ring R, the following statements hold.*

1. *Every element of R is a unit or central nilpotent. In particular, if R is reduced, then R is a divison ring;*

2. *If the group of units $U(R)$ is torsion or Abelian, then R is commutative;*

3. *If R is not reduced, then $R/J(R)$ is a field. In addition, involutions (elements $u \in R$ such that $u^2 = 1$), unipotents (elements $u \in R$ such that $u - 1 \in \mathrm{Nil}(R)$), additive commutators (elements of the form $xy - yx$, where $x, y \in R$) and multiplicative commutators (elements of the form $uvu^{-1}v^{-1}$, where $u, v \in U(R)$) are all central.*

Proof. (1) This follows from Proposition 2.

(2) If $U(R)$ is an Abelian group, then the units of R commute with each other. Since every nonunit element is central, it follows that R is commutative. On the other hand, if $U(R)$ is a torsion group, then for each $u \in U(R)$, we have $u^n = 1$, thereby $u^{n+1} = u$ for some positive integer n. Also, if $x \in R$ is not a unit, then it is a central nilpotent. Therefore, each element $x \in R$ satisfies $x - x^n \in \mathrm{C}(R)$ for some positive integer $n > 1$. According to a well-known theorem of Herstein [7, Theorem 21], such rings are commutative, completing the proof.

(3) Since R is not reduced, for any nonzero nilpotent $q \in R$ and units $u, v \in R$, it is clear that $vq \in \mathrm{C}(R)$. Moreover,

$$(uv - vu)q = u(vq) - vuq = vqu - vuq = vuq - vuq = 0.$$

This implies that $uv - vu \in J(R)$, so $R/J(R) \simeq F$ for some field F.

For the final part, it is straightforward that additive commutators and unipotents are central. Furthermore, since $uv - vu$ is a central nilpotent, it follows that $uvu^{-1}v^{-1} - 1$ is also nilpotent, which implies that multiplicative commutators are central. Finally, if $u^2 = 1$, then $(u-1)(u+1) = 0$, so either $u - 1$ or $u + 1$ is nilpotent. This completes the proof. □

It is known that in the commutative case, Artinian or perfect rings are isomorphic to a finite direct product of local rings. However, in general, this is not the case. As a result of Theorem 3, in what follows, we generalize the mentioned property from commutative rings to centrally π-regular ones. Recall that Artinian rings are always semilocal.

Corollary 8. *Every Artinian centrally π-regular ring is isomorphic to a finite direct product of local Artinian centrally π-regular rings.*

Of course, it is evident that in Corollary 8, we can replace Artinian with perfect and obtain a similar result. Moreover, it is worth noting that not all centrally π-regular rings have such a decomposition. For example, $\prod_{n=1}^{\infty} D_n$, where each D_n is a division ring, is centrally π-regular by Proposition 6.

Corollary 9. *Let R be a finite ring. Then, R is centrally π-regular if and only if it is commutative.*

Proof. ($\Rightarrow$) Considering Corollary 8, it is sufficient to show that every finite local centrally π-regular ring is commutative and this follows from Proposition 9 (2).

($\Leftarrow$) Follows by Example 1. $\square$

Indeed, Corollary 9 does not hold for Artinian rings (consider division rings). However, it seems that Artinian centrally π-regular rings can be decomposed into $S \times T$, where S is reduced and T is commutative. To either prove or disprove this, by Corollary 8, it suffices to investigate whether a local Artinian ring whose nilpotents are central is commutative or reduced. Consider such a ring R, that is, an Artinian ring in which every element is a unit or central nilpotent. If R is not reduced, then by Proposition 9 (3), we know that $R/J(R)$ is commutative. However, we have not been able to prove that R itself is commutative.

Problem 1. *Does every local Artinian ring, whose nilpotent elements are central, have to be either reduced or commutative?*

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