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Characterizations of algebraic and vertex connectivity of power graph of finite cyclic groups

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Abstract. The Power graph of a group G is a graph $\mathcal{P}(G)$ with vertex set G and two vertices x and y, $x \neq y$ are adjacent if there exists some integer k such that $x = y^k$ or $y = x^k$. We denote the vertex connectivity of power graph $\mathcal{P}(G)$ by $\mathcal{K}(\mathcal{P}(G))$ and the algebraic connectivity of power graph $\mathcal{P}(G)$ by $\lambda_{n-1}(\mathcal{P}(G))$. This paper investigates the upper bound for the vertex connectivity and the algebraic connectivity of $\mathcal{P}(\mathbb{Z}_n)$. Moreover, we discuss the equivalent conditions for $\mathcal{P}(\mathbb{Z}_n)$ to be Laplacian integral. Further the conjecture for an upper bound of the algebraic connectivity of $\mathcal{P}(\mathbb{Z}_n)$ is posed in this article.

Keywords: Power graph, Algebraic Connectivity, Vertex Connectivity, Laplacian Integral, Finite cyclic group.
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1 Introduction and preliminaries

The concept of directed power graph of a semigroup S was first introduced by [8]. Motivated by this, [3] defined the undirected power graph $\mathcal{P}(G)$ of a group G as the undirected graph whose vertex set is a set of elements G and any two vertices $a, b \in G$ are adjacent in $\mathcal{P}(G)$ if and only if there exists some integer k such that either $a = b^k$ or $b = a^k$. [9], [1] contains a detailed survey on the power graphs of groups. For a graph Γ, the set of vertices and the edges is denoted by $V(\Gamma)$ and $E(\Gamma)$ respectively. A simple graph is a graph without loops and the parallel edges. A null graph is a graph with no vertices and no edges. A graph with one vertex and no edges is called as a trivial graph. A graph is connected if and only if there is a path between every pair of vertices. A component of a graph Γ is the maximal connected subgraph of Γ. The vertex connectivity of a graph Γ is denoted by $\mathcal{K}(\Gamma)$ and it is the minimum number of vertices whose removal results in a disconnected graph or a trivial graph. For any finite simple undirected

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graph Γ with the ordered vertex set $\{v_1, v_2, \ldots, v_n\}$, the Laplacian matrix $L(\Gamma)$ of graph Γ is defined by $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $A(\Gamma)$ is the adjacency matrix of Γ whose $(i,j)^{th}$ entry is 1, if v_i is adjacent to v_j and 0 otherwise, and $D(\Gamma)$ is the diagonal matrix whose $(i,i)^{th}$ entry is degree of v_i. We denote the Laplacian characteristic polynomial $det(xI - L(\Gamma))$ of a graph Γ by $\Theta(\Gamma, x)$ instead of $\Theta(L(\Gamma), x)$. The principal submatrix of $L(\Gamma)$ formed by deleting the rows and the columns corresponding to the vertices $v_1, v_2, \ldots, v_i$ of a graph Γ is denoted by $L_{v_1,v_2,\ldots,v_i}(\Gamma)$. As per convention, if $i = n$, then $\Theta(L_{v_1,v_2,\ldots,v_n}(\Gamma), x) = 1$ [4]. The matrix $L(\Gamma)$ is a real symmetric, singular and a positive semi-definite, so all of it's eigenvalues are real and non-negative. Furthermore, the sum of each row (column) of $L(\Gamma)$ is zero, which means the smallest eigenvalue of $L(\Gamma)$ is 0. The eigenvalues of $L(\Gamma)$ are called the Laplacian eigenvalues of Γ. We denote the Laplacian eigenvalues of Γ by $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma) = 0$ always arranged in a non-increasing order and repeated according to their multiplicities. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct Laplacian eigenvalues of Γ with corresponding multiplicities $n_1, n_2, \ldots, n_k$. Then the Laplacian spectrum of Γ is denoted by $\begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ n_1 & n_2 & \ldots & n_k \end{pmatrix}$. The Laplacian spectrum of a graph has a several applications like random walks, expansion properties, statistical efficiency and optimality properties [2]. The algebraic connectivity $\lambda_{n-1}(\Gamma)$ of a graph Γ is the second smallest Laplacian eigenvalue of Γ, which is considered as a measure of connectivity of Γ [6]. Moreover, the largest Laplacian eigenvalue $\lambda_1(\Gamma)$ of a graph Γ is called the Laplacian spectral radius of Γ. A graph Γ is called as the Laplacian integral, if all of its Laplacian eigenvalues are integers. A discussion related to the Laplacian eigenvalues of a graph and it's complement is in [10], [6], [5]. The Laplacian spectrums of $\mathcal{P}(\mathbb{Z}_n)$ and $\mathcal{P}(D_{2n})$ for particular values of n along with the relationship between the Laplacian spectrums of power graphs $\mathcal{P}(\mathbb{Z}_n)$ and $\mathcal{P}(D_{2n})$ is studied in [4]. Moreover, [4] contains a discussion on the lower and the upper bounds for the algebraic connectivity of $\mathcal{P}(\mathbb{Z}_n)$. Various results on the Laplacian specta of the power graphs of finite cyclic, dicyclic and finite p-groups are studied in [11].

Theorem 1 ([11]). *The power graph of finite p-group is always Laplacian integral.*

Theorem 2 ([4]). *For each non-prime positive integer $n > 3$, the multiplicity of n as a Laplacian eigenvalue of $\mathcal{P}(\mathbb{Z}_n)$ is at least $\phi(n) + 1$.*

Theorem 3 ([4]). *For $n = p^\alpha q^\beta$, where p and q are distinct primes and $\alpha, \beta \in \mathbb{N}$, the algebraic connectivity $\lambda_{n-1}(\mathcal{P}(\mathbb{Z}_n)) \leq \phi(n) + p^{\alpha-1}q^{\beta-1}$, equality holds if $\alpha = 1 = \beta$.*

Theorem 4 ([4]). *For each positive integer $n \geq 2$, the algebraic connectivity $\lambda_{n-1}(\mathcal{P}(\mathbb{Z}_n))$ of $\mathcal{P}(\mathbb{Z}_n)$ satisfies the inequality $\lambda_{n-1}(\mathcal{P}(\mathbb{Z}_n)) \geq \phi(n) + 1$. Equality holds if n is either a prime or the product of two distinct primes.*

Theorem 5 ([11]). *For any integer $n > 1$, $\mathcal{K}(\mathcal{P}(\mathbb{Z}_n)) = \lambda_{n-1}(\mathcal{P}(\mathbb{Z}_n))$ if and only if n is a product of two distinct primes.*

Theorem 6 ([11]). *For any integer $n > 1$, the algebraic connectivity of $\mathcal{P}(\mathbb{Z}_n)$ is $\phi(n) + 1$ if and only if n is a prime number or product of two distinct primes.*

2 Algebraic and vertex connectivity of $\mathcal{P}(\mathbb{Z}_n)$

The upper bound for the algebraic and the vertex connectivity of $\mathcal{P}(\mathbb{Z}_n)$ is obtained for $n = p^\alpha q^\beta$, where $\alpha, \beta \in \mathbb{N}$ and p, q are distinct primes, and for the values of n, where n is a product of two or three distinct primes in [4]. In this section, we obtain the upper bounds for the algebraic and the vertex connectivity of $\mathcal{P}(\mathbb{Z}_n)$, where n is a product of $4, 5$ and 6 distinct primes. Hence we obtain the upper bound for the algebraic and the vertex connectivity of a power graph of a finite cyclic group G of order n, where n is a product of $4, 5$ and 6 distinct primes.

Proposition 1. *For $n = \prod_{i=1}^{4} p_i$, where $p_i, i = 1, 2, 3, 4$ are distinct primes with $p_1 < p_2 < p_3 < p_4$, the vertex connectivity $\mathcal{K}(\mathcal{P}(\mathbb{Z}_n))$ of $\mathcal{P}(\mathbb{Z}_n)$ satisfies the inequality $\mathcal{K}(\mathcal{P}(\mathbb{Z}_n)) \leq \phi(n) + \sum_{i=1}^{4} p_i + \sum_{\substack{i,j=1,2 \\ i \neq j}}^{2,3} p_i p_j - 6$.*

Proof. Let S be the subset of $\mathbb{Z}_n$ consisting of $\bar{0}$ and all the generators, $P_1 = \{a\bar{p_1} \in V(\mathcal{P}(\mathbb{Z}_n));\ p_2 \nmid a, p_3 \nmid a, p_4 \nmid a\}$, $P_2 = \{b\bar{p_2} \in V(\mathcal{P}(\mathbb{Z}_n));\ p_1 \nmid b, p_3 \nmid b, p_4 \nmid b\}$, $P_3 = \{c\bar{p_3} \in V(\mathcal{P}(\mathbb{Z}_n));\ p_1 \nmid c, p_2 \nmid c, p_4 \nmid c\}$, $P_4 = \{d\bar{p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ p_1 \nmid d, p_2 \nmid d, p_3 \nmid d\}$, $U_1 = \{u_1\overline{p_1p_2} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_1 \leq p_3p_4 - 1\}$, $U_2 = \{u_2\overline{p_1p_3} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_2 \leq p_2p_4 - 1\}$, $U_3 = \{u_3\overline{p_1p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_3 \leq p_2p_3 - 1\}$, $U_4 = \{u_4\overline{p_2p_3} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_4 \leq p_1p_4 - 1\}$, $U_5 = \{u_5\overline{p_2p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_5 \leq p_1p_3 - 1\}$, $U_6 = \{u_6\overline{p_3p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_6 \leq p_1p_2 - 1\}$, $T_1 = \{t_1\overline{p_1p_2p_3} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_1 \leq p_4 - 1\}$, $T_2 = \{t_2\overline{p_1p_2p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_2 \leq p_3 - 1\}$, $T_3 = \{t_3\overline{p_1p_3p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_3 \leq p_2 - 1\}$ and $T_4 = \{t_4\overline{p_2p_3p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_4 \leq p_1 - 1\}$. Then all the sets $S, P_i, U_j, T_i, i = 1, 2, 3, 4$ and $j = 1, 2, \ldots, 6$ are pairwise disjoint sets of vertices of $\mathcal{P}(\mathbb{Z}_n)$ whose union is $V(\mathcal{P}(\mathbb{Z}_n))$. Even though every vertex of the set S is adjacent to all other vertices of $\mathcal{P}(\mathbb{Z}_n)$, $\mathcal{P}(\mathbb{Z}_n) - S$ is connected. The connectedness diagram among the sets P_i, U_j and T_i, where $i = 1, 2, 3, 4$ and $j = 1, 2, \ldots, 6$ can be obtained as in Figure 1. Now to make the graph $\mathcal{P}(\mathbb{Z}_n) - S$ disconnected, we need to remove the sets T_1, T_2, T_3, T_4 and the three sets from U_1, U_2, U_3, U_4, U_5 and U_6. To make the upper bound of $\mathcal{K}(\mathcal{P}(\mathbb{Z}_n))$ sharp, we need to remove the sets T_1, T_2, T_3, T_4 along with the three sets with minimum cardinality from $U_1, \ldots, U_6$, which are U_3, U_5 and U_6. Therefore the graph $\mathcal{P}(\mathbb{Z}_n) - S - T_1 - T_2 - T_3 - T_4 - U_3 - U_5 - U_6$ is disconnected and thus $\mathcal{K}(\mathcal{P}(\mathbb{Z}_n)) \leq |S| + \sum_{i=1}^{4} |T_i| + |U_3| + |U_5| + |U_6| = \phi(n) + 1 + \sum_{i=1}^{4} p_i - 4 + p_1p_2 + p_1p_3 + p_2p_3 - 3 = \phi(n) + \sum_{i=1}^{4} p_i + \sum_{\substack{i,j=1,2 \\ i \neq j}}^{2,3} p_i p_j - 6$. □

Corollary 1. *For $n = \prod_{i=1}^{4} p_i$, where p_i, $i = 1, 2, 3, 4$ are distinct primes with $p_1 < p_2 < p_3 < p_4$, the algebraic connectivity $\lambda_{n-1}(\mathcal{P}(\mathbb{Z}_n))$ of $\mathcal{P}(\mathbb{Z}_n)$ satisfies the inequality $\lambda_{n-1}(\mathcal{P}(\mathbb{Z}_n)) \leq \phi(n) + \sum_{i=1}^{4} p_i + \sum_{\substack{i,j=1,2 \\ i \neq j}}^{2,3} p_i p_j - 6$.*

Proof. For any graph G, the algebraic connectivity $\lambda_{n-1}(G)$ and the vertex connectivity $\mathcal{K}(G)$ of G satisfies the inequality $\lambda_{n-1}(G) \leq \mathcal{K}(G)$ [6]. Using this fact and the upper bound obtained

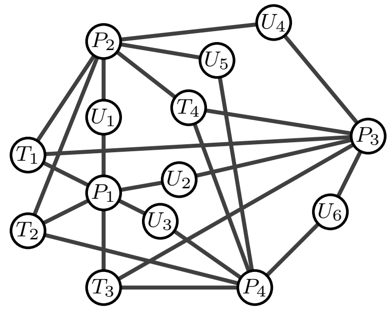

Figure 1: Connectedness diagram of $\mathcal{P}(\mathbb{Z}_n) - S$, where $n = \prod_{i=1}^{4} p_i$

for $\mathcal{K}(\mathcal{P}(\mathbb{Z}_n))$, where $n = \prod_{i=1}^{4} p_i$, $i = 1, 2, 3, 4$ are distinct primes with $p_1 < p_2 < p_3 < p_4$ in proposition 1, we can conclude the result. $\square$

Proposition 2. *For $n = \prod_{i=1}^{5} p_i$, where p_i, $i = 1, 2, 3, 4, 5$ are distinct primes with $p_1 < p_2 < p_3 < p_4 < p_5$, the vertex connectivity $\mathcal{K}(\mathcal{P}(\mathbb{Z}_n))$ of $\mathcal{P}(\mathbb{Z}_n)$ satisfies the inequality $\mathcal{K}(\mathcal{P}(\mathbb{Z}_n)) \leq \phi(n) + \sum_{i=1}^{5} p_i + \sum_{\substack{i,j=1,2 \\ i \neq j}}^{4,5} p_i p_j + \sum_{\substack{i,j,k=1,2,3 \\ i \neq j \neq k}}^{2,3,4} p_i p_j p_k - 18$.*

Proof. Let S be the subset of $\mathbb{Z}_n$ consisting of $\bar{0}$ and all the generators, $P_1 = \{a\bar{p_1} \in V(\mathcal{P}(\mathbb{Z}_n));\ p_2, p_3, p_4, p_5 \nmid a\}, P_2 = \{b\bar{p_2} \in V(\mathcal{P}(\mathbb{Z}_n));\ p_1, p_3, p_4, p_5 \nmid b\}, P_3 = \{c\bar{p_3} \in V(\mathcal{P}(\mathbb{Z}_n));\ p_1, p_2, p_4, p_5 \nmid c\}, P_4 = \{d\bar{p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ p_1, p_2, p_3, p_5 \nmid d\}, P_5 = \{e\bar{p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ p_1, p_2, p_3, p_4 \nmid e\}, U_1 = \{u_1\overline{p_1p_2} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_1 \leq p_3p_4p_5 - 1\}, U_2 = \{u_2\overline{p_1p_3} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_2 \leq p_2p_4p_5 - 1\}, U_3 = \{u_3\overline{p_1p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_3 \leq p_2p_3p_5 - 1\}, U_4 = \{u_4\overline{p_1p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_4 \leq p_2p_3p_4 - 1\}, U_5 = \{u_5\overline{p_2p_3} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_5 \leq p_1p_4p_5 - 1\}, U_6 = \{u_6\overline{p_2p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_6 \leq p_1p_3p_5 - 1\}, U_7 = \{u_7\overline{p_2p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_7 \leq p_1p_3p_4 - 1\}, U_8 = \{u_8\overline{p_3p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_8 \leq p_1p_2p_5 - 1\}, U_9 = \{u_9\overline{p_3p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_9 \leq p_1p_2p_4 - 1\}, U_{10} = \{u_{10}\overline{p_4p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_{10} \leq p_1p_2p_3 - 1\}, T_1 = \{t_1\overline{p_1p_2p_3} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_1 \leq p_4p_5 - 1\}, T_2 = \{t_2\overline{p_1p_2p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_2 \leq p_3p_5 - 1\}, T_3 = \{t_3\overline{p_1p_2p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_3 \leq p_3p_4 - 1\}, T_4 = \{t_4\overline{p_1p_3p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_4 \leq p_2p_5 - 1\}, T_5 = \{t_5\overline{p_1p_3p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_5 \leq p_2p_4 - 1\}, T_6 = \{t_6\overline{p_1p_4p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_6 \leq p_2p_3 - 1\}, T_7 = \{t_7\overline{p_2p_3p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_7 \leq p_1p_5 - 1\}, T_8 = \{t_8\overline{p_2p_3p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_8 \leq p_1p_4 - 1\}, T_9 = \{t_9\overline{p_2p_4p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_9 \leq p_1p_3 - 1\}, T_{10} = \{t_{10}\overline{p_3p_4p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{10} \leq p_1p_2 - 1\}, L_1 = \{l_1\overline{p_1p_2p_3p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_1 \leq p_5 - 1\}, L_2 = \{l_2\overline{p_1p_2p_3p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_2 \leq p_4 - 1\}, L_3 = \{l_3\overline{p_1p_2p_4p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_3 \leq p_3 - 1\}, L_4 = \{l_4\overline{p_1p_3p_4p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_4 \leq p_2 - 1\}$ and $L_5 = \{l_5\overline{p_2p_3p_4p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_5 \leq p_1 - 1\}$. Then all the sets S, P_i, U_j, T_j, L_i, $i = 1, \ldots, 5$ and $j = 1, 2, \ldots, 10$ are pairwise disjoint sets of vertices of $\mathcal{P}(\mathbb{Z}_n)$ whose union is $V(\mathcal{P}(\mathbb{Z}_n))$. Even though every vertex of the set S is adjacent to all other vertices of $\mathcal{P}(\mathbb{Z}_n)$, $\mathcal{P}(\mathbb{Z}_n) - S$ is connected. Moreover, $\mathcal{P}(\mathbb{Z}_n) - S - \sum_{i=1}^{5} L_i - \sum_{j=1}^{10} T_j$ is also connected. The connectedness diagram for $\mathcal{P}(\mathbb{Z}_n) - S - \sum_{i=1}^{5} L_i - \sum_{j=1}^{10} T_j$ can be obtained as in Figure 2. Now to make the graph $\mathcal{P}(\mathbb{Z}_n) - S - \sum_{i=1}^{5} L_i - \sum_{j=1}^{10} T_j$ disconnected, we need to remove the four sets from $U_j's$, $j = 1, 2, \ldots, 10$ which are adjacent to the same $P_i, i = 1, 2, \ldots, 5$. To make the upper bound of $\mathcal{K}(\mathcal{P}(\mathbb{Z}_n))$ sharp, we need to remove the sets S, L_i, T_j, $i = 1, 2, \ldots, 5$, $j = 1, 2, \ldots, 10$ along with the sets U_4, U_7, U_9, U_{10} with minimum cardinality. Therefore the graph $\mathcal{P}(\mathbb{Z}_n) - S - \sum_{j=1}^{10} T_j - \sum_{i=1}^{5} L_i - U_4 - U_7 - U_9 - U_{10}$ is disconnected and we have

$\mathcal{K}(\mathcal{P}(\mathbb{Z}_n)) \leq |S| + \sum_{i=1}^{5} |L_i| + \sum_{j=1}^{10} |T_j| + |U_4| + |U_7| + |U_9| + |U_{10}| = \phi(n) + \sum_{i=1}^{5} p_i + \sum_{\substack{i,j=1,2 \\ i \neq j}}^{4,5} p_i p_j + \sum_{\substack{i,j,k=1,2,3 \\ i \neq j \neq k}}^{2,3,4} p_i p_j p_k - 18.$ □

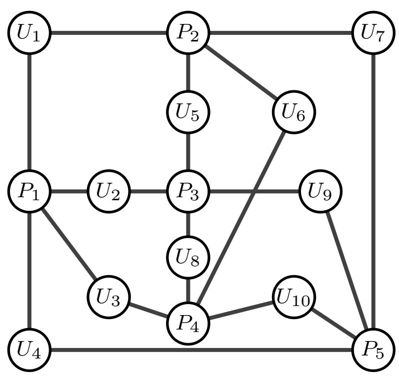

Figure 2: Connectedness diagram of $\mathcal{P}(\mathbb{Z}_n) - S - \sum_{i=1}^{5} L_i - \sum_{j=1}^{10} T_j$, where $n = \prod_{i=1}^{5} p_i$

Corollary 2. *For $n = \prod_{i=1}^{5} p_i$, where p_i, $i = 1, 2, 3, 4, 5$ are distinct primes with $p_1 < p_2 < p_3 < p_4 < p_5$, the algebraic connectivity $\lambda_{n-1}(\mathcal{P}(\mathbb{Z}_n))$ of $\mathcal{P}(\mathbb{Z}_n)$ satisfies the inequality $\lambda_{n-1}(\mathcal{P}(\mathbb{Z}_n)) \leq \phi(n) + \sum_{i=1}^{5} p_i + \sum_{\substack{i,j=1,2 \\ i \neq j}}^{4,5} p_i p_j + \sum_{\substack{i,j,k=1,2,3 \\ i \neq j \neq k}}^{2,3,4} p_i p_j p_k - 18$.*

Proof. For any graph G, the algebraic connectivity $\lambda_{n-1}(G)$ and the vertex connectivity $\mathcal{K}(G)$ of G satisfies the inequality $\lambda_{n-1}(G) \leq \mathcal{K}(G)$ [6]. Using this fact and the upper bound obtained for $\mathcal{K}(\mathcal{P}(\mathbb{Z}_n))$, where $n = \prod_{i=1}^{5} p_i$, $i = 1, 2, 3, 4, 5$ are distinct primes with $p_1 < p_2 < p_3 < p_4 < p_5$ in proposition 2, we can conclude the result. □

Proposition 3. *For $n = \prod_{i=1}^{6} p_i$, where p_i, $i = 1, 2, 3, 4, 5, 6$ are distinct primes with $p_1 < p_2 < p_3 < p_4 < p_5 < p_6$, the vertex connectivity $\mathcal{K}(\mathcal{P}(\mathbb{Z}_n))$ of $\mathcal{P}(\mathbb{Z}_n)$ satisfies the inequality* $\mathcal{K}(\mathcal{P}(\mathbb{Z}_n)) \leq \phi(n) + \sum_{i=1}^{6} p_i + \sum_{\substack{i,j=1,2 \\ i\neq j}}^{5,6} p_ip_j + \sum_{\substack{i,j,k=1,2,3 \\ i\neq j\neq k}}^{4,5,6} p_ip_jp_k + \sum_{\substack{i,j,k,l=1,2,3,4 \\ i\neq j\neq k\neq l}}^{2,3,4,5} p_ip_jp_kp_l - 45.$

Proof. Let S be the subset of $\mathbb{Z}_n$ consisting of $\bar{0}$ and all the generators, $P_1 = \{a_1\bar{p_1} \in V(\mathcal{P}(\mathbb{Z}_n));\ p_2, p_3, p_4, p_5, p_6 \nmid a_1\}, P_2 = \{a_2\bar{p_2} \in V(\mathcal{P}(\mathbb{Z}_n));\ p_1, p_3, p_4, p_5, p_6 \nmid a_2\}, P_3 = \{a_3\bar{p_3} \in V(\mathcal{P}(\mathbb{Z}_n));\ p_1, p_2, p_4, p_5, p_6 \nmid a_3\}, P_4 = \{a_4\bar{p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ p_1, p_2, p_3, p_5, p_6 \nmid a_4\}, P_5 = \{a_5\bar{p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ p_1, p_2, p_3, p_4, p_6 \nmid a_5\}, P_6 = \{a_6\bar{p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ p_1, p_2, p_3, p_4, p_5 \nmid a_6\}, U_{12} = \{u_{12}\overline{p_1p_2} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_{12} \leq p_3p_4p_5p_6 - 1\}, U_{13} = \{u_{13}\overline{p_1p_3} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_{13} \leq p_2p_4p_5p_6 - 1\}, U_{14} = \{u_{14}\overline{p_1p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_{14} \leq p_2p_3p_5p_6 - 1\}, U_{15} = \{u_{15}\overline{p_1p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_{15} \leq p_2p_3p_4p_6 - 1\}, U_{16} = \{u_{16}\overline{p_1p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_{16} \leq p_2p_3p_4p_5 - 1\}, U_{23} = \{u_{23}\overline{p_2p_3} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_{23} \leq p_1p_4p_5p_6 - 1\}, U_{24} = \{u_{24}\overline{p_2p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_{24} \leq p_1p_3p_5p_6 - 1\}, U_{25} = \{u_{25}\overline{p_2p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_{25} \leq p_1p_3p_4p_6 - 1\}, U_{26} = \{u_{26}\overline{p_2p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_{26} \leq p_1p_3p_4p_5 - 1\}, U_{34} = \{u_{34}\overline{p_3p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_{34} \leq p_1p_2p_5p_6 - 1\}, U_{35} = \{u_{35}\overline{p_3p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_{35} \leq p_1p_2p_4p_6 - 1\}, U_{36} = \{u_{36}\overline{p_3p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_{36} \leq p_1p_2p_4p_5 - 1\}, U_{45} = \{u_{45}\overline{p_4p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_{45} \leq p_1p_2p_3p_6 - 1\}, U_{46} = \{u_{46}\overline{p_4p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_{46} \leq p_1p_2p_3p_5 - 1\}, U_{56} = \{u_{56}\overline{p_5p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < u_{56} \leq p_1p_2p_3p_4 - 1\}, T_{123} = \{t_{123}\overline{p_1p_2p_3} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{123} \leq p_4p_5p_6 - 1\}, T_{124} = \{t_{124}\overline{p_1p_2p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{124} \leq p_3p_5p_6 - 1\}, T_{125} = \{t_{125}\overline{p_1p_2p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{125} \leq p_3p_4p_6 - 1\}, T_{126} = \{t_{126}\overline{p_1p_2p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{126} \leq p_3p_4p_5 - 1\}, T_{134} = \{t_{134}\overline{p_1p_3p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{134} \leq p_2p_5p_6 - 1\}, T_{135} = \{t_{135}\overline{p_1p_3p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{135} \leq p_2p_4p_6 - 1\}, T_{136} = \{t_{136}\overline{p_1p_3p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{136} \leq p_2p_4p_5 - 1\}, T_{145} = \{t_{145}\overline{p_1p_4p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{145} \leq p_2p_3p_6 - 1\}, T_{146} = \{t_{146}\overline{p_1p_4p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{146} \leq p_2p_3p_5 - 1\}, T_{156} = \{t_{156}\overline{p_1p_5p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{156} \leq p_2p_3p_4 - 1\}, T_{234} = \{t_{234}\overline{p_2p_3p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{234} \leq p_1p_5p_6 - 1\}, T_{235} = \{t_{235}\overline{p_2p_3p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{235} \leq p_1p_4p_6 - 1\}, T_{236} = \{t_{236}\overline{p_2p_3p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{236} \leq p_1p_4p_5 - 1\}, T_{245} = \{t_{245}\overline{p_2p_4p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{245} \leq p_1p_3p_6 - 1\}, T_{246} = \{t_{246}\overline{p_2p_4p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{246} \leq p_1p_3p_5 - 1\}, T_{256} = \{t_{256}\overline{p_2p_5p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{256} \leq p_1p_3p_4 - 1\}, T_{345} = \{t_{345}\overline{p_3p_4p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{345} \leq p_1p_2p_6 - 1\}, T_{346} = \{t_{346}\overline{p_3p_4p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{346} \leq p_1p_2p_5 - 1\}, T_{356} = \{t_{356}\overline{p_3p_5p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{356} \leq p_1p_2p_4 - 1\}, T_{456} = \{t_{456}\overline{p_4p_5p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < t_{456} \leq p_1p_2p_3 - 1\}, L_{1234} = \{l_{1234}\overline{p_1p_2p_3p_4} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_{1234} \leq p_5p_6 - 1\}, L_{1235} = \{l_{1235}\overline{p_1p_2p_3p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_{1235} \leq p_4p_6 - 1\}, L_{1236} = \{l_{1236}\overline{p_1p_2p_3p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_{1236} \leq p_4p_5 - 1\}, L_{1245} = \{l_{1245}\overline{p_1p_2p_4p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_{1245} \leq p_3p_6 - 1\}, L_{1246} = \{l_{1246}\overline{p_1p_2p_4p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_{1246} \leq p_3p_5 - 1\}, L_{1256} = \{l_{1256}\overline{p_1p_2p_5p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_{1256} \leq p_3p_4 - 1\}, L_{1345} = \{l_{1345}\overline{p_1p_3p_4p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_{1345} \leq p_2p_6 - 1\}, L_{1346} = \{l_{1346}\overline{p_1p_3p_4p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_{1346} \leq p_2p_5 - 1\}, L_{1356} = \{l_{1356}\overline{p_1p_3p_5p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_{1356} \leq p_2p_4 - 1\}, L_{2345} = \{l_{2345}\overline{p_2p_3p_4p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_{2345} \leq p_1p_6 - 1\}, L_{2346} = \{l_{2346}\overline{p_2p_3p_4p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_{2346} \leq p_1p_5 - 1\}, L_{2356} = \{l_{2356}\overline{p_2p_3p_5p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_{2356} \leq p_1p_4 - 1\}, L_{2456} = \{l_{2456}\overline{p_2p_4p_5p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_{2456} \leq p_1p_3 - 1\}, L_{3456} = \{l_{3456}\overline{p_3p_4p_5p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_{3456} \leq p_1p_2 - 1\}, L_{1456} = \{l_{1456}\overline{p_1p_4p_5p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < l_{1456} \leq p_2p_3 - 1\}, J_{12345} = \{j_{12345}\overline{p_1p_2p_3p_4p_5} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < j_{12345} \leq p_6 - 1\}, J_{12346} = \{j_{12346}\overline{p_1p_2p_3p_4p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < j_{12346} \leq p_5 - 1\}, J_{12356} = \{j_{12356}\overline{p_1p_2p_3p_5p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < j_{12356} \leq p_4 - 1\}, J_{12456} = \{j_{12456}\overline{p_1p_2p_4p_5p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < j_{12456} \leq p_3 - 1\}, J_{13456} = \{j_{13456}\overline{p_1p_3p_4p_5p_6} \in V(\mathcal{P}(\mathbb{Z}_n));\ 0 < j_{13456} \leq p_2 - 1\}, J_{23456} = \{j_{23456}\overline{p_2p_3p_4p_5p_6} \in$

$V(\mathcal{P}(\mathbb{Z}_n))$; $0 < j_{23456} \le p_1 - 1\}$ be pairwise disjoint sets of vertices of $\mathcal{P}(\mathbb{Z}_n)$ whose union is $V(\mathcal{P}(\mathbb{Z}_n))$. Even though every vertex of the set S is adjacent to all other vertices of $\mathcal{P}(\mathbb{Z}_n)$, $\mathcal{P}(\mathbb{Z}_n) - S$ is connected. Moreover, $\mathcal{P}(\mathbb{Z}_n) - S - \sum\limits_{\substack{i,j,k=1,2,3 \\ i\neq j\neq k}}^{4,5,6} T_{ijk} - \sum\limits_{\substack{i,j,k,l=1,2,3,4 \\ i\neq j\neq k\neq l}}^{3,4,5,6} L_{ijkl} - \sum\limits_{\substack{i,j,k,l,m=1,2,3,4,5 \\ i\neq j\neq k\neq l\neq m}}^{2,3,4,5,6} J_{ijklm}$ is also connected. The connectedness diagram of $\mathcal{P}(\mathbb{Z}_n) - S - \sum\limits_{\substack{i,j,k=1,2,3 \\ i\neq j\neq k}}^{4,5,6} T_{ijk} - \sum\limits_{\substack{i,j,k,l=1,2,3,4 \\ i\neq j\neq k\neq l}}^{3,4,5,6} L_{ijkl} - \sum\limits_{\substack{i,j,k,l,m=1,2,3,4,5 \\ i\neq j\neq k\neq l\neq m}}^{2,3,4,5,6} J_{ijklm}$ can be obtained as shown in Figure 3. Now to make the graph $\mathcal{P}(\mathbb{Z}_n) - S - \sum\limits_{\substack{i,j,k=1,2,3 \\ i\neq j\neq k}}^{4,5,6} T_{ijk} - \sum\limits_{\substack{i,j,k,l=1,2,3,4 \\ i\neq j\neq k\neq l}}^{3,4,5,6} L_{ijkl} - \sum\limits_{\substack{i,j,k,l,m=1,2,3,4,5 \\ i\neq j\neq k\neq l\neq m}}^{2,3,4,5,6} J_{ijklm}$ disconnected, we need to remove the five sets from U_{ij}, where $i = 1, 2, \ldots, 5$, $j = 2, \ldots, 6, i \neq j$ which are adjacent to the same $P_i, i = 1, 2, \ldots, 6$. To make the upper bound of $\mathcal{P}(\mathbb{Z}_n)$ sharp , we need to remove the sets $U_{46}, U_{26}, U_{36}, U_{56}, U_{16}$ with minimum cardinality from $\mathcal{P}(\mathbb{Z}_n) - S - \sum\limits_{\substack{i,j,k=1,2,3 \\ i\neq j\neq k}}^{4,5,6} T_{ijk} - \sum\limits_{\substack{i,j,k,l=1,2,3,4 \\ i\neq j\neq k\neq l}}^{3,4,5,6} L_{ijkl} - \sum\limits_{\substack{i,j,k,l,m=1,2,3,4,5 \\ i\neq j\neq k\neq l\neq m}}^{2,3,4,5,6} J_{ijklm}$. Therefore the graph

$$\mathcal{P}(\mathbb{Z}_n) - S - \sum_{\substack{i,j,k=1,2,3 \\ i\neq j\neq k}}^{4,5,6} T_{ijk} - \sum_{\substack{i,j,k,l=1,2,3,4 \\ i\neq j\neq k\neq l}}^{3,4,5,6} L_{ijkl} - \sum_{\substack{i,j,k,l,m=1,2,3,4,5 \\ i\neq j\neq k\neq l\neq m}}^{2,3,4,5,6} J_{ijklm} - U_{46} - U_{26} - U_{36} - U_{56} - U_{16}$$

is disconnected and thus

$$\mathcal{K}(\mathcal{P}(\mathbb{Z}_n)) \le |S| + \sum_{\substack{i,j,k=1,2,3 \\ i\neq j\neq k}}^{4,5,6} |T_{ijk}| + \sum_{\substack{i,j,k,l=1,2,3,4 \\ i\neq j\neq k\neq l}}^{3,4,5,6} |L_{ijkl}| + \sum_{\substack{i,j,k,l,m=1,2,3,4,5 \\ i\neq j\neq k\neq l\neq m}}^{2,3,4,5,6} |J_{ijklm}| + |U_{46}| + |U_{26}| + |U_{36}| + |U_{56}| + |U_{16}| = \phi(n) + \sum_{i=1}^{6} p_i + \sum_{\substack{i,j=1,2 \\ i\neq j}}^{5,6} p_i p_j + \sum_{\substack{i,j,k=1,2,3 \\ i\neq j\neq k}}^{4,5,6} p_i p_j p_k + \sum_{\substack{i,j,k,l=1,2,3,4 \\ i\neq j\neq k\neq l}}^{2,3,4,5} p_i p_j p_k p_l - 45.$$

□

Corollary 3. *For $n = \prod_{i=1}^{6} p_i$, where p_i, $i = 1, 2, 3, 4, 5, 6$ are distinct primes with $p_1 < p_2 < p_3 < p_4 < p_5 < p_6$, the algebraic connectivity $\lambda_{n-1}(\mathcal{P}(\mathbb{Z}_n))$ of $\mathcal{P}(\mathbb{Z}_n)$ satisfies the inequality* $\lambda_{n-1}(\mathcal{P}(\mathbb{Z}_n)) \le \phi(n) + \sum\limits_{i=1}^{6} p_i + \sum\limits_{\substack{i,j=1,2 \\ i\neq j}}^{5,6} p_i p_j + \sum\limits_{\substack{i,j,k=1,2,3 \\ i\neq j\neq k}}^{4,5,6} p_i p_j p_k + \sum\limits_{\substack{i,j,k,l=1,2,3,4 \\ i\neq j\neq k\neq l}}^{2,3,4,5} p_i p_j p_k p_l - 45.$

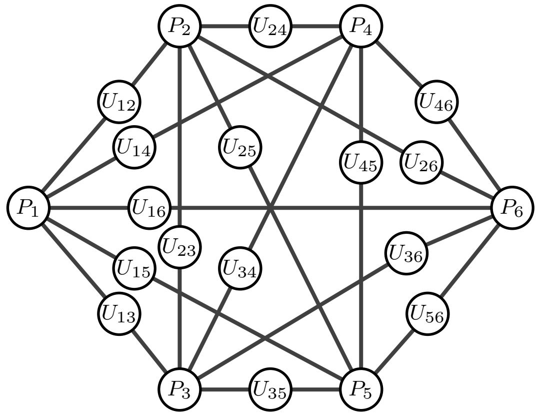

Figure 3: Connectedness diagram of $\mathcal{P}(\mathbb{Z}_n) - S - \sum\limits_{\substack{i,j,k=1,2,3 \\ i\neq j\neq k}}^{4,5,6} T_{ijk} - \sum\limits_{\substack{i,j,k,l=1,2,3,4 \\ i\neq j\neq k\neq l}}^{3,4,5,6} L_{ijkl} - \sum\limits_{\substack{i,j,k,l,m=1,2,3,4,5 \\ i\neq j\neq k\neq l\neq m}}^{2,3,4,5,6} J_{ijklm}$, where $n = \prod\limits_{i=1}^{6} p_i$

Proposition 4. *Let $n = \prod\limits_{i=1}^{4} p_i$, where $p_i, i = 1, 2, 3, 4$ are distinct primes with $p_1 < p_2 < p_3 < p_4$. Then the vertex connectivity $\mathcal{K}(\mathcal{P}(G))$ of $\mathcal{P}(G)$, where G is a finite abelian group of order n satisfies the inequality $\mathcal{K}(\mathcal{P}(G)) \leq \phi(n) + \sum\limits_{i=1}^{4} p_i + \sum\limits_{\substack{i,j=1,2 \\ i\neq j}}^{2,3} p_i p_j - 6$.*

Proof. Let G be a finite abelian group of order $n = \prod\limits_{i=1}^{4} p_i$, where $p_i, i = 1, 2, 3, 4$ are distinct primes with $p_1 < p_2 < p_3 < p_4$. By the Fundamental Theorem of finite abelian groups [7], G is isomorphic to the direct product $\mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \mathbb{Z}_{p_3} \oplus \mathbb{Z}_{p_4}$. Since $p_i's$ are distinct primes, $\mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \mathbb{Z}_{p_3} \oplus \mathbb{Z}_{p_4}$ is isomorphic to $\mathbb{Z}_{p_1p_2p_3p_4}$. Thus G is isomorphic to $\mathbb{Z}_{p_1p_2p_3p_4}$. Hence the result, by Proposition 1. □

Corollary 4. *Let $n = \prod\limits_{i=1}^{4} p_i$, where $p_i, i = 1, 2, 3, 4$ are distinct primes with $p_1 < p_2 < p_3 < p_4$. Then the algebraic connectivity $\lambda_{n-1}(\mathcal{P}(G))$ of $\mathcal{P}(G)$, where G is a finite abelian group of order n satisfies the inequality $\lambda_{n-1}(\mathcal{P}(G)) \leq \phi(n) + \sum\limits_{i=1}^{4} p_i + \sum\limits_{\substack{i,j=1,2 \\ i\neq j}}^{2,3} p_i p_j - 6$.*

Proof. For any graph G, the algebraic connectivity $\lambda_{n-1}(G)$ and the vertex connectivity $\mathcal{K}(G)$ of G satisfies the inequality $\lambda_{n-1}(G) \leq \mathcal{K}(G)$ [6]. Using this fact and the proposition 4, we can conclude the result. □

On the similar lines, we can prove the following Propositions 5, 6 and their respective corollaries 5, 6.

Proposition 5. *Let* $n = \prod_{i=1}^{5} p_i$*, where* $p_i, i = 1, 2, \ldots, 5$ *are distinct primes with* $p_1 < p_2 < p_3 < p_4 < p_5$*. Then the vertex connectivity* $\mathcal{K}(\mathcal{P}(G))$ *of* $\mathcal{P}(G)$*, where* G *is a finite abelian group of order* n *satisfies the inequality* $\mathcal{K}(\mathcal{P}(G)) \leq \phi(n) + \sum_{i=1}^{5} p_i + \sum_{\substack{i,j=1,2\\ i\neq j}}^{4,5} p_i p_j + \sum_{\substack{i,j,k=1,2,3\\ i\neq j\neq k}}^{2,3,4} p_i p_j p_k - 18.$

Corollary 5. *Let* $n = \prod_{i=1}^{5} p_i$*, where* $p_i, i = 1, 2, \ldots, 5$ *are distinct primes with* $p_1 < p_2 < p_3 < p_4 < p_5$*. Then the algebraic connectivity* $\lambda_{n-1}(\mathcal{P}(G))$ *of* $\mathcal{P}(G)$*, where* G *is a finite abelian group of order* n *satisfies the inequality* $\lambda_{n-1}(\mathcal{P}(G)) \leq \phi(n) + \sum_{i=1}^{5} p_i + \sum_{\substack{i,j=1,2\\ i\neq j}}^{4,5} p_i p_j + \sum_{\substack{i,j,k=1,2,3\\ i\neq j\neq k}}^{2,3,4} p_i p_j p_k - 18.$

Proposition 6. *Let* $n = \prod_{i=1}^{6} p_i$*, where* $p_i, i = 1, 2, \ldots, 6$ *are distinct primes with* $p_1 < p_2 < p_3 < p_4 < p_5 < p_6$*. Then the vertex connectivity* $\mathcal{K}(\mathcal{P}(G))$ *of* $\mathcal{P}(G)$*, where* G *is a finite abelian group of order* n *satisfies the inequality* $\mathcal{K}(\mathcal{P}(G)) \leq \phi(n) + \sum_{i=1}^{6} p_i + \sum_{\substack{i,j=1,2\\ i\neq j}}^{5,6} p_i p_j + \sum_{\substack{i,j,k=1,2,3\\ i\neq j\neq k}}^{4,5,6} p_i p_j p_k + \sum_{\substack{i,j,k,l=1,2,3,4\\ i\neq j\neq k\neq l}}^{2,3,4,5} p_i p_j p_k p_l - 45.$

Corollary 6. *Let* $n = \prod_{i=1}^{6} p_i$*, where* $p_i, i = 1, 2, \ldots, 6$ *are distinct primes with* $p_1 < p_2 < p_3 < p_4 < p_5 < p_6$*. Then the algebraic connectivity* $\lambda_{n-1}(\mathcal{P}(G))$ *of* $\mathcal{P}(G)$*, where* G *is a finite abelian group of order* n *satisfies the inequality* $\lambda_{n-1}(\mathcal{P}(G)) \leq \phi(n) + \sum_{i=1}^{6} p_i + \sum_{\substack{i,j=1,2\\ i\neq j}}^{5,6} p_i p_j + \sum_{\substack{i,j,k=1,2,3\\ i\neq j\neq k}}^{4,5,6} p_i p_j p_k + \sum_{\substack{i,j,k,l=1,2,3,4\\ i\neq j\neq k\neq l}}^{2,3,4,5} p_i p_j p_k p_l - 45.$

Proposition 7 ([4])**.** *For any integer* $n \geq 2$*, if* n *is a prime power or the product of two primes, then a power graph* $\mathcal{P}(\mathbb{Z}_n)$ *is a Laplacian integral.*

Proposition 8. *For any integer* $n \geq 2$*, if a power graph* $\mathcal{P}(\mathbb{Z}_n)$ *is a Laplacian integral, then the algebraic connectivity of a power graph* $\mathcal{P}(\mathbb{Z}_n)$ *is an integer.*

Proof. The algebraic connectivity of $\mathcal{P}(\mathbb{Z}_n)$ is the second smallest Laplacian eigenvalue of $\mathcal{P}(\mathbb{Z}_n)$. Moreover, $\mathcal{P}(\mathbb{Z}_n)$ is Laplacian integral if and only if each of it's Laplacian eigenvalue is an integer. Hence the result. □

Proposition 9. *For any integer* $n \geq 2$*, if* n *is a prime power or the product of two primes, then the algebraic connectivity of a power graph* $\mathcal{P}(\mathbb{Z}_n)$ *is an integer.*

Proof. If n is a prime power, then $\mathcal{P}(\mathbb{Z}_n)$ is Laplacian integral, by Theorem 1. Hence the algebraic connectivity of $\mathcal{P}(\mathbb{Z}_n)$ is an integer. Also, if n is the product of two distinct primes, then $\mathcal{K}(\mathcal{P}(\mathbb{Z}_n)) = \lambda_{n-1}(\mathcal{P}(\mathbb{Z}_n))$, which is an integer, by Theorem 5. Hence the result. □

Proposition 10. *For any integer $n \geq 2$, if the algebraic connectivity of a power graph $\mathcal{P}(\mathbb{Z}_n)$ is an integer, then n is a prime power or the product of two primes.*

Proof. Assume that the algebraic connectivity of $\mathcal{P}(\mathbb{Z}_n)$ is an integer for all values of n. If n is a prime power or the product of two primes, then the algebraic connectivity of $\mathcal{P}(\mathbb{Z}_n)$ is an integer, see Proposition 9. Let us consider values of n, where n is neither a prime power nor the product of two primes. Then n will include the values of the form $p^\alpha q^\beta$, with $\alpha, \beta \geq 1$, but not both equal to 1. Thus $\phi(n) + 1 < \lambda_{n-1}(\mathcal{P}(\mathbb{Z}_n)) < \phi(n) + p^{\alpha-1}q^{\beta-1}$, by Theorem 3,4. Thus $\lambda_{n-1}(\mathcal{P}(\mathbb{Z}_n))$ is not necessarily an integer. In particular, if we consider $n = 12$, then $5 < \lambda_{n-1}(\mathcal{P}(\mathbb{Z}_{12})) < 6$, which is not an integer. Therefore we get a contradiction to our assumption. Hence the result. □

Example 1. Consider $\mathcal{P}(\mathbb{Z}_{18})$. The Laplacian characteristic polynomial of $\mathcal{P}(\mathbb{Z}_{18})$ is given by

$$\Theta(\mathcal{P}(\mathbb{Z}_{18}), x) = \frac{x(x-18)^7}{(x-7)} \Theta(L_{\bar{0},\bar{1},\bar{5},\bar{7},\bar{1}1,\bar{1}3,\bar{1}7}(\mathcal{P}(\mathbb{Z}_{18})), x) \tag{1}$$

where $L_{\bar{0},\bar{1},\bar{5},\bar{7},\bar{1}1,\bar{1}3,\bar{1}7}(\mathcal{P}(\mathbb{Z}_{18}))$ is the principal submatrix of $L(\mathcal{P}(\mathbb{Z}_{18}))$ formed by deleting rows and columns corresponding to $\bar{0}, \bar{1}, \bar{5}, \bar{7}, \bar{1}1, \bar{1}3, \bar{1}7$ [4]. Since $\bar{0}$ and the generators of $\mathbb{Z}_{18}$ are adjacent to all other vertices in $\mathcal{P}(\mathbb{Z}_{18})$, the degree of $\bar{0}, \bar{1}, \bar{5}, \bar{7}, \bar{1}1, \bar{1}3, \bar{1}7$ is 17 in $\mathcal{P}(\mathbb{Z}_{18})$. Moreover, if $x \in \mathbb{Z}_{18}$ is non-generator, then $deg(x)$ is equal to $\phi(18) + 1+$ number of elements other than $\bar{0}, \bar{1}, \bar{5}, \bar{7}, \bar{1}1, \bar{1}3, \bar{1}7$ adjacent to x in $\mathcal{P}(\mathbb{Z}_{18})$. Also, $a, b \in \mathbb{Z}_{18}$ are adjacent in $\mathcal{P}(\mathbb{Z}_{18})$ if and only if $ax \equiv b(mod\, 18)$ or $bx \equiv a(mod\, 18)$. Therefore we have the matrix $L_{\bar{0},\bar{1},\bar{5},\bar{7},\bar{1}1,\bar{1}3,\bar{1}7}(\mathcal{P}(\mathbb{Z}_{18}))$, whose eigenvalues are the roots of characteristic polynomial $\Theta(L_{\bar{0},\bar{1},\bar{5},\bar{7},\bar{1}1,\bar{1}3,\bar{1}7}(\mathcal{P}(\mathbb{Z}_{18})), x)$, see matrix 1. The eigenvalues of the matrix $L_{\bar{0},\bar{1},\bar{5},\bar{7},\bar{1}1,\bar{1}3,\bar{1}7}(\mathcal{P}(\mathbb{Z}_{18}))$ are $7, 12, 17, 15, 15, 15, 15, 15,\ 8.159, 10.768, 17.073$. By equation 1, the Laplacian spectrum of $\mathcal{P}(\mathbb{Z}_{18})$ is given by $\begin{pmatrix} 18 & 17.073 & 17 & 15 & 12 & 10.768 & 8.159 & 0 \\ 7 & 1 & 1 & 5 & 1 & 1 & 1 & 1 \end{pmatrix}$. Hence we conclude that the algebraic connectivity of $\mathcal{P}(\mathbb{Z}_{18})$ is 8.519, which is not an integer. Moreover, $\mathcal{P}(\mathbb{Z}_{18})$ is not a Laplacian integral.

$$L_{\bar{0},\bar{1},\bar{5},\bar{7},\bar{11},\bar{13},\bar{17}}(\mathcal{P}(\mathbb{Z}_{18})) = \begin{bmatrix} 14 & 0 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & -1 \\ 0 & 11 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 14 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 16 & -1 & 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & 14 & 0 & -1 & -1 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 & 14 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & -1 & 0 & -1 & 16 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & -1 & 0 & -1 & -1 & 14 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 11 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 14 \end{bmatrix}$$

Example 2. Using the same method as that of example 1, the Laplacian spectrum of $\mathcal{P}(\mathbb{Z}_{12})$ is obtained as

$$\begin{pmatrix} 12 & 10.68 & 10 & 9 & 8.64 & 8 & 5.67 & 0 \\ 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

. Thus we conclude that the algebraic connectivity of $\mathcal{P}(\mathbb{Z}_{12})$ is 5.67, which is not an integer. Moreover, $\mathcal{P}(\mathbb{Z}_{12})$ is not a Laplacian integral.

Using propositions 7, 8, 9 and 10, we can conclude the following conjecture posed in [11]; For any integer $n \geq 2$, the following statements are equivalent:

(i) The algebraic connectivity of $\mathcal{P}(\mathbb{Z}_n)$ is an integer.

(ii) $\mathcal{P}(\mathbb{Z}_n)$ is Laplacian integral.

(iii) n is a prime power or product of two primes.

3 Conclusion

In this article, we have obtained the upper bounds for the algebraic and the vertex connectivity of $\mathcal{P}(\mathbb{Z}_n)$, where n is a product of 4, 5 and 6 distinct primes. Moreover, we proved the equivalent conditions for $\mathcal{P}(\mathbb{Z}_n)$ to be Laplacian integral and hence settled the conjecture posed in [11]. Based on our observations, we state the following for $\mathbb{Z}_n$:

Conjecture 1. *Let $n = \prod_{j=1}^{k} p_{i_j}$, where $p_{i_{m_1}} < p_{i_{m_2}}$ for $m_1 < m_2$ are distinct primes and $k, m_1, m_2 \in \mathbb{N}$. Then the algebraic connectivity $\lambda_{n-1}(\mathcal{P}(\mathbb{Z}_n))$ of power graph $\mathcal{P}(\mathbb{Z}_n)$ satisfies the inequality*

$$\lambda_{n-1}(\mathcal{P}(\mathbb{Z}_n)) \leq \phi(n)+1+\sum_{j=1}^{k} p_{i_j} - \binom{k}{1} + \sum_{\substack{j_1,j_2=1,2 \\ j_1\neq j_2}}^{k-1,k} p_{i_{j_1}}p_{i_{j_2}} - \binom{k}{2} + \sum_{\substack{j_1,j_2,j_3=1,2,3 \\ j_1\neq j_2\neq j_3}}^{k-2,k-1,k} p_{i_{j_1}}p_{i_{j_2}}p_{i_{j_3}} - \binom{k}{3} + \ldots + \sum_{\substack{j_1,j_2,\ldots,j_{k-3}=1,2,\ldots,k-3 \\ j_1\neq j_2\neq\ldots\neq j_{k-3}}}^{4,5,\ldots,k} p_{i_{j_1}}p_{i_{j_2}}\cdots p_{i_{j_{k-3}}} - \binom{k}{k-3} + \sum_{\substack{j_1,j_2,\ldots,j_{k-2}=1,2,\ldots,k-2 \\ j_1\neq j_2\neq\ldots\neq j_{k-2}}}^{2,3,\ldots,k-1} p_{i_{j_1}}p_{i_{j_2}}\cdots p_{i_{j_{k-2}}} - (k-1).$$

The eigenvalues of the matrices in example 1 and 2 are calculated using WX-Maxima.

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