Journal of Algebra and Related Topics
Vol. 14, No. S1, 2026, pp. 257-265. Research Article

Identification Gorenstein rings via special semidualizing modules

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Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be a finitely generated R-module such that $\operatorname{Hom}_R(M, R) \cong \overset{n}{\underset{i=1}{\oplus}} C$ for some positive integer n. We try to present new characterizations of Gorenstein rings via M and C. It is proved that if $\operatorname{depth} R = 0$ and $\operatorname{id}_R(M) < \infty$ then R is Gorenstein. Also, it is shown that if M is a Cohen-Macaulay R-module with finite injective dimension, then R is Gorenstein.

Keywords: The Auslander-Reiten conjecture, Semidualizing modules, Free modules.
AMS Subject Classification 2010: 13C05, 13D07,13H10.

1 Introduction

Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. A finitely generated R-module C is called a semidualizing module if $\operatorname{Hom}_R(C, C) \cong R$ and $\operatorname{Ext}^i_R(C, C) = 0$ for all $i > 0$. For instance R always is a semidualizing module and over a Cohen-Macaulay local ring, the canonical module, is a semidualizing module.

As a generalization of the ring, a semidualizing module behaves like R in some properties, like the Krull dimension, depth, base change and etc. Also, when a semidualizing R-module has finite projective dimension, it must be isomorphic to R and if a semidualizing R-module C has finite injective dimension then R is Cohen-Macaulay with the canonical module C. When the canonical module of a Cohen-Macaulay ring R is isomorphic to the ring itself, then R must be Gorenstein, and this maybe a good motivation to identify Gorenstein ring via semidualizing module.

The properties of a semidualizing R-module C, when C is a dual (i.e., $C \cong M^*$ for some finitely

*Corresponding author
Received: 30 September 2024/ Revised: 12 May 2025/ Accepted: 14 May 2025
DOI: 10.22124/JART.2025.28567.1720

generated R-module M) is studied in [1]. For instance, if C is a dual, then it must be reflexive [1, Theorem 3.8]. Also, over a one dimensional Cohen-Macaulay ring, a reflexive semidualizing R-module is isomorphic to R [1, Theorem 4.9]. Another result over a one dimensional Cohen-Macaulay ring R, says that if $M^* \cong C$, where C is a semidualizing R-module and M is a finitely generated R-module with finite injective dimension, then R must be Gorenstein [1, Corollary 3.11].

In this paper, our main goal is to characterize Gorenstein rings, via some special semidaulizing modules. These class of semidualizing R-modules satisfy the condition $M^* \cong \overset{r}{\underset{i=1}{\oplus}} C$, where M is a finitely generated R-module and n is a positive integer. One of the main result is:

Theorem A. Let $\operatorname{depth} R = 0$ and C be a semidualizing R-module. Suppose that M is a finitely generated R-module such that $\mathrm{id}_R(M) < \infty$ and $M^* \cong \overset{n}{\underset{i=1}{\oplus}} C$ for some $n \in \mathbb{N}$. Then R is Gorenstein.

If we replace the condition $\operatorname{depth} R = 0$ by the condition "M is a Cohen-Macaulay R-module" then the assertion holds true. In other words:

Theorem B. Let C be a semidualizing R-module and M be a finitely generated Cohen-Macaulay R-module with finite injective dimension. If $M^* \cong \overset{n}{\underset{i=1}{\oplus}} C$ for some $n \in \mathbb{N}$, then R is Gorenstein and thus $C \cong R$.

Indeed, according to Theorem B, R is Gorenstein if, and only if there exist a semidualizing R module C and a Cohen-Macaulay R-module M with $\mathrm{id}_R M < \infty$, such that M^* is isomorphic to a direct sum of copies of C.

Also, there is a characterization of Gorenstein rings via semidualizing modules and their duals (Corollary 4).

2 Preliminaries

Throughout this paper all rings are commutative with $1 \neq 0$, Noetherian and local.

Definition 1. *The injective dimension of an R-module M, is denoted by $\mathrm{id}_R(M)$ and is defined as*

$$\mathrm{id}_R(M) := \inf\{n \,:\, \mathrm{Ext}^i_R(N, M) = 0, \textit{ for all } i > 0 \textit{ and } R\textit{-modules} \quad N\}.$$

This notion generalize the concept of injective modules and plays an imortant role in the studying of homological properties of modules. For instance, a ring R is called a Gorenstien ring, whenever $\mathrm{id}_R R < \infty$. There are many useful results about finitely generated modules of finite injective dimension in [3, Section 3.1].

For a finitely generated R-module M, $\mu(M)$ is the number of minimal generators of M and $r_R(M)$ denotes the type of M which is defined as $\mathrm{vdim}_{\mathrm{R}/\mathfrak{m}} \mathrm{Ext}_R^{\operatorname{depth} M}(R/\mathfrak{m}, M))$. We say that M has rank r, whenever $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module of rank r for all $\mathfrak{p} \in \mathrm{Ass}_R(R)$. We use M^* to denote $\mathrm{Hom}_R(M, R)$. Finally, R is called a generically Gorenstein ring, whenever $R_\mathfrak{p}$ is a Gorenstein ring for all $\mathfrak{p} \in \mathrm{Ass}_R R$.

Now, we recall the notion of semidualizing modules and some results which are needed in the rest of the paper.

Definition 2. *Let C be a finitely generated R-module. Then C is called a semidualizing module, when $\alpha_C : R \to \mathrm{Hom}_R(C, C)$ which is defined as $f(r)(c) = rc$ for any $r \in R$ and $c \in C$, is an isomorphism and $\mathrm{Ext}^i_R(C, C) = 0$ for each $i > 0$.*

Corollary 1. *Every semidualizing R-module is an indecomposable R-module.*

Proof. Let C be a semidualizing R-module and let $C \cong L \oplus N$ for some (finitely generated) R-modules L and N. As $\mathrm{Hom}_R(C, C) \cong R$, one has

$$\mathrm{Hom}_R(L, L) \oplus \mathrm{Hom}_R(L, N) \oplus \mathrm{Hom}_R(N, L) \oplus \mathrm{Hom}_R(N, N) \cong R.$$

But one of the R-modules $\mathrm{Hom}_R(L, L)$ or $\mathrm{Hom}_R(N, N)$ must be zero, since R is indecomposable. Therefore $L = 0$ or $N = 0$, which completes the proof. □

Definition 3. *Let C be a semidualizing R-module. If $\mathrm{id}_R C < \infty$, then C is called a dualizing module.*

Remark 1. *By Remark 2, if the dualizing module exists, then R is Cohen-Macaulay. One can check that the the canonical module of a Cohen-Macaulay (see [3, Definition 3.3.1]) is coincide with the dualizing module. Also, when R is an Artinian ring, then $\mathrm{E}_R(R/\mathfrak{m})$ is the canonical module of R.*

Proposition 1. *The following statements are hold for a semidualizing module C:*

(i) *One has $dim\, R = dim\, C$, $\mathrm{depth}\, R = \mathrm{depth}\, C$ and $\mathrm{Ass}_R R = \mathrm{Ass}_R C$. Consequently, $\mathrm{Spec}(R) = \mathrm{Supp}\, C$.*

(ii) *C is faithful, that is $\mathrm{Ann}_R(C) = 0$.*

(iii) *The sequence $x_1, \ldots, x_n$ ($n \in \mathbb{N}$) is an R-sequence if, and only if it is a C-sequnece.*

Proof. See [8, Proposition 2.1.16]. □

Remark 2. *Let M be a finitely generated R-module such that $\mathrm{id}_R M < \infty$. Then, by [3, Corollary 9.6.2 and Remark 9.6.4], R is a Cohen-Macaulay ring.*

3 Main results

Theorem 1. *Let W be a finitely generated R-module with $\mathrm{depth}\, W = \mathrm{depth}\, R = d$ and let $\underline{x} = x_1, \ldots, x_d$ be a sequence of elements of R which is both R and W-sequence. Suppose that $\mathrm{Ext}^{d+1}(k, W) = 0$. Then R is Cohen-Macaulay.*

Proof. By [6, Lemma 2, Page 140], we have

$$\mathrm{Ext}^1_{R/\underline{x}}(k, W/\underline{x}W) \cong \mathrm{Ext}^{d+1}_R(k, W) = 0.$$

Hence we may assume that, $\mathrm{depth}\, R = \mathrm{depth}\, W = 0$ and $\mathrm{Ext}^1_R(k, W) = 0$. Our aim is to show that R is Artinian. To do this, let M be a finite length module. First, we use induction on $l_R(M)$ to show that $l_R(\mathrm{Hom}_R(M,W)) = l_R(M) \cdot l_R(\mathrm{Hom}_R(k,W))$ and $\mathrm{Ext}^1_R(M,W) = 0$. When $l_R(M) = 0$ or 1, we have nothing to prove. So, let $L_R(M) > 1$ and suppose that the assertion holds for all positive integers less than $l_R(M)$. There exists an exact sequence $0 \to N \to M \to k \to 0$, where N is a submodule of M with $l_R(N) = l_R(M) - 1$, so that $\mathrm{Ext}^1_R(N,W) = 0$ and $l_R(\mathrm{Hom}_R(N,W)) = l_R(N) \cdot l_R(\mathrm{Hom}_R(k,W))$, by induction hypothesis. Applying $\mathrm{Hom}_R(-,W)$ to the above exact sequence, yields the following exact sequence:

$$0 \to \mathrm{Hom}_R(k,W) \to \mathrm{Hom}_R(M,W) \to \mathrm{Hom}_R(N,W) \to \mathrm{Ext}^1_R(k,W) \to \cdots .$$

As $\mathrm{Ext}^1_R(N,W) = \mathrm{Ext}^1_R(k,W) = 0$, we deduce that $\mathrm{Ext}^1_R(M,W) = 0$. Hence the above exact sequence yields into the following:

$$0 \to \mathrm{Hom}_R(k,W) \to \mathrm{Hom}_R(M,W) \to \mathrm{Hom}_R(N,W) \to 0.$$

Therefore,

$$\begin{aligned} l_R(\mathrm{Hom}_R(M,W)) &= l_R(\mathrm{Hom}_R(N,W)) + l_R(\mathrm{Hom}_R(k,W)) \\ &= l_R(N) \cdot l_R(\mathrm{Hom}_R(k,W)) + l_R(\mathrm{Hom}_R(k,W)) \\ &= (L_R(N) + 1) \cdot l_R(\mathrm{Hom}_R(N,W)) \\ &= l_R(M) \cdot l_R(\mathrm{Hom}_R(M,W)). \end{aligned}$$

Now, suppose in contrary, $\dim R > 0$. Then by Nakayama's Lemma, for any positive integer t, $l_R(m^{t-1}/m^t) > 0$. Hence for any positive integer n, the following equality holds:

$$l_R(R/m^n) = l_R(R/m) + l_R(m/m^2) + \cdots + l_R(m^{n-1}/m^n) \geq n.$$

As $\mathrm{depth_R}\, W = \mathrm{depth_R}\, R = 0$, it follows that $\mathfrak{m} \in \mathrm{Ass}_R(W)$, so that $l_R(\mathrm{Hom}_R(k,W)) > 0$. Hence,

$$l_R(\mathrm{Hom}_R(R/m^n,W)) = l_R(R/m^n) l_R(\mathrm{Hom}_R(k,W)) \geq n$$

and therefore,

$$\lim_{n\to\infty} l_R(\mathrm{Hom}_R(R/m^n,W)) = \infty. \tag{1}$$

On the other hand there exists an ascending chain of submodules of W as follows:

$$(0 :_W \mathfrak{m}) \subseteq (0 :_W \mathfrak{m}^2) \subseteq \cdots$$

which is stable. Note that $\mathrm{Hom}_R(R/\mathfrak{m}^i,W) \cong (0 :_W \mathfrak{m}^i)$ for each non-negative values of i, therefore the set $\{l_R(\mathrm{Hom}_R(R/\mathfrak{m}^n,W)) \mid n > 0\}$ is bounded which contradicts (1). Hence R is Artinian. □

Corollary 2. *Let C be a semidualizing module such that $\mathrm{Ext}^{d+1}(k,C) = 0$, where $d = \mathrm{depth}\, R$. Then R is Cohen-Macaulay and C is the canonical module for R.*

Proof. Let $\underline{x}$ be a maximal R-sequence, then by [8, Theorem 2.2.6] $\underline{x}$ is an C-sequence. So that, by Theorem 1, we conclude that R is Cohen-Macaulay and therefore $R/\underline{x}$ is an Artinian ring. Therefore, to show that C is the canonical module for R, it is enough to show that $C/\underline{x}C \cong$

$\mathrm{E}_{R/\underline{x}}(k)$ by [3, Exercise 3.3.23]. Hence we may assume that R is an Artinian ring and C is a semidualizing module such that $\mathrm{Ext}^1(k,C)=0$. Therefore $\mathrm{Spec}\,(R)=\{\mathfrak{m}\}$ and $\mathrm{Ext}^1(k,C)=0$. It follows from [6, Lemma 1, page 139] that $\mathrm{id}_R(C)<\infty$ and therefore $C\cong \bigoplus_{i=1}^{r}\mathrm{E}_R(k)$ for some $r\in\mathbb{N}$. However, C is an indecomposable R-module, so $r=1$, which shows that $C\cong \mathrm{E}_R(k)$. □

By taking $C=R$ in Corollary 2, we get the following result.

Corollary 3. *Let* $\operatorname{depth} R=d$ *and suppose that* $\mathrm{Ext}_R^{d+1}(k,R)=0$. *Then* R *is Gorenstein.*

Next we give a characterization of Gorenstein local rings, via semidaulizing modules:

Corollary 4. *Let* $\operatorname{depth} R=d$ *and suppose that* C *is semidualizing* R*-module. The following statements are equivalent:*

(1) $r_R(R)=1$ *and* $\mathrm{Ext}_R^{d+1}(k,C)=0$.

(2) C *is dualizing and* C^* *is semidualizing* R*-module.*

(3) C^* *is dualizing.*

(4) $\mathrm{id}_R(C^*)<\infty$ *and for each* $1\le i\le d$, $\mathrm{Ext}_R^i(C,R)=0$.

(5) R *is Gorenstein.*

In particular if one of the conditions hold, then $C\cong R$

Proof. $(5)\Rightarrow(1),(2),(3),(4)$: As R is Gorenstein, one has $r_R(R)=1$ and also the only semidualizing R-module is R and therefore all parts hold true by these two facts.
For completing the proof, we show that R is Gorenstein under each condition:
If (1) holds then R is Cohen-Macaulay by Corollary 2 and so that R is Gorenstein by [3, Theorem 3.2.10].
If (2) holds, then R is Cohen-Macaulay and so that, $C\cong R$ by [1, Corollary 4.3] and therefore $\mathrm{id}_R(R)<\infty$ because C is dualizing.
If (3) holds, a similar argument as in (2) leads us to the result.
If (4) holds, then R is Gorenstein, by [5, Theorem 2.5].
It is clear that the only semidualizing R-module over a Gorenstein ring is the ring itself, hence all equivalent conditions in this result, yield that $C\cong R$. □

Example 1. Let K be a field, and consider the Artinian local ring $R=K[x,y]/(x,y)^2$ with the maximal ideal $\mathfrak{m}=(x,y)/(x,y)^2$. Of course $\mathfrak{m}^2=0$ and $R/\mathfrak{m}\cong K$. Set $C:=\mathrm{Hom}_K(R,K)$, thus by [3, Theorem 3.3.7(b)] C is a dualizing module for R, hence $\mathrm{id}_R(C)=0$ which yields that $\mathrm{Ext}_R^1(K,C)=0$. On the other hand

$$\begin{aligned} r_R(R) &= \mathrm{vdim}_K(C)\\ &= \mathrm{vdim}_{R/\mathfrak{m}}(C)\\ &= \mathrm{vdim}_{R/\mathfrak{m}}(0:_R\mathfrak{m})\\ &= \mathrm{vdim}_{R/\mathfrak{m}}(\mathfrak{m}). \end{aligned}$$

However, since $\{\overline{x}, \overline{y}\}$ is a minimal basis for $\mathfrak{m}$, hence $r_R(R) = \operatorname{vdim}_{R/\mathfrak{m}}(\mathfrak{m}) = 2$ which shows that R is not Gorenstein. This example shows that the condition $r_R(R) = 1$ in Corollary 4 can not be omitted.

Corollary 5. *Let* $\operatorname{depth} R = 0$ *and let* M *is a finitely generated* R*-module such that* $M^* \cong \overset{n}{\underset{i=1}{\oplus}} R$ *for some* $n \in \mathbb{N}$*. If* $\operatorname{Ext}^1_R(k, M) = 0$*, then* R *is Gorenstein.*

Proof. As M^* has a rank, it follows from [3, Exercise 1.4.23] that M has a rank, therefore M is free since $\operatorname{depth} R = 0$. Using this fact and the hypothesis $\operatorname{Ext}^1_R(k, M) = 0$ we deduce that $\operatorname{Ext}^1_R(k, R) = 0$. Hence by Corollary 3, R Gorenstein. □

Corollary 6. *Let* R *be a* d*-dimensional Cohen-Macaulay ring and let* C *be a semidualizing* R*-module which has a rank. Suppose that* M *is a finitely generated* R*-module such that* $M^* \cong \overset{n}{\underset{i=1}{\oplus}} C$ *for some* $n \in \mathbb{N}$*. If* $\operatorname{Ext}^{d+1}_R(k, M) = 0$*, then* R *is generically Gorenstein.*

Proof. As R is Cohen-Macaulay, one has $\dim R/\mathfrak{p} = d$ for each $\mathfrak{p} \in \operatorname{Ass} R$. On the other hand, by using $\operatorname{Ext}^{d+1}_R(k, M) = 0$ and [6, Lemma 4, page 141], one has $\operatorname{Ext}^1_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, M_\mathfrak{p}) = 0$ for each $\mathfrak{p} \in \operatorname{Ass} R$. As C has a rank, it follows that $C_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module, hence by Corollary 5, we get $R_\mathfrak{p}$ is a Gorenstein ring. □

To prove the next theorem, we need the following lemma. However, with a similar argument, one can prove the lemma for any two indecomposbale finitely generated R-modules.

Lemma 1. *Let* C *and* C' *be semidualizing* R*-modules and suppose that* $C \otimes_R C'$ *is a free* R*-module. Then* $C \cong C' \cong R$

Proof. One can apply $- \otimes_R C'$ to the following exact sequence

$$0 \to K \to \overset{n}{\underset{i=1}{\oplus}} R \to C \to 0$$

to build an exact sequence as follows

$$0 \to K' \to \overset{n}{\underset{i=1}{\oplus}} C' \to C \otimes_R C' \to 0$$

By the hypothesis $C \otimes_R C' \cong \overset{t}{\underset{i=1}{\oplus}} R$ for some $t \in \mathbb{N}$, therefore the above exact sequence splits and hence $K' \oplus (\overset{t}{\underset{i=1}{\oplus}} R) \cong \overset{n}{\underset{i=1}{\oplus}} C'$. Now, by using [7, Proposition 4.47] we can find a decomposition of K' which every component is indecomposable, then use the Krull–Schmidt theorem to see that $C' \cong R$. □

Theorem 2. *Let* $\operatorname{depth} R = 0$ *and* C *be a semidualizing* R*-module. Suppose that* M *is a finitely generated* R*-module such that* $\operatorname{id}_R(M) < \infty$ *and* $M^* \cong \overset{n}{\underset{i=1}{\oplus}} C$ *for some* $n \in \mathbb{N}$*. Then* R *is Gorenstein.*

Proof. First of all, we note that R is Cohen-Macaulay by Remark 2, therefore by the hypothesis $\dim R = 0$ which means that R is Artinian. On the other hand, by [6, Theorem 18.9], $\mathrm{id}_R(M) = \operatorname{depth} R = 0$. Hence M is an injective R-module and thereofre $M \cong \overset{m}{\underset{i=1}{\oplus}} \mathrm{E}_R(k)$ for some $m \in \mathbb{N}$, and so by using the hyppthesis $M^* \cong \overset{n}{\underset{i=1}{\oplus}} C$, we get $\mathrm{Hom}_R(\mathrm{E}_R(k), F) \cong \overset{n}{\underset{i=1}{\oplus}} C$, where F is a finitely generated free R-module. Applying $\mathrm{Hom}_R(C, -)$ on the last isomorphism and using the Hom-tensor adjointness, one has the

$$\mathrm{Hom}_R(\mathrm{E}_R(k), \mathrm{Hom}_R(C, F)) \cong \overset{n}{\underset{i=1}{\oplus}} R$$

which yields that $\mathrm{Hom}_R(C, F) \cong \overset{n}{\underset{i=1}{\oplus}} \mathrm{E}_R(k)$. Now apply $\mathrm{Hom}_R(-, \mathrm{E}_R(k))$ on the both sides of the last isomorphism. Of course the right hand side is isomorphic to $\overset{n}{\underset{i=1}{\oplus}} R$ and in view of [2, Lemma 10.2.16], the left hand side is isomorphic to $C \otimes_R \mathrm{Hom}_R(F, \mathrm{E}_R(k))$. Therefore $C \otimes_R \mathrm{Hom}_R(F, \mathrm{E}_R(k))$ is free. As F is a free R-module, it follows that $F \cong \overset{l}{\underset{i=1}{\oplus}} R$ and therefore $C \otimes_R \mathrm{Hom}_R(F, \mathrm{E}_R(k)) \cong \overset{l}{\underset{i=1}{\oplus}} (C \otimes \mathrm{E}_R(k))$. Hence $C \otimes \mathrm{E}_R(k)$ is a free R-module. Thus $C \cong \mathrm{E}_R(k) \cong R$ by Lemma 1 which shows that R is Gorenstein. □

Corollary 7. *Let C be a semidualizing R-module and suppose that M is a finitely generated R-module such that $M^* \cong \overset{n}{\underset{i=1}{\oplus}} C$ for some $n \in \mathbb{N}$. If $\mathrm{id}_R(M) < \infty$, then R is generically Gorenstein (and so C has a rank).*

Proof. Note that R is Cohen-Macaulay by Remark 2. Let $\mathfrak{p} \in \mathrm{Ass}_R(R)$. Then $M_{\mathfrak{p}}^* \cong \overset{n}{\underset{i=1}{\oplus}} C_{\mathfrak{p}}$ and $\operatorname{depth}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0$. Hence, $R_{\mathfrak{p}}$ is Gorenstein by Theorem 2 and therefore $R_{\mathfrak{p}} \cong C_{\mathfrak{p}}$ which shows that C has a rank. □

If in Theorem 2 $\operatorname{depth}_R R \neq 0$, then should be R Gorenstein?

The following result shows that the above question has positive answer when M is a Cohen-Macaulay R-module.

Theorem 3. *Let C be a semidualizing R-module and M be a finitely generated Cohen-Macaualay R-module with finite injective dimension. If $M^* \cong \overset{n}{\underset{i=1}{\oplus}} C$ for some $n \in \mathbb{N}$, then R is Gorenstein and thus $C \cong R$.*

Proof. By Remark 2, R is a Cohen-Macaulay ring and using the basic properties of $\hat{R}$, we assume that R is complete and therefore ω_R exists. Also, as $\mathrm{Spec}(R) = \mathrm{Supp}(C) = \mathrm{Supp}(M^*) \subseteq \mathrm{Supp}(M)$, one has $\dim M = \dim R$ and therefore M is a maximal Cohen-Macaulay R-module. Hence by [3, Exercise 3.3.28], $M \cong \overset{s}{\underset{i=1}{\oplus}} \omega_R$ for some $s \in \mathbb{N}$. Now we apply $\mathrm{Hom}_R(C, -)$ on the both sides of $\overset{s}{\underset{i=1}{\oplus}} (\omega_R)^* \cong \overset{n}{\underset{i=1}{\oplus}} C$, to get $\overset{s}{\underset{i=1}{\oplus}} \mathrm{Hom}_R(C, (\omega_R)^*) \cong \overset{n}{\underset{i=1}{\oplus}} R$. Hence $\mathrm{Hom}_R(C, (\omega_R)^*)$ is a free R-module and therefore $(\omega_R)^* \cong \overset{t}{\underset{i=1}{\oplus}} C$ for some $t \in \mathbb{N}$ (see [1, Lemma 3.5]). But R is generically Gorenstein by Corollary 7, hence for all $\mathfrak{p} \in \operatorname{Ass} R$, $C_{\mathfrak{p}} \cong \omega_{R_{\mathfrak{p}}} \cong R_{\mathfrak{p}}$, so

$$R_{\mathfrak{p}} \cong \omega_{R_{\mathfrak{p}}} \cong ((\omega_R)^*)_{\mathfrak{p}} \cong \overset{t}{\underset{i=1}{\oplus}} C_{\mathfrak{p}} \cong \overset{t}{\underset{i=1}{\oplus}} R_{\mathfrak{p}}$$

which yields that $t = 1$ and therefore $(\omega_R)^* \cong C$. Thus R is Gorenstein by Corollary 4. □

Theorem 4. *Let C be a semidualizing R-module and M be a finitely generated R-module such that $\operatorname{id}_R(M) < \infty$. If $M^* \cong \overset{n}{\underset{i=1}{\oplus}} C$ for some $n \in \mathbb{N}$, then C is reflexive. In particular, if $\dim R \leq 1$, then $C \cong R$.*

Proof. Again by Remark 2, R is a Cohen-Macaulay ring. Set $D := \overset{n}{\underset{i=1}{\oplus}} C$, by [4, Lemma 1.1.9 (a)], there exists a finitely generated R-module H such that $D^{**} \cong D \bigoplus H$. By Corollary 7, C has a rank, therefore $R_{\mathfrak{p}} \cong C_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Ass} R$. Hence for all $\mathfrak{p} \in \operatorname{Ass} R$,

$$(D^{**})_{\mathfrak{p}} \cong D_{\mathfrak{p}} \cong \overset{n}{\underset{i=1}{\oplus}} R_{\mathfrak{p}}.$$

It follows that $\overset{n}{\underset{i=1}{\oplus}} R_{\mathfrak{p}} \cong \overset{n}{\underset{i=1}{\oplus}} R_{\mathfrak{p}} \bigoplus H_{\mathfrak{p}}$, which implies that $H_{\mathfrak{p}} = 0$. Thus $\operatorname{Ass} R \bigcap \operatorname{Supp} H = \emptyset$. As $\operatorname{Ass} H \subseteq \operatorname{Ass} D^{**} \subseteq \operatorname{Ass} R$ (see [3, Exercise 1.2.27]), we deduce that $\operatorname{Ass} H = \emptyset$ and so that $H = 0$. Therefore $D^{**} \cong D$ and thus $\overset{n}{\underset{i=1}{\oplus}} C^{**} \cong \overset{n}{\underset{i=1}{\oplus}} C$. Applying $\operatorname{Hom}_R(C, -)$ on the both sides of the last isomorphism, we get

$$\overset{n}{\underset{i=1}{\oplus}} \operatorname{Hom}_R(C, C^{**}) \cong \overset{n}{\underset{i=1}{\oplus}} R$$

which implies that $\operatorname{Hom}_R(C, C^{**})$ is a free R-module and therefore is isomorphic to $\overset{t}{\underset{i=1}{\oplus}} R$ for some $t \in \mathbb{N}$. Using this R-module instead of $\operatorname{Hom}_R(C, C^{**})$ in the isomorphism $\overset{n}{\underset{i=1}{\oplus}} \operatorname{Hom}_R(C, C^{**}) \cong \overset{n}{\underset{i=1}{\oplus}} R$, we get that $t = 1$ by the Krull–Schmidt theorem, which shows that

$$\operatorname{Hom}_R(C, C^{**}) \cong R.$$

Now by [1, Lemma 3.5], $C \cong C^{**}$. In particular, if $\dim R \leq 1$, we deduce that $C \cong R$ by [1, Theorem 4.9], since C is reflexive. □

Acknowledgments

The authors would like to thank the referee for careful reading.

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