

Identification Gorenstein rings via special semidualizing modules

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Abstract. Let (R, \mathfrak{m}) be a Noetherian local ring and M be a finitely generated R -module such that $\text{Hom}_R(M, R) \cong \bigoplus_{i=1}^n C$ for some positive integer n . We try to present new characterizations of Gorenstein rings via M and C . It is proved that if $\text{depth } R = 0$ and $\text{id}_R(M) < \infty$ then R is Gorenstein. Also, it is shown that if M is a Cohen-Macaulay R -module with finite injective dimension, then R is Gorenstein.

Keywords: The Auslander-Reiten conjecture, Semidualizing modules, Free modules.

AMS Subject Classification 2010: 13C05, 13D07, 13H10.

1 Introduction

Let R be a Noetherian local ring with maximal ideal \mathfrak{m} . A finitely generated R -module C is called a semidualizing module if $\text{Hom}_R(C, C) \cong R$ and $\text{Ext}_R^i(C, C) = 0$ for all $i > 0$. For instance R always is a semidualizing module and over a Cohen-Macaulay local ring, the canonical module, is a semidualizing module.

As a generalization of the ring, a semidualizing module behaves like R in some properties, like the Krull dimension, depth, base change and etc. Also, when a semidualizing R -module has finite projective dimension, it must be isomorphic to R and if a semidualizing R -module C has finite injective dimension then R is Cohen-Macaulay with the canonical module C . When the canonical module of a Cohen-Macaulay ring R is isomorphic to the ring itself, then R must be Gorenstein, and this maybe a good motivation to identify Gorenstein ring via semidualizing module.

The properties of a semidualizing R -module C , when C is a dual (i.e., $C \cong M^*$ for some finitely

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Received: 30 September 2024/ Revised: 12 May 2025/ Accepted: 14 May 2025

DOI: [10.22124/JART.2025.28567.1720](https://doi.org/10.22124/JART.2025.28567.1720)

generated R -module M) is studied in [1]. For instance, if C is a dual, then it must be reflexive [1, Theorem 3.8]. Also, over a one dimensional Cohen-Macaulay ring, a reflexive semidualizing R -module is isomorphic to R [1, Theorem 4.9]. Another result over a one dimensional Cohen-Macaulay ring R , says that if $M^* \cong C$, where C is a semidualizing R -module and M is a finitely generated R -module with finite injective dimension, then R must be Gorenstein [1, Corollary 3.11].

In this paper, our main goal is to characterize Gorenstein rings, via some special semidualizing modules. These class of semidualizing R -modules satisfy the condition $M^* \cong \bigoplus_{i=1}^r C$, where M is a finitely generated R -module and n is a positive integer. One of the main result is:

Theorem A. Let $\text{depth } R = 0$ and C be a semidualizing R -module. Suppose that M is a finitely generated R -module such that $\text{id}_R(M) < \infty$ and $M^* \cong \bigoplus_{i=1}^n C$ for some $n \in \mathbb{N}$. Then R is Gorenstein.

If we replace the condition $\text{depth } R = 0$ by the condition " M is a Cohen-Macaulay R -module " then the assertion holds true. In other words:

Theorem B. Let C be a semidualizing R -module and M be a finitely generated Cohen-Macaulay R -module with finite injective dimension. If $M^* \cong \bigoplus_{i=1}^n C$ for some $n \in \mathbb{N}$, then R is Gorenstein and thus $C \cong R$.

Indeed, according to Theorem B, R is Gorenstein if, and only if there exist a semidualizing R module C and a Cohen-Macaulay R -module M with $\text{id}_R M < \infty$, such that M^* is isomorphic to a direct sum of copies of C .

Also, there is a characterization of Gorenstein rings via semidualizing modules and their duals (Corollary 4).

2 Preliminaries

Throughout this paper all rings are commutative with $1 \neq 0$, Noetherian and local.

Definition 1. The injective dimension of an R -module M , is denoted by $\text{id}_R(M)$ and is defined as

$$\text{id}_R(M) := \inf\{n : \text{Ext}_R^i(N, M) = 0, \text{ for all } i > 0 \text{ and } R\text{-modules } N\}.$$

This notion generalize the concept of injective modules and plays an important role in the studying of homological properties of modules. For instance, a ring R is called a Gorenstien ring, whenever $\text{id}_R R < \infty$. There are many useful results about finitely generated modules of finite injective dimension in [3, Section 3.1].

For a finitely generated R -module M , $\mu(M)$ is the number of minimal generators of M and $r_R(M)$ denotes the type of M which is defined as $\text{vdim}_{R/\mathfrak{m}} \text{Ext}_R^{\text{depth } M}(R/\mathfrak{m}, M)$. We say that M has rank r , whenever $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module of rank r for all $\mathfrak{p} \in \text{Ass}_R(R)$. We use M^* to denote $\text{Hom}_R(M, R)$. Finally, R is called a generically Gorenstein ring, whenever $R_{\mathfrak{p}}$ is a Gorenstein ring for all $\mathfrak{p} \in \text{Ass}_R R$.

Now, we recall the notion of semidualizing modules and some results which are needed in the rest of the paper.

Definition 2. Let C be a finitely generated R -module. Then C is called a *semidualizing module*, when $\alpha_C : R \rightarrow \text{Hom}_R(C, C)$ which is defined as $f(r)(c) = rc$ for any $r \in R$ and $c \in C$, is an isomorphism and $\text{Ext}_R^i(C, C) = 0$ for each $i > 0$.

Corollary 1. Every semidualizing R -module is an indecomposable R -module.

Proof. Let C be a semidualizing R -module and let $C \cong L \oplus N$ for some (finitely generated) R -modules L and N . As $\text{Hom}_R(C, C) \cong R$, one has

$$\text{Hom}_R(L, L) \oplus \text{Hom}_R(L, N) \oplus \text{Hom}_R(N, L) \oplus \text{Hom}_R(N, N) \cong R.$$

But one of the R -modules $\text{Hom}_R(L, L)$ or $\text{Hom}_R(N, N)$ must be zero, since R is indecomposable. Therefore $L = 0$ or $N = 0$, which completes the proof. \square

Definition 3. Let C be a semidualizing R -module. If $\text{id}_R C < \infty$, then C is called a *dualizing module*.

Remark 1. By Remark 2, if the dualizing module exists, then R is Cohen-Macaulay. One can check that the canonical module of a Cohen-Macaulay (see [3, Definition 3.3.1]) is coincide with the dualizing module. Also, when R is an Artinian ring, then $\mathbf{E}_R(R/\mathfrak{m})$ is the canonical module of R .

Proposition 1. The following statements are hold for a semidualizing module C :

- (i) One has $\dim R = \dim C$, $\text{depth } R = \text{depth } C$ and $\text{Ass}_R R = \text{Ass}_R C$. Consequently, $\text{Spec}(R) = \text{Supp } C$.
- (ii) C is faithful, that is $\text{Ann}_R(C) = 0$.
- (iii) The sequence x_1, \dots, x_n ($n \in \mathbb{N}$) is an R -sequence if, and only if it is a C -sequence.

Proof. See [8, Proposition 2.1.16]. \square

Remark 2. Let M be a finitely generated R -module such that $\text{id}_R M < \infty$. Then, by [3, Corollary 9.6.2 and Remark 9.6.4], R is a Cohen-Macaulay ring.

3 Main results

Theorem 1. Let W be a finitely generated R -module with $\text{depth } W = \text{depth } R = d$ and let $\underline{x} = x_1, \dots, x_d$ be a sequence of elements of R which is both R and W -sequence. Suppose that $\text{Ext}_R^{d+1}(k, W) = 0$. Then R is Cohen-Macaulay.

Proof. By [6, Lemma 2, Page 140], we have

$$\text{Ext}_{R/\underline{x}}^1(k, W/\underline{x}W) \cong \text{Ext}_R^{d+1}(k, W) = 0.$$

Hence we may assume that, $\text{depth } R = \text{depth } W = 0$ and $\text{Ext}_R^1(k, W) = 0$. Our aim is to show that R is Artinian. To do this, let M be a finite length module. First, we use induction

on $l_R(M)$ to show that $l_R(\text{Hom}_R(M, W)) = l_R(M) \cdot l_R(\text{Hom}_R(k, W))$ and $\text{Ext}_R^1(M, W) = 0$. When $l_R(M) = 0$ or 1 , we have nothing to prove. So, let $l_R(M) > 1$ and suppose that the assertion holds for all positive integers less than $l_R(M)$. There exists an exact sequence $0 \rightarrow N \rightarrow M \rightarrow k \rightarrow 0$, where N is a submodule of M with $l_R(N) = l_R(M) - 1$, so that $\text{Ext}_R^1(N, W) = 0$ and $l_R(\text{Hom}_R(N, W)) = l_R(N) \cdot l_R(\text{Hom}_R(k, W))$, by induction hypothesis. Applying $\text{Hom}_R(-, W)$ to the above exact sequence, yields the following exact sequence:

$$0 \rightarrow \text{Hom}_R(k, W) \rightarrow \text{Hom}_R(M, W) \rightarrow \text{Hom}_R(N, W) \rightarrow \text{Ext}_R^1(k, W) \rightarrow \cdots$$

As $\text{Ext}_R^1(N, W) = \text{Ext}_R^1(k, W) = 0$, we deduce that $\text{Ext}_R^1(M, W) = 0$. Hence the above exact sequence yields into the following:

$$0 \rightarrow \text{Hom}_R(k, W) \rightarrow \text{Hom}_R(M, W) \rightarrow \text{Hom}_R(N, W) \rightarrow 0.$$

Therefore,

$$\begin{aligned} l_R(\text{Hom}_R(M, W)) &= l_R(\text{Hom}_R(N, W)) + l_R(\text{Hom}_R(k, W)) \\ &= l_R(N) \cdot l_R(\text{Hom}_R(k, W)) + l_R(\text{Hom}_R(k, W)) \\ &= (l_R(N) + 1) \cdot l_R(\text{Hom}_R(k, W)) \\ &= l_R(M) \cdot l_R(\text{Hom}_R(k, W)). \end{aligned}$$

Now, suppose in contrary, $\dim R > 0$. Then by Nakayama's Lemma, for any positive integer t , $l_R(m^{t-1}/m^t) > 0$. Hence for any positive integer n , the following equality holds:

$$l_R(R/m^n) = l_R(R/m) + l_R(m/m^2) + \cdots + l_R(m^{n-1}/m^n) \geq n.$$

As $\text{depth}_R W = \text{depth}_R R = 0$, it follows that $\mathfrak{m} \in \text{Ass}_R(W)$, so that $l_R(\text{Hom}_R(k, W)) > 0$. Hence,

$$l_R(\text{Hom}_R(R/m^n, W)) = l_R(R/m^n) l_R(\text{Hom}_R(k, W)) \geq n$$

and therefore,

$$\lim_{n \rightarrow \infty} l_R(\text{Hom}_R(R/m^n, W)) = \infty. \quad (1)$$

On the other hand there exists an ascending chain of submodules of W as follows:

$$(0 :_W \mathfrak{m}) \subseteq (0 :_W \mathfrak{m}^2) \subseteq \cdots$$

which is stable. Note that $\text{Hom}_R(R/\mathfrak{m}^i, W) \cong (0 :_W \mathfrak{m}^i)$ for each non-negative values of i , therefore the set $\{l_R(\text{Hom}_R(R/\mathfrak{m}^n, W)) \mid n > 0\}$ is bounded which contradicts (1). Hence R is Artinian. \square

Corollary 2. *Let C be a semidualizing module such that $\text{Ext}^{d+1}(k, C) = 0$, where $d = \text{depth } R$. Then R is Cohen-Macaulay and C is the canonical module for R .*

Proof. Let \underline{x} be a maximal R -sequence, then by [8, Theorem 2.2.6] \underline{x} is an C -sequence. So that, by Theorem 1, we conclude that R is Cohen-Macaulay and therefore R/\underline{x} is an Artinian ring. Therefore, to show that C is the canonical module for R , it is enough to show that $C/\underline{x}C \cong$

$E_{R/\underline{x}}(k)$ by [3, Exercise 3.3.23]. Hence we may assume that R is an Artinian ring and C is a semidualizing module such that $\text{Ext}^1(k, C) = 0$. Therefore $\text{Spec}(R) = \{\mathfrak{m}\}$ and $\text{Ext}^1(k, C) = 0$. It follows from [6, Lemma 1, page 139] that $\text{id}_R(C) < \infty$ and therefore $C \cong \bigoplus_{i=1}^r E_R(k)$ for some $r \in \mathbb{N}$. However, C is an indecomposable R -module, so $r = 1$, which shows that $C \cong E_R(k)$. \square

By taking $C = R$ in Corollary 2, we get the following result.

Corollary 3. *Let $\text{depth } R = d$ and suppose that $\text{Ext}_R^{d+1}(k, R) = 0$. Then R is Gorenstein.*

Next we give a characterization of Gorenstein local rings, via semidualizing modules:

Corollary 4. *Let $\text{depth } R = d$ and suppose that C is semidualizing R -module. The following statements are equivalent:*

- (1) $r_R(R) = 1$ and $\text{Ext}_R^{d+1}(k, C) = 0$.
- (2) C is dualizing and C^* is semidualizing R -module.
- (3) C^* is dualizing.
- (4) $\text{id}_R(C^*) < \infty$ and for each $1 \leq i \leq d$, $\text{Ext}_R^i(C, R) = 0$.
- (5) R is Gorenstein.

In particular if one of the conditions hold, then $C \cong R$

Proof. (5) \Rightarrow (1), (2), (3), (4): As R is Gorenstein, one has $r_R(R) = 1$ and also the only semidualizing R -module is R and therefore all parts hold true by these two facts.

For completing the proof, we show that R is Gorenstein under each condition:

If (1) holds then R is Cohen-Macaulay by Corollary 2 and so that R is Gorenstein by [3, Theorem 3.2.10].

If (2) holds, then R is Cohen-Macaulay and so that, $C \cong R$ by [1, Corollary 4.3] and therefore $\text{id}_R(R) < \infty$ because C is dualizing.

If (3) holds, a similar argument as in (2) leads us to the result.

If (4) holds, then R is Gorenstein, by [5, Theorem 2.5].

It is clear that the only semidualizing R -module over a Gorenstein ring is the ring itself, hence all equivalent conditions in this result, yield that $C \cong R$. \square

Example 1. Let K be a field, and consider the Artinian local ring $R = K[x, y]/(x, y)^2$ with the maximal ideal $\mathfrak{m} = (x, y)/(x, y)^2$. Of course $\mathfrak{m}^2 = 0$ and $R/\mathfrak{m} \cong K$. Set $C := \text{Hom}_K(R, K)$, thus by [3, Theorem 3.3.7(b)] C is a dualizing module for R , hence $\text{id}_R(C) = 0$ which yields that $\text{Ext}_R^1(K, C) = 0$. On the other hand

$$\begin{aligned} r_R(R) &= \text{vdim}_K(C) \\ &= \text{vdim}_{R/\mathfrak{m}}(C) \\ &= \text{vdim}_{R/\mathfrak{m}}(0 :_R \mathfrak{m}) \\ &= \text{vdim}_{R/\mathfrak{m}}(\mathfrak{m}). \end{aligned}$$

However, since $\{\bar{x}, \bar{y}\}$ is a minimal basis for \mathfrak{m} , hence $r_R(R) = \text{vdim}_{R/\mathfrak{m}}(\mathfrak{m}) = 2$ which shows that R is not Gorenstein. This example shows that the condition $r_R(R) = 1$ in Corollary 4 can not be omitted.

Corollary 5. *Let $\text{depth } R = 0$ and let M is a finitely generated R -module such that $M^* \cong \bigoplus_{i=1}^n R$ for some $n \in \mathbb{N}$. If $\text{Ext}_R^1(k, M) = 0$, then R is Gorenstein.*

Proof. As M^* has a rank, it follows from [3, Exercise 1.4.23] that M has a rank, therefore M is free since $\text{depth } R = 0$. Using this fact and the hypothesis $\text{Ext}_R^1(k, M) = 0$ we deduce that $\text{Ext}_R^1(k, R) = 0$. Hence by Corollary 3, R Gorenstein. \square

Corollary 6. *Let R be a d -dimensional Cohen-Macaulay ring and let C be a semidualizing R -module which has a rank. Suppose that M is a finitely generated R -module such that $M^* \cong \bigoplus_{i=1}^n C$ for some $n \in \mathbb{N}$. If $\text{Ext}_R^{d+1}(k, M) = 0$, then R is generically Gorenstein.*

Proof. As R is Cohen-Macaulay, one has $\dim R/\mathfrak{p} = d$ for each $\mathfrak{p} \in \text{Ass } R$. On the other hand, by using $\text{Ext}_R^{d+1}(k, M) = 0$ and [6, Lemma 4, page 141], one has $\text{Ext}_{R_{\mathfrak{p}}}^1(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}}) = 0$ for each $\mathfrak{p} \in \text{Ass } R$. As C has a rank, it follows that $C_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module, hence by Corollary 5, we get $R_{\mathfrak{p}}$ is a Gorenstein ring. \square

To prove the next theorem, we need the following lemma. However, with a similar argument, one can prove the lemma for any two indecomposable finitely generated R -modules.

Lemma 1. *Let C and C' be semidualizing R -modules and suppose that $C \otimes_R C'$ is a free R -module. Then $C \cong C' \cong R$*

Proof. One can apply $- \otimes_R C'$ to the following exact sequence

$$0 \rightarrow K \rightarrow \bigoplus_{i=1}^n R \rightarrow C \rightarrow 0$$

to build an exact sequence as follows

$$0 \rightarrow K' \rightarrow \bigoplus_{i=1}^n C' \rightarrow C \otimes_R C' \rightarrow 0$$

By the hypothesis $C \otimes_R C' \cong \bigoplus_{i=1}^t R$ for some $t \in \mathbb{N}$, therefore the above exact sequence splits and hence $K' \oplus (\bigoplus_{i=1}^t R) \cong \bigoplus_{i=1}^n C'$. Now, by using [7, Proposition 4.47] we can find a decomposition of K' which every component is indecomposable, then use the Krull-Schmidt theorem to see that $C' \cong R$. \square

Theorem 2. *Let $\text{depth } R = 0$ and C be a semidualizing R -module. Suppose that M is a finitely generated R -module such that $\text{id}_R(M) < \infty$ and $M^* \cong \bigoplus_{i=1}^n C$ for some $n \in \mathbb{N}$. Then R is Gorenstein.*

Proof. First of all, we note that R is Cohen-Macaulay by Remark 2, therefore by the hypothesis $\dim R = 0$ which means that R is Artinian. On the other hand, by [6, Theorem 18.9], $\text{id}_R(M) = \text{depth } R = 0$. Hence M is an injective R -module and therefore $M \cong \bigoplus_{i=1}^m E_R(k)$ for some $m \in \mathbb{N}$, and so by using the hypothesis $M^* \cong \bigoplus_{i=1}^n C$, we get $\text{Hom}_R(E_R(k), F) \cong \bigoplus_{i=1}^n C$, where F is a finitely generated free R -module. Applying $\text{Hom}_R(C, -)$ on the last isomorphism and using the Hom-tensor adjointness, one has the

$$\text{Hom}_R(E_R(k), \text{Hom}_R(C, F)) \cong \bigoplus_{i=1}^n R$$

which yields that $\text{Hom}_R(C, F) \cong \bigoplus_{i=1}^n E_R(k)$. Now apply $\text{Hom}_R(-, E_R(k))$ on the both sides of the last isomorphism. Of course the right hand side is isomorphic to $\bigoplus_{i=1}^n R$ and in view of [2, Lemma 10.2.16], the left hand side is isomorphic to $C \otimes_R \text{Hom}_R(F, E_R(k))$. Therefore $C \otimes_R \text{Hom}_R(F, E_R(k))$ is free. As F is a free R -module, it follows that $F \cong \bigoplus_{i=1}^l R$ and therefore $C \otimes_R \text{Hom}_R(F, E_R(k)) \cong \bigoplus_{i=1}^l (C \otimes E_R(k))$. Hence $C \otimes E_R(k)$ is a free R -module. Thus $C \cong E_R(k) \cong R$ by Lemma 1 which shows that R is Gorenstein. \square

Corollary 7. *Let C be a semidualizing R -module and suppose that M is a finitely generated R -module such that $M^* \cong \bigoplus_{i=1}^n C$ for some $n \in \mathbb{N}$. If $\text{id}_R(M) < \infty$, then R is generically Gorenstein (and so C has a rank).*

Proof. Note that R is Cohen-Macaulay by Remark 2. Let $\mathfrak{p} \in \text{Ass}_R(R)$. Then $M_{\mathfrak{p}}^* \cong \bigoplus_{i=1}^n C_{\mathfrak{p}}$ and $\text{depth}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0$. Hence, $R_{\mathfrak{p}}$ is Gorenstein by Theorem 2 and therefore $R_{\mathfrak{p}} \cong C_{\mathfrak{p}}$ which shows that C has a rank. \square

If in Theorem 2 $\text{depth}_R R \neq 0$, then should be R Gorenstein?

The following result shows that the above question has positive answer when M is a Cohen-Macaulay R -module.

Theorem 3. *Let C be a semidualizing R -module and M be a finitely generated Cohen-Macaulay R -module with finite injective dimension. If $M^* \cong \bigoplus_{i=1}^n C$ for some $n \in \mathbb{N}$, then R is Gorenstein and thus $C \cong R$.*

Proof. By Remark 2, R is a Cohen-Macaulay ring and using the basic properties of \hat{R} , we assume that R is complete and therefore ω_R exists. Also, as $\text{Spec}(R) = \text{Supp}(C) = \text{Supp}(M^*) \subseteq \text{Supp}(M)$, one has $\dim M = \dim R$ and therefore M is a maximal Cohen-Macaulay R -module. Hence by [3, Exercise 3.3.28], $M \cong \bigoplus_{i=1}^s \omega_R$ for some $s \in \mathbb{N}$. Now we apply $\text{Hom}_R(C, -)$ on the both sides of $\bigoplus_{i=1}^s (\omega_R)^* \cong \bigoplus_{i=1}^n C$, to get $\bigoplus_{i=1}^s \text{Hom}_R(C, (\omega_R)^*) \cong \bigoplus_{i=1}^n R$. Hence $\text{Hom}_R(C, (\omega_R)^*)$ is

a free R -module and therefore $(\omega_R)^* \cong \bigoplus_{i=1}^t C$ for some $t \in \mathbb{N}$ (see [1, Lemma 3.5]). But R is generically Gorenstein by Corollary 7, hence for all $\mathfrak{p} \in \text{Ass } R$, $C_{\mathfrak{p}} \cong \omega_{R_{\mathfrak{p}}} \cong R_{\mathfrak{p}}$, so

$$R_{\mathfrak{p}} \cong \omega_{R_{\mathfrak{p}}} \cong ((\omega_R)^*)_{\mathfrak{p}} \cong \bigoplus_{i=1}^t C_{\mathfrak{p}} \cong \bigoplus_{i=1}^t R_{\mathfrak{p}}$$

which yields that $t = 1$ and therefore $(\omega_R)^* \cong C$. Thus R is Gorenstein by Corollary 4. \square

Theorem 4. *Let C be a semidualizing R -module and M be a finitely generated R -module such that $\text{id}_R(M) < \infty$. If $M^* \cong \bigoplus_{i=1}^n C$ for some $n \in \mathbb{N}$, then C is reflexive. In particular, if $\dim R \leq 1$, then $C \cong R$.*

Proof. Again by Remark 2, R is a Cohen-Macaulay ring. Set $D := \bigoplus_{i=1}^n C$, by [4, Lemma 1.1.9 (a)], there exists a finitely generated R -module H such that $D^{**} \cong D \bigoplus H$. By Corollary 7, C has a rank, therefore $R_{\mathfrak{p}} \cong C_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Ass } R$. Hence for all $\mathfrak{p} \in \text{Ass } R$,

$$(D^{**})_{\mathfrak{p}} \cong D_{\mathfrak{p}} \cong \bigoplus_{i=1}^n R_{\mathfrak{p}}.$$

It follows that $\bigoplus_{i=1}^n R_{\mathfrak{p}} \cong \bigoplus_{i=1}^n R_{\mathfrak{p}} \bigoplus H_{\mathfrak{p}}$, which implies that $H_{\mathfrak{p}} = 0$. Thus $\text{Ass } R \cap \text{Supp } H = \emptyset$. As $\text{Ass } H \subseteq \text{Ass } D^{**} \subseteq \text{Ass } R$ (see [3, Exercise 1.2.27]), we deduce that $\text{Ass } H = \emptyset$ and so that $H = 0$. Therefore $D^{**} \cong D$ and thus $\bigoplus_{i=1}^n C^{**} \cong \bigoplus_{i=1}^n C$. Applying $\text{Hom}_R(C, -)$ on the both sides of the last isomorphism, we get

$$\bigoplus_{i=1}^n \text{Hom}_R(C, C^{**}) \cong \bigoplus_{i=1}^n R$$

which implies that $\text{Hom}_R(C, C^{**})$ is a free R -module and therefore is isomorphic to $\bigoplus_{i=1}^t R$ for some $t \in \mathbb{N}$. Using this R -module instead of $\text{Hom}_R(C, C^{**})$ in the isomorphism $\bigoplus_{i=1}^n \text{Hom}_R(C, C^{**}) \cong \bigoplus_{i=1}^n R$, we get that $t = 1$ by the Krull–Schmidt theorem, which shows that

$$\text{Hom}_R(C, C^{**}) \cong R.$$

Now by [1, Lemma 3.5], $C \cong C^{**}$. In particular, if $\dim R \leq 1$, we deduce that $C \cong R$ by [1, Theorem 4.9], since C is reflexive. \square

Acknowledgments

The authors would like to thank the referee for careful reading.

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