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Non-identity order divisor graphs of groups

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Abstract. Let G be a group with identity e. In this paper, we define and study the non-identity order divisor graph of G, where the vertex set is $G - \{e\}$ and two distinct vertices x and y are adjacent if and only if either $o(x)|o(y)$ or $o(y)|o(x)$. We denote the order divisor graph of group G by $\Gamma_{niod}(G)$. We study some basic properties of $\Gamma_{niod}(G)$ such as connectedness, completeness, bipartiteness and Eulerian property. The lower bound as well as the number of edges of $\Gamma_{niod}(G)$ are also calculated for some group G and some characterizations for fundamental properties of $\Gamma_{niod}(G)$ have been obtained. Finally, we explore the relation between the order prime graph and the non-identity order divisor graph of some group G.

Keywords: Non-identity order divisor graph, Order prime graph, Eulerian graph, Finite group.
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1 Introduction

The study of algebraic structures using the properties of graphs has become an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring or a group and thereby investigating the algebraic properties of the ring or the group using the associated graph, for instance, refer to [1, 2, 10, 11]. In the present article, for any group G, we assign a graph and investigate algebraic properties of the group using the graph theoretical concepts.

Throughout this paper, we consider undirected, simple and finite graphs and finite groups. We recall few definitions from graph theory and group theory. For a graph $\Gamma = (V, E)$, V denote the set of all vertices and E denote the set of all edges in Γ. The degree of a vertex v in Γ is the number of edges incident to v and we denote it by $deg_\Gamma(v)$. The order of Γ is $|V(\Gamma)|$, the

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maximum and the minimum degree of Γ is denoted by $\Delta(\Gamma)$ and $\delta(\Gamma)$, respectively. A graph Γ is regular if the degree of all vertices are equal. A vertex of degree 0 is known as an isolated vertex of Γ. A graph Ω is called a subgraph of Γ if either $V(\Omega) \subseteq V(\Gamma)$ or $E(\Omega) \subseteq E(\Gamma)$ or both. For $\Gamma = (V, E)$, let $S \subseteq V$. A subgraph Ω of Γ is said to be an induced subgraph of Γ induced by S, if $V(\Omega) = S$ and each edge of Γ having its ends in S is also an edge of Ω. A simple graph Γ is said to be complete if every pair of distinct vertices of Γ are adjacent. An Eulerian graph has an Eulerian trail, a closed trail containing all vertices and edges. Let G be a group with identity e. The order of the group G is the number of elements in G and is denoted by $o(G)$. The order of an element a in a group G is the smallest positive integer k such that $a^k = e$. If no such integer exists, we say a has infinite order. The order of an element a is denoted by $o(a)$. The notation $< a >$ represents the subgroup generated by the element a. Let p be a prime number. A group G with $o(G) = p^k$ for some $k \in \mathbb{Z}^+$, is called a p-group. For undefined terms and definitions in graph theory and in algebra, we refer to [4] and [5], respectively.

The order prime graph of a group, denoted as $OP(G)$ was introduced in 2009 [7]. In this graph vertices are the elements of the group and any two distinct vertices are adjacent if and only if their orders are relatively prime. In a similar way, 2-order prime graph is defined in [8] as follows: Let G be a group with identity e. The 2-order prime graph Γ_{2op} of G is a graph with $V(\Gamma_{2op}) = G - e$ and two distinct vertices x and y are adjacent in Γ_{2op} if and only if $gcd(o(x), o(y)) = 2$. In 2016, the concept of an order divisor graph of a group, denoted by $OD(G)$, was introduced in [6], where the vertices are the elements of G and two distinct vertices x and y having different orders are adjacent provided that $o(x)|o(y)$ or $o(y)|o(x)$. In 2017, the order divisor graphs of finite groups was defined and studied in [3], where they considered the subgroups of G as the vertices and two distinct vertices H, K are adjacent if and only if either $o(H)|o(K)$ or $o(K)|o(H)$, where $o(H)$, $o(K)$ denote the orders of H and K, respectively. In 2021, the order prime graph of group was defined and studied in [9], where the authors considered the group elements as the vertex set and two distinct vertices are adjacent if and only if $o(ab) = 1$ or $o(ab) = p$ for some prime p.

2 Non-identity order divisor graphs of groups and their properties

Motivated by the above concepts, in this paper, we define and study the non-identity order divisor graphs.

Definition 1. *The non-identity order divisor graph, denoted by $\Gamma_{niod}(G)$, of a group G is a graph where $V(\Gamma_{niod}(G)) = G - \{e\}$ and two distinct vertices x and y are adjacent in $\Gamma_{niod}(G)$ if and only if either $o(x)|o(y)$ or $o(y)|o(x)$.*

Next, we observe some basic properties of non-identity order divisor graphs.

Proposition 1. *Let G be a finite group with identity e. For any $x \in G - \{e\}$, x and x^{-1} are adjacent in $\Gamma_{niod}(G)$.*

Proof. Let $x \in G - \{e\}$. The adjacency follows as $o(x) = o(x^{-1})$. □

Proposition 2. *Let G be a finite group with identity e. For any $x \in G - \{e\}$, $deg_{\Gamma_{niod}(G)}(x) \geq o(x) - 2$.*

Proof. Let $x \in G - \{e\}$. Clearly, x is adjacent to $x^2, x^3, \ldots, x^{o(x)-1}$ and so $deg_{\Gamma_{niod}(G)}(x) \geq o(x) - 2$ for all $x \in G - \{e\}$. □

Proposition 3. *For any finite group G, the isolated vertices of $\Gamma_{niod}(G)$ are self inverse elements in G.*

Proof. Suppose a is not a self inverse element in G, then $o(a) > 2$. Since $o(a) = o(a^{-1})$, a and a^{-1} are adjacent in $\Gamma_{niod}(G)$ and so a is not an isolated vertex of $\Gamma_{niod}(G)$. □

Remark 1. The converse of Proposition 3 is not true. For example, consider $G = (\mathbb{Z}_6, +_6)$ and $a = 3$. In G, 3 is a self inverse element, but $deg_{\Gamma_{niod}(G)}(3) = 2$, implying that 3 is not an isolated vertex.

Theorem 1. *Let G be a finite group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$, where $p_1, p_2, \ldots, p_n$ are prime numbers and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive integers. If G has unique subgroups of orders $p_1, p_2, \ldots, p_n$, then $\Gamma_{niod}(G)$ is connected.*

Proof. Since each p_i divides $o(G)$, G contain elements a_i such that $o(a_i) = p_i$, for $1 \leq i \leq n$. Let $x_i, x_j \in G$ such that $o(x_i) = p_i$ and $o(x_j) = p_j$ for some $1 \leq i, j \leq n$ and $i \neq j$. Consider the subgroups $H_i =< x_i >$ and $H_j =< x_j >$ of G. By the assumption, H_i and H_j are unique subgroups of order p_i and p_j, respectively. This implies H_i and H_j are normal subgroups of G. Therefore, H_iH_j is also a normal subgroup in G and so $o(H_iH_j) = p_ip_j$. Clearly, the subgroup H_iH_j is cyclic and so it contains an element y of order p_ip_j. Therefore, x_iyx_j is a path in $\Gamma_{niod}(G)$. Suppose $a, b \in G$. Then, there exist p_i and p_j for some i, j with $1 \leq i, j \leq n$ such that $p_i | o(a)$ and $p_j | o(b)$. This gives, the path ax_iyx_jb between a and b. Hence $\Gamma_{niod}(G)$ is connected. □

Theorem 2. *Let G be a finite group. Then $\Gamma_{niod}(G)$ is a complete graph if and only if $o(G) = p^m$, where p is a prime number and m is a positive integer.*

Proof. Let G be a group of order n. Assume that $\Gamma_{niod}(G)$ is a complete graph. Suppose, if possible, n is not a power of a prime, then there exist at least two prime divisors p and q of n. By Cauchy's theorem, G has two elements a and b such that $o(a) = p$ and $o(b) = q$. Clearly, $o(a)$ does not divide $o(b)$ so a and b are non adjacent in $\Gamma_{niod}(G)$, which is a contradiction. Hence $o(G) = p^m$, for some prime number p. Conversely, assume that $o(G) = p^m$ for some prime p. Since $o(G) = p^m$, for any $x \in G - \{e\}$, $o(x) = p^k$ where k is an integer with $1 \leq k \leq m$. Therefore, $o(x) | o(y)$ or $o(y) | o(x)$ for all $x, y \in G - \{e\} = V(\Gamma_{niod}(G))$ and hence $\Gamma_{niod}(G)$ is complete. □

This results implies that:

Remark 2. *Let G be a group of order p^m, where p is a prime number. Then, $\Gamma_{niod}(G) \cong K_{p^m-1}$*

Proposition 4. *Let G be a finite group and q be the number of edges in $\Gamma_{niod}(G)$. Then $q \geq \frac{\sum\limits_{x \in G - \{e\}} (o(x)-2)}{2}$. Moreover, this bound is sharp.*

Proof. By Proposition 2, for all $x \in G - \{e\}$, $deg_{\Gamma_{niod}(G)}(x) \geq o(x) - 2$.
Thus, $\sum\limits_{x \in G-\{e\}} deg_{\Gamma_{niod}(G)}(x) \geq \sum\limits_{x \in G-\{e\}} o(x) - 2$. Hence $q \geq \frac{\sum\limits_{x \in G-\{e\}} (o(x)-2)}{2}$. □

Remark 3. For a prime p and a group $\mathbb{Z}_p$, $\Gamma_{niod}(\mathbb{Z}_p) \cong K_{p-1}$ and the bound is sharp for this graph.

We now characterize the groups G for which the associated graphs $\Gamma_{niod}(G)$ attains this bound.

Theorem 3. *Let G be a finite group. The number of edges in $\Gamma_{niod}(G)$, denoted by q, is equal to $\frac{\sum\limits_{x \in G-\{e\}} o(x)-2}{2}$ if and only if G is a group of prime order.*

Proof. Let G be a group of prime order p. Clearly, every element of $G - \{e\}$ has order p. Therefore, the degree of each element in $\Gamma_{niod}(G)$ is $o(x) - 2$. Hence the result follows.

Conversely, let $q = \frac{\sum\limits_{x \in G-\{e\}} o(x)-2}{2}$. If possible, assume that, the order of G has atleast two distinct prime divisors p_1 and p_2. Then by Cauchy's theorem, G has two elements a and b such that $o(a) = p_1$ and $o(b) = p_2$. We know that the number of elements of order d in a finite group is divisible by $\phi(d)$, where $\phi(d)$ is the Euler's ϕ function of d. Therefore, if there are two integers r and s, then the number of elements of order p_1 is equal to $r\phi(p_1) = r(p_1 - 1)$ and the number of elements of order p_2 is $s(p_2 - 1)$. By our assumption $deg(a)$ in $\Gamma_{niod}(G)$ is $o(a) - 2 = p_1 - 2$. Since every element of order p_1 is adjacent to a, $r(p_1 - 1) - 1 = p_1 - 2$ implies $r(p_1 - 1) = p_1 - 1$ and so $r = 1$. Similarly $s = 1$. Therefore, G has unique subgroups of order p_1 and p_2 and hence, $o(ab) = p_1 p_2$. Since $ab \notin < a >$, $deg(a) > o(a) - 2$, which is a contradiction. Therefore, $o(G)$ must be divisible by a unique prime p. That is, $o(G) = p^n$, where $n \geq 1$. Suppose $n > 1$, by Theorem 2, $\Gamma_{niod}(G)$ is complete and thus, $deg(a) > o(a) - 2$, which is again a contradiction. Hence, G is a group of prime order p. □

Theorem 4. *For any finite group G, $\Gamma_{niod}(G)$ is a tree if and only if G is isomorphic to one of the groups $\mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof. Clearly $\Gamma_{niod}(\mathbb{Z}_2) = K_1$ and $\Gamma_{niod}(\mathbb{Z}_3) = K_2$ and hence $\Gamma_{niod}(G)$ is a tree where G is either $\mathbb{Z}_2$ or $\mathbb{Z}_3$.
Conversely, assume that $\Gamma_{niod}(G)$ is a tree. Suppose $p|o(G)$ for some prime number $p \geq 5$. Then, G has an element of order p. By Remark 2, $\Gamma_{niod}(G)$ has a subgraph $K_{p-1}; p \geq 5$ and so $\Gamma_{niod}(G)$ has a cycle, which is a contradiction. Thus, p must be either 2 or 3. Now, there arise two cases:

Case 1: Suppose G has elements of orders 2 and 3, $o(G)$ must be greater than or equal to 6. If G has unique subgroups of orders 2 and 3, then G must contain an element of order 6 and these three elements form a cycle, which is a contradiction. Therefore, G must contain at least 3 elements of order 2 or 3 and so $K_3 \subset \Gamma_{niod}(G)$, which is again a contradiction.

Case 2: Suppose every element of $G - \{e\}$ is of order either 2 or 3. Then, $o(G) = 2^m$ or $3^m; m \in \mathbb{Z}^+$. Suppose $m > 1$, by Theorem2, $\Gamma_{niod}(G)$ is complete and so $K_3 \subset \Gamma_{niod}(G)$, again a contradiction to the assumption. Hence, either $G \cong \mathbb{Z}_2$ or $G \cong \mathbb{Z}_3$. □

Since every star is a tree, from the above theorem, we get the following corollary.

Corollary 1. *$\Gamma_{niod}(G)$ is a star if and only if $G \cong \mathbb{Z}_3$.*

Theorem 5. *$\Gamma_{niod}(G)$ is cycle C_n if and only if $o(G) = 4$.*

Proof. Let $o(G) = 4$. Therefore, either $G \cong \mathbb{Z}_4$ or $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence $\Gamma_{niod}(G) \cong C_3$. Conversely, assume that $\Gamma_{niod}(G)$ is a cycle C_n. Since every element a of a cycle has degree 2 and in $\Gamma_{niod}(G)$, $deg(a) \geq o(a)-2$, $o(a) = 4$ or 2. Therefore $o(G) = 2^n$. By Theorem 2, $\Gamma_{niod}(G)$ is complete. The graph which is both cycle and complete is C_3 only. Hence, $o(G) = 4$. □

Theorem 6. *$\Gamma_{niod}(G)$ is a bipartite graph if and only if $G \cong \mathbb{Z}_3$.*

Proof. Let $G \cong \mathbb{Z}_3$. Clearly, $\Gamma_{niod}(\mathbb{Z}_3) = K_2$ and hence $\Gamma_{niod}(G)$ is a bipartite graph.
Conversely, assume that $\Gamma_{niod}(G)$ is a bipartite graph. Therefore, $\Gamma_{niod}(G)$ has no odd cycle. Using the same proof technique as in the converse part of Theorem 4, for any possible graph other than $\mathbb{Z}_3$, we obtain K_3 as a subgraph of $\Gamma_{niod}(G)$, which is an odd cycle. Hence, we conclude that $G \cong \mathbb{Z}_3$. □

Theorem 7. *Let G be a group of order pq, where p and q are distinct primes such that $p < q$ and ϕ be the Euler's ϕ function. Then:*

$$\Gamma_{niod}(G) \cong \begin{cases} (K_{p-1} \cup K_{q-1}) + K_{\phi(pq)}, & \text{if } G \text{ is cyclic;} \\ K_{q(p-1)} \cup K_{q-1}, & \text{if } G \text{ is non cyclic.} \end{cases}$$

Proof. Let G be a group of order pq, where p and q are two distinct primes such that $p < q$ and ϕ be the Euler's ϕ function. We have to consider the following two cases:

Case 1: Let G be a cyclic group. Then, G has a unique p-Sylow subgroup, namely, H_p and another unique q-Sylow subgroup, namely, H_q. Clearly, $\Gamma_{niod}(H_p) \cong K_{p-1}$ and $\Gamma_{niod}(H_q) \cong K_{q-1}$. Note that all elements in $G-(H_p \cup H_q)$ are generators of G and so $|G-(H_p \cup H_q)| = \phi(pq)$. The generators of G are adjacent to all other vertices in $\Gamma_{niod}(G)$, so $\Gamma_{niod}(G) \cong (K_{p-1} \cup K_{q-1}) + K_{\phi(pq)}$.

Case 2: Let G be a non-cyclic group. This implies, G has q p-Sylow subgroups and a unique q-Sylow subgroup. Hence $\Gamma_{niod}(G) \cong K_{q(p-1)} \cup K_{q-1}$. □

Theorem 8. *Let G be a finite group. Then $\Gamma_{niod}(G)$ is Eulerian if and only if $o(G) = 2^n$, where n is a positive integer.*

Proof. Assume that G is a group of order 2^n, where n is a positive integer. By Theorem 2, $\Gamma_{niod}(G) \cong K_{2^n-1}$ and so $\Gamma_{niod}(G)$ is Eulerian. Conversely, suppose that $\Gamma_{niod}(G)$ is Eulerian. If possible, let $o(G) \neq 2^n; n \in \mathbb{Z}^+$, then there exists a prime $p \neq 2$ such that $p|o(G)$. Let a be the element of G such that $o(a) = p$. Then, we have two cases:

Case 1: Let $deg(a) = o(a) - 2$ in $\Gamma_{niod}(G)$, then $deg(a)$ is odd, which is a contradiction that $\Gamma_{niod}(G)$ is Eulerian.

Case 2: Let $deg(a) > o(a) - 2$. Then there exists an element $b \in G$ such that $b \notin < a >$ and b is adjacent to a. Note that $o(b) = o(b^{-1})$ and a is adjacent to both b and b^{-1}. Since $p \neq 2$ and a is adjacent to b, b is not a self inverse element. Hence, $b \neq b^{-1}$. Therefore, whenever b is adjacent to a, b^{-1} is also adjacent to a. Since $o(a)$ is odd, a is not adjacent to any self-inverse element of G and so $deg(a)$ is odd in $\Gamma_{niod}(G)$, which is a contradiction to $\Gamma_{niod}(G)$ is Eulerian. Hence $o(G) = 2^n$. □

Theorem 9. *Let G be a group of order $n = p_1^2 p_2$, where p_1 and p_2 be distinct prime numbers. Let k_1, k_2, k_3, k_4 and k_5 be the number of elements of orders p_1, p_2, p_1p_2, p_1^2 and $p_1^2 p_2$ respectively. Then the number of edges in $\Gamma_{niod}(G)$ is $\frac{n(n-1)-2t}{2}$, where $t = k_1k_2 + k_2k_4 + k_3k_4$.*

Proof. Let G be a group of order $n = p_1^2 p_2$, where p_1 and p_2 are distinct primes. Let k_1, k_2, k_3, k_4 and k_5 be the number of elements of orders p_1, p_2, p_1p_2, p_1^2 and $p_1^2p_2$ respectively. Let A, B, C, D and E be the partition of the elements of $G - \{e\}$ having order p_1, p_2, p_1p_2, p_1^2 and $p_1^2p_2$, respectively. Therefore, $|A| = k_1, |B| = k_2, |C| = k_3, |D| = k_4, |E| = k_5$.

Case 1: Let G be a cyclic group. Since A, B, C, D and E are the partitions of $G - \{e\}$, $n = k_1 + k_2 + k_3 + k_4 + k_5$. Clearly, the graph induced by each of the sets A, B, C, D and E is complete. Every element of the set A is adjacent to every element of the sets C, D and E. Then, the degree of each element of the set A is equal to $k_1 - 1 + k_3 + k_4 + k_5 = n - k_2 - 1$. Thus, the sum of the degrees of all elements in the set A is equal to $nk_1 - k_1k_2 - k_1$. Similarly, the sum of the degrees of all elements in the sets B, C, D and E are $nk_2 - k_1k_2 - k_2k_4 - k_2$, $nk_3 - k_3k_4 - k_3$, $nk_4 - k_2k_4 - k_3k_4 - k_4$ and $nk_5 - k_5$, respectively. Thus, the sum of the degrees of all elements is $n^2 - n - 2t$, where $t = k_1k_2 + k_2k_4 + k_3k_4$. Hence the number of edges in $\Gamma_{niod}(G)$ is $\frac{n(n-1)-2t}{2}$.

Case 2: Let G be a non-cyclic group. Then, G can not have elements of the order $p_1^2p_2$. Thus, A, B, C, D are the partitions of $G - \{e\}$ and $n = k_1 + k_2 + k_3 + k_4$. Clearly, the graph induced by each of the sets A, B, C and D is complete. Every element of the set A is adjacent to every element of the sets C and D. Then, the degree of each element of the set A is equal to $k_1 - 1 + k_3 + k_4 = n - k_2 - 1$. Thus, the sum of the degrees of all elements in the set A is equal to $nk_1 - k_1k_2 - k_1$. Similarly, the sum of the degrees of all elements in the sets B, C and D are $nk_2 - k_1k_2 - k_2k_4 - k_2$, $nk_3 - k_3k_4 - k_3$ and $nk_4 - k_2k_4 - k_3k_4 - k_4$, respectively. Therefore, the sum of the degrees of all elements is $n^2 - n - 2t$, where $t = k_1k_2 + k_2k_4 + k_3k_4$. Hence the number of edges in $\Gamma_{niod}(G)$ is $\frac{n(n-1)-2t}{2}$.

From both cases we conclude that the number edges of $\Gamma_{niod}(G)$ is $\frac{n(n-1)-2t}{2}$, where $n = p_1^2p_2$ and $t = k_1k_2 + k_2k_4 + k_3k_4$. □

3 Relation between the order prime graphs and non-identity order divisor graphs of groups

Next, we discuss the relation between the order prime graphs $OP(G)$ and non-identity order divisor graphs of groups $\Gamma_{niod}(G)$.

Theorem 10. *Let G be a group of order $2p$, where p is an odd prime. Then, $OP(G) - \{e\} \cong \overline{\Gamma_{niod}(G)}$.*

Proof. Here, we have to consider the following two cases:

Case 1: Let G be a cyclic group. Then $G - \{e\}$ has an element of order 2, $p - 1$ elements of order p and $p - 1$ elements of order $2p$. In $OP(G)$, the element of order 2 is adjacent to only the elements of order p and remaining elements are non-adjacent to each other. Therefore, $OP(G) - \{e\} \cong K_{1,p-1} \cup \overline{K_{p-1}}$. In $\Gamma_{niod}(G)$, the elements of order p and $2p$ are adjacent to each other. The element of order 2 is adjacent only to the elements of order $2p$. Therefore, $\Gamma_{niod}(G) \cong K_{p-1} + (K_1 \cup K_{p-1})$ and so $\overline{\Gamma_{niod}(G)} \cong K_{1,p-1} \cup \overline{K_{p-1}}$. Hence, $OP(G) - \{e\} \cong \overline{\Gamma_{niod}(G)}$.

Case 2: Let G be a non-cyclic group. Then, $G - \{e\}$ can be partitioned into two sets A and B such that A is the set of elements of order 2 and B is the set of elements of order p. Let $|A| = m$ and $|B| = n$. In $OP(G)$, each element of A is adjacent to all the elements of B and no elements of A and no elements of B are adjacent to each other. Therefore, $OP(G) - \{e\} \cong K_{m,n}$. In $\Gamma_{niod}(G)$, each element of A is adjacent to each other and each element of B is adjacent to each other and no element of A is adjacent to any element of B. Therefore, $\Gamma_{niod}(G) \cong K_m \cup K_n$ and so $\overline{\Gamma_{niod}(G)} \cong K_{m,n}$. Hence, $OP(G) - \{e\} \cong \overline{\Gamma_{niod}(G)}$.

□

Theorem 11. *Let G be a group of order p^m, where p is a prime number. Then, $OP(G) - \{e\} \cong \overline{\Gamma_{niod}(G)}$.*

Proof. Let G be a group such that $o(G) = p^m$. Let $a, b \in G - \{e\}$ such that a and b are non-adjacent in $OP(G) \iff gcd(o(a), o(b)) = p^k \iff o(b)$ is a multiple of $o(a) \iff a$ and b are adjacent in $\Gamma_{niod}(G) \iff a$ and b are non-adjacent in $\overline{\Gamma_{niod}(G)}$. Therefore, $OP(G) - \{e\} \cong \overline{\Gamma_{niod}(G)}$. □

Proposition 5. *Let G be a group. Let $a, b \in G$ such that $o(a) \leq o(b)$. Then, a and b are non-adjacent in $\Gamma_{niod}(G)$ if and only if $gcd(o(a), o(b)) < o(a)$.*

Proof. Let G be a group and $a, b \in G$ such that $o(a) \leq o(b)$. Assume that a and b are non-adjacent in $\Gamma_{niod}(G)$. Therefore, $o(a) \nmid o(b)$ and hence, $gcd(o(a), o(b)) < o(a)$. Conversely, assume that $gcd(o(a), o(b)) < o(a)$. Suppose a and b are adjacent and $o(a)|o(b)$. Therefore, $o(b) = ko(a)$, which implies $gcd(o(a), o(b)) = o(a)$, which is a contradiction to our assumption. Hence, a and b are non-adjacent. □

Theorem 12. *Let G be a group. The order prime graph of $G - \{e\}$ is a subgraph of the complement of non-identity order divisor graph.*

Proof. Let G be a group. Let $a, b \in G - \{e\}$ such that $o(a) \leq o(b)$. If a and b are adjacent in $OP(G)$, then $gcd(o(a), o(b)) = 1$. Therefore, $gcd(o(a), o(b)) < o(a)$. By Proposition 5, a and b are non-adjacent in $\Gamma_{niod}(G)$ and hence a and b are adjacent in its complement. □

Remark 4. *The converse of the above theorem need not be true. For example, consider the group $(\mathbb{Z}_{12}, +_{12})$. Here, $o(2) = 6$ and $o(3) = 4$. Thus, $gcd(o(2), o(3)) = 2$. Therefore, the vertices 2 and 3 are non-adjacent in $OP(G)$ and in non-identity order divisor graph.*

Theorem 13. *Let G be a group of order $p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are distinct prime numbers and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers. Let β_0 be the independence number of $\Gamma_{niod}(G)$. Then, $\beta_0(\Gamma_{niod}(G)) \geq k$.*

Proof. Let G be a group and $o(G) = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_k^{\alpha_k}$. Let β_0 be the independence number of $\Gamma_{niod}(G)$. Since each $p_i | o(G)$, by Cauchy's theorem, there exists $a_1, a_2, \ldots, a_k \in G$ such that $o(a_1) = p_1, o(a_2) = p_2, \ldots, o(a_k) = p_k$. Clearly, the set $\{a_1, a_2, \ldots, a_k\}$ is an independent set of $\Gamma_{niod}(G)$. Therefore, $\beta_0(\Gamma_{niod}(G)) \geq k$. □

4 Conclusion

In this paper, we have defined the non-identity order divisor graph $\Gamma_{niod}(G)$ on a group G with identity e. We have studied connectedness, completeness, bipartiteness, Eulerian property, and independence number of $\Gamma_{niod}(G)$ and we have characterized the groups for which $\Gamma_{niod}(G)$ is a tree. Furthermore, we established a relationship between the non-identity order divisor graph of group G and the order prime graph. There is further scope for studying the domination aspects, the chromatic number and other related properties of $\Gamma_{niod}(G)$.

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