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The algebraic classification of 7-dimensional nilpotent 3-Lie algebras

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Abstract. This paper focuses on the classification of 7-dimensional nilpotent 3-Lie algebras. We employ a systematic approach by considering the structure of these algebras through the central ideals. Specifically, we divide the 7-dimensional nilpotent 3-Lie algebra by a 1-dimensional central ideal, resulting in a 6- dimensional nilpotent 3-Lie algebra. Our findings reveal the relationships between 7-dimensional structures and their 6-dimensional counterparts, contributing to a deeper understanding of the properties and classifications of nilpotent 3-Lie algebras.

Keywords: Nilpotent n-Lie algebra, Algebraic classification, Low dimensions.
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1 Introduction

The classification of Lie algebras is one of the important and classical issues in the theory of finite-dimensional Lie algebras. The first research on the classification of nilpotent Lie algebras was done by Umlauf [31] for 6-dimensional, in his thesis at the late 19th century, and then continued by Morozov [25]. Since then, many efforts have been made to complete the classification of nilpotent Lie algebras over different fields in the literature [5]. In dimension 7, the first classification was done based on Morozov's method [30]. However, due to the incompleteness of the results and the complexity of these calculations, the issue of computing the complete lists was a recurring problem. Seeley [30] and Gong [11] presented the classification of nilpotent Lie algebras of dimension 7 on complex ($\mathbb{C}$) and real ($\mathbb{R}$) spaces. In dimension 8, there are only partial results [27]. The classification of 9-dimensional 2-step nilpotent 3-Lie algebras is done in [7]. For the 2-step nilpotent case, Ren and Zhu [28] provided a complete classification of 2-step nilpotent Lie algebras of dimension 8 with a 2-dimensional center. A complete classification of 2-step nilpotent Lie algebras of dimension 8, and in the field of complex numbers is given in [32].

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In addition, the classification of 2-step nilpotent Lie algebras of dimension 9 with 2-dimensional center is presented in [29].

There are important results in geometric classification of Lie algebras [4, 13]. The geometric classification is developed for many other varieties of (non-Lie) algebras [16, 24]). Moreover, the study of rigid 2-step nilpotent in a variety of 2-step nilpotent anticommutative algebras has been done in [1]. Some other important works in algebraic classifications of Lie and anticommutative algebras could be found in [12, 19, 22]. Moreover, there are many results with algebraic classifications in non-Lie varieties of algebras [14, 20].

In 1985, Filippov [10] introduced the concept of n-Lie algebras. An anticommutative algebra A with operation $[x_1, \dots, x_n]$ is called n-Lie algebra, when it satisfies the following generalized Jacoby identity:

$$[[x_1, x_2, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^{n} [x_1, \dots, x_{i-1}, [x_i, y_2, \dots, y_n], x_{i+1}, \dots, x_n].$$

Assume that $A_1, \dots, A_n$ are subalgebras of an n-Lie algebra A. Then, the subalgebra of A generated by all vectors $[x_1, \dots, x_n]$ $(x_i \in A_i)$ will be represented by the symbol $[A_1, \dots, A_n]$. The subalgebra $A^2 = [A, \dots, A]$ is called the derived n-Lie algebra of A. The center of n-Lie algebra A is defined as follows:

$$Z(A) = \{x \in A : [x, A, \dots, A] = 0\}.$$

Assume that $Z_0(A) = 0$. Then ith center of A is defined inductively as

$$Z_i(A)/Z_{i-1}(A) = Z(A/Z_{i-1}(A)) \qquad \text{for all } i \geq 1.$$

The notion of nilpotent n-Lie algebras was defined by Kasymov [18]. An n-Lie algebra A is nilpotent if $A^s = 0$, where s is a nonnegative integer number. Note that A^i is defined as inducted by $A^1 = A, A^{i+1} = [A^i, A, \dots, A]$. An n-Lie algebra A is nilpotent of class c if $A^{c+1} = 0$ and $A^i \neq 0$ for each $i \leq c$.

The algebraic classification of n-Lie algebras is also discussed in the literature. Some new examples of n-Lie algebras over positive characteristics and in infinite-dimensional case are given in [26], and a classification of simples n-Lie algebras is given in [23]. The study of the variety of n-Lie algebra structures on an $(n+1)$-dimensional vector space in complex and finite-dimensional cases is done in [21], and the classification of simple n-Lie algebras and superalgebras in a more general case is done in [3]. On the other hand, determining the structure of d-dimensional n-Lie algebras is a classic problem, which is answered for dimension of at most $n + 2$ in [2, 10]. The $(n + 3)$-dimensional nilpotent n-Lie algebras over arbitrary field are classified in [6]. The d-dimensional 2-step nilpotent n-Lie algebras with $d \leq n + 6$ is classified in [7, 15, 17].

In this paper, we have classified 7-dimensional nilpotent 3-Lie algebras, building upon the foundational work established by Darabi et al. [6]. Previous classifications have provided significant insights into the structure and properties of nilpotent algebras, particularly in lower dimensions. Our results extend these findings by introducing new classifications. This not only enhances our understanding of 3-Lie algebras but also opens avenues for further research in related fields.

2 7-dimensional nilpotent 3-Lie algebras with the derived subalgebra of dimension 2

In this section, we classify 7-dimensional nilpotent 3-Lie algebras with the derived subalgebra of dimension 2. An important category of nilpotent Lie algebras, which plays an important role in the classification of 2-step nilpotent Lie algebras, is the Heisenberg algebras. We call an n-Lie algebra A, as a generalized Heisenberg of rank k, if $A^2 = Z(A)$ and $\dim A^2 = k$. The authors in [9] studied the case when $k = 1$, which is called later special Heisenberg n-Lie algebras.

Theorem 1 ([9]). *Every special Heisenberg n-Lie algebra has dimension $mn+1$ for some natural number m, and it is isomorphic to*

$$H(n,m) = \left\langle x, x_1, \ldots, x_{nm} : \left[x_{n(i-1)+1}, x_{n(i-1)+2}, \ldots, x_{ni}\right] = x, i = 1, \ldots, m \right\rangle.$$

The following lemmas is an immediate consequence of [6, Theorems 3.3 and 4.3].

Lemma 1. *The only nonabelian nilpotent 3-Lie algebras of dimension 6 are $H(3,1) \oplus F(2)$, $A_{3,5,1} \oplus F(1)$, $A_{3,6,i}$ $(1 \leq i \leq 5)$.*

Lemma 2. *The only 7-dimensional 2-step nilpotent 3-Lie algebras are $H(3,1) \oplus F(3)$, $H(3,2)$, $A_{3,6,1} \oplus F(1)$, $A_{3,7,1}$, and $A_{3,7,2}$.*

Now, we present the classification of the 7-dimensional nilpotent 3-Lie algebra with the derived subalgebra of dimension 2. Let A be a 7-dimensional nilpotent 3-Lie algebra of class three with the derived subalgebra of dimension t2 with basis $\{e_1, \ldots, e_7\}$. Since $0 \neq A^3 \subset Z(A)$, we have $A^3 \subset A^2 \cap Z(A)$. Let e_7 be a central element of A^2. Then $A/\langle e_7 \rangle$ is a nilpotent 3-Lie algebra of dimension 6 with the derived algebra of dimension 1. By using Lemma 1, we have $A/\langle e_7 \rangle \cong H(3,1) \oplus F(2)$.

In this case, the multiplication table in A can be written as

$$[e_1, e_2, e_3] = e_6 + \alpha e_7, \quad [e_i, e_j, e_k] = \alpha_{ijk} e_7,$$

where $1 \leq i < j < k \leq 6$, and $\{i, j, k\} \notin \{\{1, 2, 3\}\}$.

Regarding a suitable change of basis, one can assume that $\alpha = 0$, and the Jacobi identity gives us $\alpha_{146} = \alpha_{246} = \alpha_{346} = \alpha_{156} = \alpha_{256} = \alpha_{356} = \alpha_{456} = 0$.

The above multiplication shows that the dimension of the center of A is at most 3. We discuss on the dimension of the center of A.

Case 1: Let $\dim Z(A) = 3$. Then, up to isomorphism, we have two possibilities for $Z(A)$:

(i) $Z(A) = \langle e_5, e_6, e_7 \rangle$. In this case, the multiplication in A can be written as follows:

$$[e_1, e_2, e_3] = e_6, \quad [e_2, e_3, e_4] = \alpha_{234} e_7, \quad [e_1, e_3, e_4] = \alpha_{134} e_7, \quad [e_1, e_2, e_4] = \alpha_{124} e_7.$$

At least one of $\alpha_{234}, \alpha_{134}, \alpha_{124}$ is not equal to zero. Without loss of generality, assume that $\alpha_{234} \neq 0$. Applying the transformations

$$e_1' = e_1 - \frac{\alpha_{134}}{\alpha_{234}} e_2 + \frac{\alpha_{124}}{\alpha_{234}} e_3, \quad e_j' = e_j, \quad 2 \leq j \leq 6, \quad e_7' = \alpha_{234} e_7,$$

we obtain

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_6, [e_2, e_3, e_4] = e_7 \rangle,$$

which is isomorphic to $A_{3,6,1} \oplus F(1)$.

(ii) $Z(A) = \langle e_4, e_5, e_7 \rangle$. In this case, the multiplication in A can be written as follows:

$$[e_1, e_2, e_3] = e_6, \quad [e_2, e_3, e_6] = \alpha_{236} e_7, \quad [e_1, e_3, e_6] = \alpha_{136} e_7, \quad [e_1, e_2, e_6] = \alpha_{126} e_7.$$

At least one of $\alpha_{236}, \alpha_{136}, \alpha_{126}$ is not equal to zero. Without loss of generality, assume that $\alpha_{236} \neq 0$. Applying the transformations

$$e'_1 = e_1 - \frac{\alpha_{136}}{\alpha_{236}} e_2 + \frac{\alpha_{126}}{\alpha_{236}} e_3, \quad e'_j = e_j, \quad 2 \leq j \leq 6, \quad e'_7 = \alpha_{236} e_7,$$

we obtain

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_6, [e_2, e_3, e_6] = e_7 \rangle,$$

which is isomorphic to $A_{3,5,1} \oplus F(2)$.

Case 2: Let $\dim Z(A) = 2$. Then, up to isomorphism, we have two possibilities for $Z(A)$:

(i) $Z(A) = \langle e_6, e_7 \rangle$. In this case, $A^2 = Z(A)$. Thus, by Theorem 3.2 of [8],

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_6, [e_3, e_4, e_5] = e_7 \rangle.$$

This algebra is nilpotent of class two.

(ii) $Z(A) = \langle e_5, e_7 \rangle$. In this case, the multiplication in A can be written as follows:

$$\begin{aligned} &[e_1, e_2, e_3] = e_6, && [e_2, e_3, e_4] = \alpha_{234} e_7, && [e_1, e_3, e_4] = \alpha_{134} e_7, && [e_1, e_2, e_4] = \alpha_{124} e_7, \\ &[e_2, e_3, e_6] = \alpha_{236} e_7, && [e_1, e_3, e_6] = \alpha_{136} e_7, && [e_1, e_2, e_6] = \alpha_{126} e_7. \end{aligned}$$

Since $\dim(A/\langle e_5 \rangle)^2 = 2$ and $\dim Z(A/\langle e_5 \rangle) = 1$, according to Lemma 1, A is isomorphic to $A_{3,6,2} \oplus F(1)$.

Case 3: Let $\dim Z(A) = 1$. Then, the multiplication table in A can be written as:

$$\begin{aligned} &[e_1, e_2, e_3] = e_6, && [e_2, e_3, e_4] = \alpha_{234} e_7, && [e_1, e_3, e_4] = \alpha_{134} e_7, && [e_1, e_2, e_4] = \alpha_{124} e_7, \\ &[e_2, e_3, e_5] = \alpha_{235} e_7, && [e_1, e_3, e_5] = \alpha_{135} e_7, && [e_1, e_2, e_5] = \alpha_{125} e_7, && [e_2, e_3, e_6] = \alpha_{236} e_7, \\ &[e_1, e_3, e_6] = \alpha_{136} e_7, && [e_1, e_2, e_6] = \alpha_{126} e_7, && [e_3, e_4, e_5] = \alpha_{345} e_7, && [e_2, e_4, e_5] = \alpha_{245} e_7, \\ &[e_1, e_4, e_5] = \alpha_{145} e_7. \end{aligned}$$

At least one of $\alpha_{236}, \alpha_{136}, \alpha_{126}$ is not equal to zero. Without loss of generality, assume that $\alpha_{236} \neq 0$. Applying the transformations

$$e'_1 = e_1 - \frac{\alpha_{136}}{\alpha_{236}} e_2 + \frac{\alpha_{126}}{\alpha_{236}} e_3, \quad e'_j = e_j, \quad 2 \leq j \leq 6, \quad e'_7 = \alpha_{236} e_7,$$

we obtain

$$\begin{aligned} &[e_1, e_2, e_3] = e_6, && [e_2, e_3, e_6] = e_7, \\ &[e_2, e_3, e_4] = \alpha_{234} e_7, && [e_1, e_3, e_4] = \alpha_{134} e_7, && [e_1, e_2, e_4] = \alpha_{124} e_7, \\ &[e_2, e_3, e_5] = \alpha_{235} e_7, && [e_1, e_3, e_5] = \alpha_{135} e_7, && [e_1, e_2, e_5] = \alpha_{125} e_7, \\ &[e_3, e_4, e_5] = \alpha_{345} e_7, && [e_2, e_4, e_5] = \alpha_{245} e_7, && [e_1, e_4, e_5] = \alpha_{145} e_7. \end{aligned}$$

Since e_4 and e_5 are not belong to $Z(A)$, we examine three cases:

(i) If $\alpha_{145} = \alpha_{245} = \alpha_{345} = 0$, then at least one of $\alpha_{234}, \alpha_{134}, \alpha_{124}$ is not equal to zero. We claim that α_{234} is zero. Let $\alpha_{234} \neq 0$. Then by applying the transformations

$$e_1' = e_1 - \frac{\alpha_{134}}{\alpha_{234}} e_2 + \frac{\alpha_{124}}{\alpha_{234}} e_3, \quad e_j' = e_j, \quad 2 \leq j \leq 7,$$

we obtain

$$\begin{array}{lll} [e_1, e_2, e_3] = e_6, & [e_2, e_3, e_6] = e_7, & [e_2, e_3, e_4] = \alpha_{234} e_7, \\ [e_2, e_3, e_5] = \alpha_{235} e_7, & [e_1, e_3, e_5] = \alpha_{135} e_7, & [e_1, e_2, e_5] = \alpha_{125} e_7. \end{array}$$

In this case, $e_6 - \frac{1}{\alpha_{234}} e_4$ is the central element, and this is a contradiction. Without loss of generality, assume that $\alpha_{134} \neq 0$. Applying the transformations

$$e_2' = e_2 - \frac{\alpha_{124}}{\alpha_{134}} e_3, \quad e_j' = e_j, \quad 1 \leq j \leq 7, j \neq 2,$$

we obtain

$$\begin{array}{lll} [e_1, e_2, e_3] = e_6, & [e_2, e_3, e_6] = e_7, & [e_1, e_3, e_4] = \alpha_{134} e_7, \\ [e_2, e_3, e_5] = \alpha_{235} e_7, & [e_1, e_3, e_5] = \alpha_{135} e_7, & [e_1, e_2, e_5] = \alpha_{125} e_7. \end{array}$$

At least one of $\alpha_{235}, \alpha_{135}, \alpha_{125}$ is not equal to zero. Similar to the previous case, $\alpha_{235} = 0$. On the other hand, if $\alpha_{135} \neq 0$, then we have a contradiction. Thus, the multiplication in A can be written as

$$[e_1, e_2, e_3] = e_6, \quad [e_2, e_3, e_6] = e_7, \quad [e_1, e_3, e_4] = \alpha_{134} e_7, \quad [e_1, e_2, e_5] = \alpha_{125} e_7.$$

By replacing e_4 by $\frac{1}{\alpha_{134}} e_4$ and e_5 by $\frac{1}{\alpha_{125}} e_5$, the algebra A is as follows:

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_6, [e_2, e_3, e_6] = [e_1, e_3, e_4] = [e_1, e_2, e_5] = e_7 \rangle,$$

which is denoted by $A_{3,7,3}$.

(ii) Let α_{345} be not zero. Then applying the transformations

$$e_1' = e_1 - \frac{\alpha_{145}}{\alpha_{345}} e_3, \quad e_2' = e_2 - \frac{\alpha_{245}}{\alpha_{345}} e_3, \quad e_j' = e_j, \quad 3 \leq j \leq 7,$$

we obtain

$$\begin{array}{lll} [e_1, e_2, e_3] = e_6, & [e_2, e_3, e_6] = e_7, & [e_3, e_4, e_5] = \alpha_{345} e_7, \\ [e_2, e_3, e_4] = \alpha_{234} e_7, & [e_1, e_3, e_4] = \alpha_{134} e_7, & [e_1, e_2, e_4] = \alpha_{124} e_7, \\ [e_2, e_3, e_5] = \alpha_{235} e_7, & [e_1, e_3, e_5] = \alpha_{135} e_7, & [e_1, e_2, e_5] = \alpha_{125} e_7. \end{array}$$

Now, by replacing e_1 by $e_1 + \frac{\alpha_{135}}{\alpha_{345}} e_4 - \frac{\alpha_{134}}{\alpha_{345}} e_5$, e_2 by $e_2 + \frac{\alpha_{235}}{\alpha_{345}} e_4 - \frac{\alpha_{234}}{\alpha_{345}} e_5$ and e_6 by $e_6 + \frac{\alpha_{234}\alpha_{135} - \alpha_{134}\alpha_{235}}{\alpha_{345}} e_7$, we obtain

$$\begin{array}{lll} [e_1, e_2, e_3] = e_6, & [e_2, e_3, e_6] = e_7, & [e_3, e_4, e_5] = \alpha_{345} e_7, \\ [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_1, e_2, e_5] = \alpha_{125} e_7. & \end{array}$$

Finally, by replacing e_1 by $\alpha_{345}(e_1 - e_3)$, e_2 by $e_2 + \frac{\alpha_{125}}{\alpha_{345}} e_4 - \frac{\alpha_{124}}{\alpha_{345}} e_5$, e_6 by $\alpha_{345} e_6$, and e_7 by $\alpha_{345} e_7$, we obtain

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_6, [e_2, e_3, e_6] = [e_3, e_4, e_5] = e_7 \rangle,$$

which is denoted by $A_{3,7,4}$.

(iii) Let α_{145} be not zero. Then by applying the transformations

$$e'_2 = e_2 - \frac{\alpha_{245}}{\alpha_{145}}e_1, \quad e'_3 = e_3 - \frac{\alpha_{345}}{\alpha_{145}}e_1, \quad e'_j = e_j, \quad 1 \leq j \leq 7, j \neq 2, 3,$$

we obtain

$$\begin{array}{lll} [e_1, e_2, e_3] = e_6, & [e_2, e_3, e_6] = e_7, & [e_1, e_4, e_5] = \alpha_{145}e_7, \\ [e_2, e_3, e_4] = \alpha_{234}e_7, & [e_1, e_3, e_4] = \alpha_{134}e_7, & [e_1, e_2, e_4] = \alpha_{124}e_7, \\ [e_2, e_3, e_5] = \alpha_{235}e_7, & [e_1, e_3, e_5] = \alpha_{135}e_7, & [e_1, e_2, e_5] = \alpha_{125}e_7. \end{array}$$

Similar to part (ii), by replacing e_2 by $e_2 - \frac{\alpha_{125}}{\alpha_{145}}e_4 + \frac{\alpha_{124}}{\alpha_{145}}e_5$, e_3 by $e_3 - \frac{\alpha_{135}}{\alpha_{145}}e_4 + \frac{\alpha_{134}}{\alpha_{145}}e_5$, and e_6 by $e_6 + \frac{\alpha_{134}\alpha_{125} - \alpha_{124}\alpha_{135}}{\alpha_{145}}e_7$, we obtain

$$\begin{array}{lll} [e_1, e_2, e_3] = e_6, & [e_2, e_3, e_6] = e_7, & [e_1, e_4, e_5] = \alpha_{145}e_7, \\ [e_2, e_3, e_4] = \alpha_{234}e_7, & [e_2, e_3, e_5] = \alpha_{235}e_7. & \end{array}$$

Finally, by replacing e_2 by $\alpha_{145}(-e_1 + e_2)$, e_3 by $e_3 + \frac{\alpha_{235}}{\alpha_{145}}e_4 - \frac{\alpha_{234}}{\alpha_{145}}e_5$, e_5 by $\alpha_{145}e_5$, e_6 by $\alpha_{145}e_6$, and e_7 by $\alpha_{145}^2e_7$, we obtain

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_6, [e_2, e_3, e_6] = [e_1, e_4, e_5] = e_7\rangle,$$

which is denoted by $A_{3,7,5}$.

The following theorem is an immediate consequence of the above results and Lemma 2.

Theorem 2. *Let A be a 7-dimensional nilpotent 3-Lie algebra with the derived subalgebra of dimension 2. Then A is isomorphic to $A_{3,5,1} \oplus F(2)$, $A_{3,6,1} \oplus F(1)$, $A_{3,6,2} \oplus F(1)$, $A_{3,7,1}$, $A_{3,7,3}$, $A_{3,7,4}$, or $A_{3,7,5}$.*

3 7-dimensional nilpotent 3-Lie algebras with the derived subalgebra of dimension 3

In this section, we classify 7-dimensional nilpotent 3-Lie algebras with the derived subalgebra of dimension 3. Let A be a 7-dimensional nilpotent 3-Lie algebra with the derived subalgebra of dimension 3 with basis $\{e_1, \ldots, e_7\}$. Let e_7 be a central element of A^2. Then $A/\langle e_7\rangle$ is a nilpotent 3-Lie algebra of dimension 6 with the derived algebra of dimension 2. By using Lemma 1, $A/\langle e_7\rangle$ is isomorphic to $A_{3,5,1} \oplus F(1)$, $A_{3,6,1}$, or $A_{3,6,2}$.

Lemma 3. *Let $A/\langle e_7\rangle \cong A_{3,5,1} \oplus F(1)$. Then A is isomorphic to $A_{3,6,3} \oplus F(1)$, $A_{3,6,4} \oplus F(1)$, $A_{3,6,5} \oplus F(1)$, $A_{3,7,6}$, $A_{3,7,7}$, $A_{3,7,8}$, $A_{3,7,9}$, $A_{3,7,10}$, or $A_{3,7,11}$.*

Proof. Let $A/\langle e_7\rangle \cong A_{3,5,1} \oplus F(1)$. In this case, the multiplication table in A can be written as

$$[e_1, e_2, e_3] = e_4 + \alpha e_7, \quad [e_2, e_3, e_4] = e_6 + \beta e_7, \quad [e_i, e_j, e_k] = \alpha_{ijk}e_7,$$

where $1 \leq i < j < k \leq 6$, and $\{i, j, k\} \notin \{\{1, 2, 3\}, \{2, 3, 4\}\}$. Regarding a suitable change of basis, one can assume that $\alpha = \beta = 0$, and the Jacobi identity gives us $\alpha_{146} = \alpha_{246} = \alpha_{346} =$

$\alpha_{156} = \alpha_{256} = \alpha_{356} = \alpha_{456} = \alpha_{145} = \alpha_{245} = \alpha_{345} = \alpha_{136} = \alpha_{126} = 0$. Thus the multiplication in A can be written as

$$\begin{array}{lll} [e_1,e_2,e_3] = e_4, & [e_2,e_3,e_4] = e_6, & [e_1,e_3,e_4] = \alpha_{134}e_7, \\ [e_1,e_2,e_4] = \alpha_{124}e_7, & [e_2,e_3,e_5] = \alpha_{235}e_7, & [e_1,e_3,e_5] = \alpha_{135}e_7, \\ [e_1,e_2,e_5] = \alpha_{125}e_7, & [e_2,e_3,e_6] = \alpha_{236}e_7. & \end{array}$$

We discuss on the dimension of the center of A. The above multiplication shows that the dimension of the center of A is at most 3.

(i) Let $\dim Z(A) = 3$. In this case, $Z(A) = \langle e_5, e_6, e_7\rangle$ and the multiplication in A can be written as follows:

$$[e_1,e_2,e_3] = e_4, \quad [e_2,e_3,e_4] = e_6, \quad [e_1,e_3,e_4] = \alpha_{134}e_7, \quad [e_1,e_2,e_4] = \alpha_{124}e_7.$$

Since the dimension of derived subalgebra is 3,

$$A = \langle e_1,\dots,e_7 : [e_1,e_2,e_3] = e_4, [e_2,e_3,e_4] = e_6, [e_1,e_3,e_4] = e_7\rangle.$$

This algebra is isomorphic to $A_{3,6,3} \oplus F(1)$.

(ii) Let $\dim Z(A) = 2$. In this case, we have two possibilities for $Z(A)$:

(a): $Z(A) = \langle e_6, e_7\rangle$. In this case, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1,e_2,e_3] = e_4, & [e_2,e_3,e_4] = e_6, & [e_1,e_3,e_4] = \alpha_{134}e_7, \\ [e_1,e_2,e_4] = \alpha_{124}e_7, & [e_2,e_3,e_5] = \alpha_{235}e_7, & [e_1,e_3,e_5] = \alpha_{135}e_7, \\ [e_1,e_2,e_5] = \alpha_{125}e_7. & & \end{array}$$

Now, $A/\langle e_6\rangle$ is a 6-dimensional nilpotent 3-Lie algebra with derived subalgebra of dimension 2 and center of dimension 2 or 1. If the dimension of center is 1, then $A/\langle e_6\rangle = A_{3,6,2}$. Thus, up to isomorphism,

$$A = \langle e_1,\dots,e_7 : [e_1,e_2,e_3] = e_4, [e_2,e_3,e_4] = e_6, [e_1,e_3,e_4] = [e_2,e_3,e_5] = e_7\rangle.$$

This algebra is nilpotent of class three and denoted by $A_{3,7,6}$.

If the dimension of center is 2, then $A/\langle e_6\rangle = A_{3,6,1}$. Thus, up to isomorphism, we have the following algebras:

$$A = \langle e_1,\dots,e_7 : [e_1,e_2,e_3] = e_4, [e_2,e_3,e_4] = e_6, [e_2,e_3,e_5] = e_7\rangle.$$

$$A = \langle e_1,\dots,e_7 : [e_1,e_2,e_3] = e_4, [e_2,e_3,e_4] = e_6, [e_1,e_3,e_5] = e_7\rangle.$$

These algebras are nilpotent of class three. The first algebra denoted by $A_{3,7,7}$ and the second algebra denoted by $A_{3,7,8}$.

(b): $Z(A) = \langle e_5, e_7\rangle$. In this case, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1,e_2,e_3] = e_4, & [e_2,e_3,e_4] = e_6, & [e_1,e_3,e_4] = \alpha_{134}e_7, \\ [e_1,e_2,e_4] = \alpha_{124}e_7, & [e_2,e_3,e_6] = \alpha_{236}e_7. & \end{array}$$

Since e_6 is not belong to $Z(A)$, so $\alpha_{236} \neq 0$. By putting $e_7' = \alpha_{236}e_7$ and relabeling, we have

$$\begin{array}{lll} [e_1,e_2,e_3] = e_4, & [e_2,e_3,e_4] = e_6, & [e_2,e_3,e_6] = e_7, \\ [e_1,e_3,e_4] = \alpha_{134}e_7, & [e_1,e_2,e_4] = \alpha_{124}e_7. & \end{array}$$

If $\alpha_{134} = \alpha_{124} = 0$, then

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_6, [e_2, e_3, e_6] = e_7 \rangle.$$

This algebra is isomorphic to $A_{3,6,4} \oplus F(1)$. Otherwise, without loss of generality, assume that $\alpha_{134} \neq 0$. By putting $e_1' = \frac{1}{\alpha_{134}} e_1, e_2' = e_2 - \frac{\alpha_{124}}{\alpha_{134}} e_3, e_3' = e_3, e_4' = \frac{1}{\alpha_{134}} e_4, e_5' = \frac{1}{\alpha_{134}} e_5, e_6' = \frac{1}{\alpha_{134}} e_6, e_7' = \frac{1}{\alpha_{134}} e_7$, and relabeling, we have

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_6, [e_2, e_3, e_6] = [e_1, e_3, e_4] = e_7 \rangle.$$

This algebra is isomorphic to $A_{3,6,5} \oplus F(1)$.

(iii) Let $\dim Z(A) = 1$. In this case, $Z(A) = \langle e_7 \rangle$ and the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_4] = e_6, & [e_1, e_3, e_4] = \alpha_{134} e_7, \\ [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_2, e_3, e_5] = \alpha_{235} e_7, & [e_1, e_3, e_5] = \alpha_{135} e_7, \\ [e_1, e_2, e_5] = \alpha_{125} e_7, & [e_2, e_3, e_6] = \alpha_{236} e_7. & \end{array}$$

Since e_6 is not belong to $Z(A)$, so $\alpha_{236} \neq 0$. On the other hand, at least one of $\alpha_{235}, \alpha_{135}, \alpha_{125}$ is not equal to zero. We claim that α_{235} is zero. Let $\alpha_{235} \neq 0$. Then by applying the transformations

$$e_1' = e_1 - \frac{\alpha_{135}}{\alpha_{235}} e_2 + \frac{\alpha_{125}}{\alpha_{235}} e_3, \quad e_j' = e_j, \quad 2 \leq j \leq 7,$$

we obtain

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_4] = e_6, & [e_2, e_3, e_6] = \alpha_{236} e_7 \\ [e_1, e_3, e_4] = \alpha_{134} e_7, & [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_2, e_3, e_5] = \alpha_{235} e_7. \end{array}$$

In this case, $\frac{1}{\alpha_{236}} e_6 - \frac{1}{\alpha_{235}} e_5$ is the central element, and this is a contradiction. Without loss of generality, assume that $\alpha_{135} \neq 0$. By applying th transformations

$$e_2' = e_2 - \frac{\alpha_{125}}{\alpha_{135}} e_3, \quad e_j' = e_j, \quad 1 \leq j \leq 7, j \neq 2,$$

we obtain

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_4] = e_6, & [e_2, e_3, e_6] = \alpha_{236} e_7, \\ [e_1, e_3, e_4] = \alpha_{134} e_7, & [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_1, e_3, e_5] = \alpha_{135} e_7. \end{array}$$

If $\alpha_{134} = \alpha_{124} = 0$, then by suitable change of basis,

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_6, [e_2, e_3, e_6] = [e_1, e_3, e_5] = e_7 \rangle.$$

This algebra is nilpotent of class four and denoted by $A_{3,7,9}$.

If $\alpha_{134} \neq 0$, then by replacing e_2 by $e_2 - \frac{\alpha_{124}}{\alpha_{134}} e_3$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_4] = e_6, & [e_2, e_3, e_6] = \alpha_{236} e_7, \\ [e_1, e_3, e_4] = \alpha_{134} e_7, & [e_1, e_3, e_5] = \alpha_{135} e_7. & \end{array}$$

This algebra is isomorphic to

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_6, [e_2, e_3, e_6] = [e_1, e_3, e_5] = [e_1, e_3, e_4] = e_7 \rangle.$$

This algebra is nilpotent of class four and denoted by $A_{3,7,10}$.

Similarly, if $\alpha_{124} \neq 0$, then by suitable change of basis,

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_6, [e_2, e_3, e_6] = [e_1, e_3, e_5] = [e_1, e_2, e_4] = e_7 \rangle.$$

This algebra is nilpotent of class four and denoted by $A_{3,7,11}$. □

Lemma 4. *Let $A/\langle e_7 \rangle \cong A_{3,6,1}$. Then A is isomorphic to $A_{3,7,2}$, $A_{3,7,7}$, $A_{3,7,8}$, $A_{3,7,12}$, $A_{3,7,13}$, $A_{3,7,14}$, $A_{3,7,15}$, or $A_{3,7,16}$.*

Proof. Let $A/\langle e_7 \rangle \cong A_{3,6,1}$. In this case, the multiplication in A can be written as

$$[e_1, e_2, e_3] = e_4 + \alpha e_7, \quad [e_2, e_3, e_5] = e_6 + \beta e_7, \quad [e_i, e_j, e_k] = \alpha_{ijk} e_7,$$

where $1 \leq i < j < k \leq 6$, and $\{i, j, k\} \notin \{\{1, 2, 3\}, \{2, 3, 5\}\}$.

Regarding a suitable change of basis, one can assume that $\alpha = \beta = 0$, and the Jacobi identity gives us $\alpha_{146} = \alpha_{246} = \alpha_{346} = \alpha_{156} = \alpha_{456} = \alpha_{145} = 0, \alpha_{345} = \alpha_{136}, \alpha_{245} = \alpha_{126}$. Thus, the multiplication table in A can be written as

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_2, e_3, e_4] = \alpha_{234} e_7, \\ [e_1, e_3, e_4] = \alpha_{134} e_7, & [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_1, e_3, e_5] = \alpha_{135} e_7, \\ [e_1, e_2, e_5] = \alpha_{125} e_7, & [e_2, e_3, e_6] = \alpha_{236} e_7, & [e_1, e_3, e_6] = \alpha_{136} e_7, \\ [e_1, e_2, e_6] = \alpha_{126} e_7, & [e_3, e_4, e_5] = \alpha_{136} e_7, & [e_2, e_4, e_5] = \alpha_{126} e_7, \\ [e_3, e_5, e_6] = \alpha_{356} e_7, & [e_2, e_5, e_6] = \alpha_{256} e_7. & \end{array}$$

We discuss on the dimension of the center of A. The above multiplication shows that the dimension of the center of A is at most 3.

(i) Let $\dim Z(A) = 3$. In this case, $Z(A) = \langle e_4, e_6, e_7 \rangle$ and the multiplication in A can be written as follows:

$$[e_1, e_2, e_3] = e_4, \quad [e_2, e_3, e_5] = e_6, \quad [e_1, e_3, e_5] = \alpha_{135} e_7, \quad [e_1, e_2, e_5] = \alpha_{125} e_7.$$

Since $\dim A^2 = 3$, at least one of $\alpha_{135}, \alpha_{125}$ is not equal to zero. Regarding a suitable change of basis, this algebra is isomorphic to $A_{3,7,2}$.

(ii) Let $\dim Z(A) = 2$. In this case, we have two possibilities for $Z(A)$:

(a): $Z(A) = \langle e_6, e_7 \rangle$. In this case, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_2, e_3, e_4] = \alpha_{234} e_7, \\ [e_1, e_3, e_4] = \alpha_{134} e_7, & [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_1, e_3, e_5] = \alpha_{135} e_7, \\ [e_1, e_2, e_5] = \alpha_{125} e_7, & [e_3, e_4, e_5] = \alpha_{136} e_7, & [e_2, e_4, e_5] = \alpha_{126} e_7. \end{array}$$

Now, $A/\langle e_6 \rangle$ is a 6-dimensional nilpotent 3-Lie algebra with derived subalgebra of dimension 2, and the center of dimension 2 or 1. If the dimension of center is 1, then $A/\langle e_6 \rangle = A_{3,6,2}$. Thus, up to isomorphism

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_2, e_3, e_4] = [e_1, e_3, e_5] = e_7 \rangle.$$

This algebra is nilpotent of class three and denoted by $A_{3,7,12}$.

If the dimension of the center is 2 and equal to the derived subalgebra, then $A/\langle e_6\rangle = A_{3,6,1}$. Thus, up to isomorphism, we have the following algebra:

$$A = \langle e_1, \dots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_2, e_3, e_4] = e_7\rangle.$$

This algebra is $A_{3,7,2}$. Finally, if the dimension of the center is 2 and not equal to the derived subalgebra, then $A/\langle e_6\rangle = A_{3,5,1} \oplus F(1)$. Thus, up to isomorphism, we have the following algebra:

$$A = \langle e_1, \dots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_2, e_3, e_4] = e_7\rangle.$$

This algebra is isomorphic to $A_{3,7,7}$.

(b): $Z(A) = \langle e_4, e_7\rangle$. In this case, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_1, e_3, e_5] = \alpha_{135}e_7, \\ [e_1, e_2, e_5] = \alpha_{125}e_7, & [e_2, e_3, e_6] = \alpha_{236}e_7, & [e_1, e_3, e_6] = \alpha_{136}e_7, \\ [e_1, e_2, e_6] = \alpha_{126}e_7, & [e_3, e_5, e_6] = \alpha_{356}e_7, & [e_2, e_5, e_6] = \alpha_{256}e_7. \end{array}$$

Now, $A/\langle e_4\rangle$ is a 6-dimensional nilpotent 3-Lie algebra with derived subalgebra of dimension 2 and the center of dimension 2 or 1. If the dimension of the center is 1, then $A/\langle e_4\rangle = A_{3,6,3}$. Thus, up to isomorphism

$$A = \langle e_1, \dots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_2, e_3, e_6] = [e_1, e_3, e_5] = e_7\rangle.$$

This algebra is isomorphic to $A_{3,7,12}$.

If the dimension of center is 2 and equal to the derived subalgebra, then $A/\langle e_4\rangle = A_{3,6,1}$. Thus, up to isomorphism, we have the following algebra:

$$A = \langle e_1, \dots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_1, e_3, e_5] = e_7\rangle.$$

This algebra is $A_{3,7,2}$. Finally, if the dimension of the center is 2 and not equal to the derived subalgebra, then $A/\langle e_4\rangle = A_{3,5,1} \oplus F(1)$. Thus, up to isomorphism, we have the following algebras:

$$A = \langle e_1, \dots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_2, e_3, e_6] = e_7\rangle,$$

$$A = \langle e_1, \dots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_3, e_5, e_6] = e_7\rangle.$$

These algebras are isomorphic to $A_{3,7,7}$ and $A_{3,7,8}$, respectively.

(iii) Let $\dim Z(A) = 1$. In this case, $Z(A) = \langle e_7\rangle$. Since e_6 is not belong to $Z(A)$, at least one of $\alpha_{236}, \alpha_{136}, \alpha_{126}, \alpha_{256}, \alpha_{356}$ is not equal to zero. Up to isomorphism, we have three possibilities.

(a): Let $\alpha_{236} \neq 0$. By applying the transformations

$$e'_1 = e_1 - \frac{\alpha_{136}}{\alpha_{236}}e_2 + \frac{\alpha_{126}}{\alpha_{236}}e_3, \quad e'_j = e_j, \quad 2 \leq j \leq 7,$$

we obtain

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_2, e_3, e_4] = \alpha_{234}e_7, \\ [e_1, e_3, e_4] = \alpha_{134}e_7, & [e_1, e_2, e_4] = \alpha_{124}e_7, & [e_1, e_3, e_5] = \alpha_{135}e_7, \\ [e_1, e_2, e_5] = \alpha_{125}e_7, & [e_2, e_3, e_6] = \alpha_{236}e_7, & [e_3, e_5, e_6] = \alpha_{356}e_7, \\ [e_2, e_5, e_6] = \alpha_{256}e_7. & & \end{array}$$

Now, by applying the transformations

$$e_5' = e_5 + \frac{\alpha_{356}}{\alpha_{236}} e_2 - \frac{\alpha_{256}}{\alpha_{236}} e_3, \quad e_j' = e_j, \quad 1 \le j \le 7, j \ne 5,$$

we obtain

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_2, e_3, e_4] = \alpha_{234} e_7, \\ [e_1, e_3, e_4] = \alpha_{134} e_7, & [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_1, e_3, e_5] = \alpha_{135} e_7, \\ [e_1, e_2, e_5] = \alpha_{125} e_7, & [e_2, e_3, e_6] = \alpha_{236} e_7. & \end{array}$$

Since e_4 is not belong to $Z(A)$, at least one of $\alpha_{234}, \alpha_{134}, \alpha_{124}$ is not equal to zero. We claim that α_{234} is zero. Let $\alpha_{234} \ne 0$. Then by applying the transformations

$$e_1' = e_1 - \frac{\alpha_{134}}{\alpha_{234}} e_2 + \frac{\alpha_{124}}{\alpha_{234}} e_3, \quad e_j' = e_j, \quad 2 \le j \le 7,$$

so $\frac{1}{\alpha_{234}} e_4 - \frac{1}{\alpha_{236}} e_6$ is the central element, and this is a contradiction. Without loss of generality, assume that $\alpha_{134} \ne 0$. By applying the transformations

$$e_2' = e_2 - \frac{\alpha_{124}}{\alpha_{134}} e_3, \quad e_j' = e_j, \quad 1 \le j \le 7, j \ne 2,$$

we obtain

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_1, e_3, e_4] = \alpha_{134} e_7, \\ [e_1, e_3, e_5] = \alpha_{135} e_7, & [e_1, e_2, e_5] = \alpha_{125} e_7, & [e_2, e_3, e_6] = \alpha_{236} e_7. \end{array}$$

By replacing e_5 by $e_5 - \frac{\alpha_{135}}{\alpha_{134}} e_4$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_1, e_3, e_4] = \alpha_{134} e_7, \\ [e_1, e_2, e_5] = \alpha_{125} e_7, & [e_2, e_3, e_6] = \alpha_{236} e_7. & \end{array}$$

By applying the transformations

$$e_2' = e_2 - \frac{\alpha_{125}}{\alpha_{134}} e_3, \quad e_5' = \frac{\alpha_{134}}{\alpha_{236}} (e_5 + e_4), \quad e_6' = \frac{\alpha_{134}}{\alpha_{236}} e_6, \quad e_7' = \alpha_{134} e_7,$$

this algebra is isomorphic to

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_2, e_3, e_6] = [e_1, e_3, e_4] = e_7 \rangle.$$

This algebra is nilpotent of class three and denoted by $A_{3,7,13}$.

(b): Let $\alpha_{136} \ne 0$. Then by replacing e_2 by $e_2 - \frac{\alpha_{236}}{\alpha_{136}} e_1 - \frac{\alpha_{126}}{\alpha_{136}} e_3$ and e_6 by $e_6 - \frac{\alpha_{236}\alpha_{135}}{\alpha_{136}} e_7$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_2, e_3, e_4] = \alpha_{234} e_7, \\ [e_1, e_3, e_4] = \alpha_{134} e_7, & [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_1, e_3, e_5] = \alpha_{135} e_7, \\ [e_1, e_2, e_5] = \alpha_{125} e_7, & [e_1, e_3, e_6] = \alpha_{136} e_7, & [e_3, e_4, e_5] = \alpha_{136} e_7, \\ [e_3, e_5, e_6] = \alpha_{356} e_7, & [e_2, e_5, e_6] = \alpha_{256} e_7. & \end{array}$$

Again, by replacing e_5 by $e_5 + \frac{\alpha_{356}}{\alpha_{136}} e_1$ and e_6 by $e_6 + \frac{\alpha_{356}}{\alpha_{136}} e_4$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_2, e_3, e_4] = \alpha_{234} e_7, \\ [e_1, e_3, e_4] = \alpha_{134} e_7, & [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_1, e_3, e_5] = \alpha_{135} e_7, \\ [e_1, e_2, e_5] = \alpha_{125} e_7, & [e_1, e_3, e_6] = \alpha_{136} e_7, & [e_3, e_4, e_5] = \alpha_{136} e_7, \\ [e_2, e_5, e_6] = \alpha_{256} e_7. & & \end{array}$$

Now, by replacing e_1 by $e_5 + \frac{\alpha_{135}}{\alpha_{136}} e_4$, e_2 with $e_2 - \frac{\alpha_{234}}{\alpha_{136}} e_5$ and e_4 by $e_4 + \frac{\alpha_{234}\alpha_{135}}{\alpha_{136}} e_7$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_1, e_3, e_4] = \alpha_{134} e_7, \\ [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_1, e_2, e_5] = \alpha_{125} e_7, & [e_1, e_3, e_6] = \alpha_{136} e_7, \\ [e_3, e_4, e_5] = \alpha_{136} e_7, & [e_2, e_5, e_6] = \alpha_{256} e_7. & \end{array}$$

By replacing e_1 by $e_1 + \frac{\alpha_{134}}{\alpha_{136}} e_5$, and e_4 by $e_4 + \frac{\alpha_{134}}{\alpha_{136}} e_6$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_1, e_2, e_4] = \alpha_{124} e_7, \\ [e_1, e_2, e_5] = \alpha_{125} e_7, & [e_1, e_3, e_6] = \alpha_{136} e_7, & [e_3, e_4, e_5] = \alpha_{136} e_7, \\ [e_2, e_5, e_6] = \alpha_{256} e_7. & & \end{array}$$

By replacing e_2 by $e_2 + e_3$, and e_5 by $e_5 + \frac{\alpha_{256}}{\alpha_{136}} e_1$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_1, e_2, e_4] = \alpha_{124} e_7, \\ [e_1, e_2, e_5] = \alpha_{125} e_7, & [e_1, e_3, e_6] = \alpha_{136} e_7, & [e_3, e_4, e_5] = \alpha_{136} e_7. \end{array}$$

Now, by replacing e_1 by $e_1 + e_4$, and e_2 by $e_2 + \frac{\alpha_{125}}{\alpha_{136}} e_3$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_1, e_2, e_4] = \alpha_{124} e_7, \\ [e_1, e_3, e_6] = \alpha_{136} e_7, & [e_3, e_4, e_5] = \alpha_{136} e_7. & \end{array}$$

Finally, by replacing e_1 by $e_1 + e_3$, e_2 by $e_2 + \frac{\alpha_{124}}{\alpha_{136}} e_5$ and e_7 by $\alpha_{136} e_7$, this algebra is isomorphic to

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_1, e_3, e_6] = [e_3, e_4, e_5] = e_7 \rangle.$$

This algebra is nilpotent of class three and denoted by $A_{3,7,14}$.

(c): Let $\alpha_{356} \neq 0$. Then by replacing e_2 by $e_2 - \frac{\alpha_{256}}{\alpha_{356}} e_3 + \frac{\alpha_{236}}{\alpha_{356}} e_5$ and e_6 by $e_6 - \frac{\alpha_{236}\alpha_{135}}{\alpha_{356}} e_7$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_2, e_3, e_4] = \alpha_{234} e_7, \\ [e_1, e_3, e_4] = \alpha_{134} e_7, & [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_1, e_3, e_5] = \alpha_{135} e_7, \\ [e_1, e_2, e_5] = \alpha_{125} e_7, & [e_1, e_3, e_6] = \alpha_{136} e_7, & [e_1, e_2, e_6] = \alpha_{126} e_7, \\ [e_3, e_4, e_5] = \alpha_{136} e_7, & [e_2, e_4, e_5] = \alpha_{126} e_7, & [e_3, e_5, e_6] = \alpha_{356} e_7. \end{array}$$

Again, by replacing e_1 by $e_1 + \frac{\alpha_{136}}{\alpha_{356}} e_5$ and e_4 by $e_4 + \frac{\alpha_{136}}{\alpha_{356}} e_6$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_2, e_3, e_4] = \alpha_{234} e_7, \\ [e_1, e_3, e_4] = \alpha_{134} e_7, & [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_1, e_3, e_5] = \alpha_{135} e_7, \\ [e_1, e_2, e_5] = \alpha_{125} e_7, & [e_1, e_2, e_6] = \alpha_{126} e_7, & [e_2, e_4, e_5] = \alpha_{126} e_7, \\ [e_3, e_5, e_6] = \alpha_{356} e_7. & & \end{array}$$

Now, by replacing e_1 by $e_1 + e_3$, e_2 by $e_2 - \frac{\alpha_{126}}{\alpha_{356}}e_5$, and e_4 by $e_4 + \frac{\alpha_{135}\alpha_{126}}{\alpha_{356}}e_7$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_2, e_3, e_4] = \alpha_{234}e_7, \\ [e_1, e_3, e_4] = \alpha_{134}e_7, & [e_1, e_2, e_4] = \alpha_{124}e_7, & [e_1, e_3, e_5] = \alpha_{135}e_7, \\ [e_1, e_2, e_5] = \alpha_{125}e_7, & [e_3, e_5, e_6] = \alpha_{356}e_7. & \end{array}$$

By replacing e_1 by $e_1 - \frac{\alpha_{135}}{\alpha_{356}}e_6$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_2, e_3, e_4] = \alpha_{234}e_7, \\ [e_1, e_3, e_4] = \alpha_{134}e_7, & [e_1, e_2, e_4] = \alpha_{124}e_7, & [e_1, e_2, e_5] = \alpha_{125}e_7, \\ [e_3, e_5, e_6] = \alpha_{356}e_7. & & \end{array}$$

By replacing e_2 by $e_2 + e_3$ and e_1 by $e_1 - \frac{\alpha_{125}}{\alpha_{356}}e_6$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_5] = e_6, & [e_2, e_3, e_4] = \alpha_{234}e_7, \\ [e_1, e_3, e_4] = \alpha_{134}e_7, & [e_1, e_2, e_4] = \alpha_{124}e_7, & [e_3, e_5, e_6] = \alpha_{356}e_7. \end{array}$$

Since e_4 is not belong to $Z(A)$, at least one of $\alpha_{234}, \alpha_{134}, \alpha_{124}$ is not equal to zero. Regarding a suitable change of basis, up to isomorphism, the following algebras are estimated:

$$\langle e_1, \dots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_3, e_5, e_6] = [e_2, e_3, e_4] = e_7\rangle,$$

$$\langle e_1, \dots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_3, e_5, e_6] = [e_1, e_3, e_4] = e_7\rangle,$$

$$\langle e_1, \dots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_3, e_5, e_6] = [e_1, e_2, e_4] = e_7\rangle.$$

These algebras are nilpotent of class three. The first algebra is isomorphic to $A_{3,7,13}$. The second and third algebras are denoted by $A_{3,7,15}$ and $A_{3,7,16}$, respectively. □

Lemma 5. *Let $A/\langle e_7\rangle \cong A_{3,6,2}$. Then A is isomorphic to $A_{3,7,6}$ or $A_{3,7,17}$.*

Proof. Let $A/\langle e_7\rangle \cong A_{3,6,2}$. In this case, the multiplication in A can be written as

$$\begin{array}{ll} [e_1, e_2, e_3] = e_4 + \alpha e_7, & [e_2, e_3, e_4] = e_6 + \beta e_7, \\ [e_1, e_3, e_5] = e_6 + \gamma e_7, & [e_i, e_j, e_k] = \alpha_{ijk}e_7, \end{array}$$

where $1 \le i < j < k \le 6$, and $\{i, j, k\} \notin \{\{1, 2, 3\}, \{2, 3, 4\}, \{1, 3, 5\}\}$.

Regarding a suitable change of basis, one can assume that $\alpha = \beta = 0$, and the Jacobi identity gives us $\alpha_{146} = \alpha_{246} = \alpha_{346} = \alpha_{156} = \alpha_{256} = \alpha_{356} = \alpha_{456} = \alpha_{145} = \alpha_{245} = \alpha_{136} = \alpha_{126} = 0, \alpha_{345} = -\alpha_{236}$. Thus, the multiplication table in A can be written as

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_4] = e_6, & [e_1, e_3, e_5] = e_6 + \gamma e_7, \\ [e_1, e_3, e_4] = \alpha_{134}e_7, & [e_1, e_2, e_4] = \alpha_{124}e_7, & [e_2, e_3, e_5] = \alpha_{235}e_7, \\ [e_1, e_2, e_5] = \alpha_{125}e_7, & [e_2, e_3, e_6] = \alpha_{236}e_7, & [e_3, e_4, e_5] = -\alpha_{236}e_7. \end{array}$$

We discuss on the dimension of the center of A. The above multiplication shows that the dimension of the center of A is at most 2.

(i) Let $\dim Z(A) = 2$. In this case, $Z(A) = \langle e_6, e_7\rangle$ and the multiplication in A can be written as follows:

$$\begin{aligned} &[e_1,e_2,e_3]=e_4, && [e_2,e_3,e_4]=e_6, && [e_1,e_3,e_5]=e_6+\gamma e_7,\\ &[e_1,e_3,e_4]=\alpha_{134}e_7, && [e_1,e_2,e_4]=\alpha_{124}e_7, && [e_2,e_3,e_5]=\alpha_{235}e_7,\\ &[e_1,e_2,e_5]=\alpha_{125}e_7. \end{aligned}$$

Since $\dim(A/\langle e_6\rangle)^2 = 2$ and $\dim Z(A/\langle e_6\rangle) = 1$, according to Lemma 1, $A/\langle e_6\rangle$ is isomorphic to $A_{3,6,2}$. Thus, up to isomorphism,

$$A = \langle e_1,\ldots,e_7 : [e_1,e_2,e_3] = e_4, [e_2,e_3,e_4] = e_6, [e_1,e_3,e_4] = [e_2,e_3,e_5] = e_7\rangle.$$

This algebra is $A_{3,7,6}$.

(ii) Let $\dim Z(A) = 1$. In this case, $Z(A) = \langle e_7\rangle$.

Since e_6 is not central element, α_{236} is not equal to zero. Regarding a suitable change of basis, the multiplication in A can be written as follows:

$$\begin{aligned} &[e_1,e_2,e_3]=e_4, && [e_2,e_3,e_4]=e_6, && [e_1,e_3,e_5]=e_6+\gamma e_7,\\ &[e_1,e_3,e_4]=\alpha_{134}e_7, && [e_1,e_2,e_4]=\alpha_{124}e_7, && [e_2,e_3,e_5]=\alpha_{235}e_7,\\ &[e_1,e_2,e_5]=\alpha_{125}e_7, && [e_2,e_3,e_6]=e_7, && [e_3,e_4,e_5]=-e_7. \end{aligned}$$

By applying the transformations $e_1' = e_1-\gamma e_6$, $e_1' = e_2-\alpha_{235}e_4$ and $e_1' = e_1-\alpha_{134}e_5$, respectively, γ, α_{235}, and α_{134} are removed. Thus, the multiplication in A can be written as follows:

$$\begin{aligned} &[e_1,e_2,e_3]=e_4, && [e_2,e_3,e_4]=e_6, && [e_1,e_3,e_5]=e_6,\\ &[e_1,e_2,e_4]=\alpha_{124}e_7, && [e_1,e_2,e_5]=\alpha_{125}e_7, && [e_2,e_3,e_6]=e_7,\\ &[e_3,e_4,e_5]=-e_7. \end{aligned}$$

Now, by replacing e_1 by e_1+e_5 and e_2 by $e_2+\alpha_{124}e_3$, the multiplication in A can be written as follows:

$$\begin{aligned} &[e_1,e_2,e_3]=e_4, && [e_2,e_3,e_4]=e_6, && [e_1,e_3,e_5]=e_6,\\ &[e_1,e_2,e_5]=\alpha_{125}e_7, && [e_2,e_3,e_6]=e_7, && [e_3,e_4,e_5]=-e_7. \end{aligned}$$

Finally, by replacing e_1 by e_1+e_3, and e_5 by $e_5+\alpha_{125}e_6$, we have

$$\begin{aligned} A = \langle e_1,\ldots,e_7 : [e_1,e_2,e_3] = e_4, [e_2,e_3,e_4] = [e_1,e_3,e_5] = e_6,\\ [e_2,e_3,e_6] = -[e_3,e_4,e_5] = e_7\rangle. \end{aligned}$$

This algebra is nilpotent of class four and denoted by $A_{3,7,17}$. □

The following theorem is an immediate consequence of Lemmas 3, 4, and 5.

Theorem 3. *Let A be a 7-dimensional nilpotent 3-Lie algebra with the derived subalgebra of dimension 3. Then A is isomorphic to $A_{3,6,3}\oplus F(1)$, $A_{3,6,4}\oplus F(1)$, $A_{3,6,5}\oplus F(1)$, $A_{3,7,2}$, $A_{3,7,6}$, $A_{3,7,7}$, $A_{3,7,8}$, $A_{3,7,9}$, $A_{3,7,10}$, $A_{3,7,11}$, $A_{3,7,12}$, $A_{3,7,13}$, $A_{3,7,14}$, $A_{3,7,15}$, $A_{3,7,16}$, or $A_{3,7,17}$.*

4 7-dimensional nilpotent 3-Lie algebras with the derived subalgebra of dimension 4

In this section, we classify 7-dimensional nilpotent 3-Lie algebras with the derived subalgebra of dimension 4. Let A be a 7-dimensional nilpotent 3-Lie algebra with the derived subalgebra of dimension 4 with basis $\{e_1, \ldots, e_7\}$. Let e_7 be a central element of A^2. Then $A/\langle e_7 \rangle$ is a nilpotent 3-Lie algebra of dimension 6 with the derived algebra of dimension 3. By using Lemma 1, $A/\langle e_7 \rangle$ is isomorphic to $A_{3,6,3}$, $A_{3,6,4}$ or $A_{3,6,5}$.

Lemma 6. *Let $A/\langle e_7 \rangle \cong A_{3,6,3}$. Then A is isomorphic to $A_{3,7,18}$, $A_{3,7,19}$, $A_{3,7,20}$, $A_{3,7,21}$, or $A_{3,7,22}$.*

Proof. Let $A/\langle e_7 \rangle \cong A_{3,6,3}$. In this case, the multiplication in A can be written as

$$[e_1, e_2, e_3] = e_4 + \alpha e_7, \quad [e_2, e_3, e_4] = e_5 + \beta e_7, \quad [e_1, e_3, e_4] = e_6 + \gamma e_7,$$
$$[e_i, e_j, e_k] = \alpha_{ijk} e_7,$$

where $1 \leq i < j < k \leq 6$, and $\{i, j, k\} \notin \{\{1, 2, 3\}, \{2, 3, 4\}, \{1, 3, 4\}\}$.

Regarding a suitable change of basis, one can assume that $\alpha = \beta = \gamma = 0$, and the Jacobi identity gives us $\alpha_{146} = \alpha_{246} = \alpha_{346} = \alpha_{156} = \alpha_{256} = \alpha_{356} = \alpha_{456} = \alpha_{145} = \alpha_{245} = \alpha_{345} = \alpha_{126} = \alpha_{125} = 0, \alpha_{135} = \alpha_{236}$. Thus, the multiplication table in A can be written as

$$\begin{aligned} &[e_1, e_2, e_3] = e_4, && [e_2, e_3, e_4] = e_5, && [e_1, e_3, e_4] = e_6, \\ &[e_1, e_2, e_4] = \alpha_{124} e_7, && [e_2, e_3, e_5] = \alpha_{235} e_7, && [e_1, e_3, e_5] = \alpha_{236} e_7, \\ &[e_2, e_3, e_6] = \alpha_{236} e_7, && [e_1, e_3, e_6] = \alpha_{136} e_7. \end{aligned}$$

We discuss on the dimension of the center of A. The above multiplication shows that the dimension of the center of A is at most 3.

(i) Let $\dim Z(A) = 3$. In this case, $Z(A) = \langle e_5, e_6, e_7 \rangle$ and the multiplication in A can be written as follows:

$$[e_1, e_2, e_3] = e_4, \quad [e_2, e_3, e_4] = e_5, \quad [e_1, e_3, e_4] = e_6, \quad [e_1, e_2, e_4] = \alpha_{124} e_7.$$

If $\alpha_{124} = 0$, then the algebra is isomorphic to $A_{3,6,3} \oplus F(1)$, and otherwise, we have the following algebra:

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_1, e_3, e_4] = e_6, [e_1, e_2, e_4] = e_7 \rangle.$$

This algebra is nilpotent of class three and denoted by $A_{3,7,18}$.

(ii) Let $\dim Z(A) = 2$. In this case, without loss of generality assume that $Z(A) = \langle e_6, e_7 \rangle$ and the multiplication in A can be written as follows:

$$\begin{aligned} &[e_1, e_2, e_3] = e_4, && [e_2, e_3, e_4] = e_5, && [e_1, e_3, e_4] = e_6, \\ &[e_1, e_2, e_4] = \alpha_{124} e_7, && [e_2, e_3, e_5] = \alpha_{235} e_7. \end{aligned}$$

Since e_5 is not central element, α_{235} is not equal to zero. Regarding a suitable change of basis, the multiplication in A can be written as follows:

$$\begin{aligned} &[e_1, e_2, e_3] = e_4, && [e_2, e_3, e_4] = e_5, && [e_1, e_3, e_4] = e_6, \\ &[e_1, e_2, e_4] = \alpha_{124} e_7, && [e_2, e_3, e_5] = e_7. \end{aligned}$$

If $\alpha_{124} = 0$, then

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_1, e_3, e_4] = e_6, [e_2, e_3, e_5] = e_7 \rangle.$$

This algebra is nilpotent of class four and denoted by $A_{3,7,19}$.

If $\alpha_{124} \neq 0$, then regarding a suitable change of basis,

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_1, e_3, e_4] = e_6, [e_1, e_2, e_4] = [e_2, e_3, e_5] = e_7\rangle.$$

This algebra is nilpotent of class four and denoted by $A_{3,7,20}$.

(iii) Let $\dim Z(A) = 1$. In this case, $Z(A) = \langle e_7\rangle$.

Since e_6 is not belong to $Z(A)$, at least one of $\alpha_{236}, \alpha_{136}$ is not equal to zero.

(a): If $\alpha_{236} \neq 0$, then by replacing e_6 by $e_6 - \frac{\alpha_{136}}{\alpha_{236}} e_5$ and e_1 by $e_1 - \frac{\alpha_{136}}{\alpha_{236}} e_2$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_4] = e_5, & [e_1, e_3, e_4] = e_6, \\ [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_2, e_3, e_5] = \alpha_{235} e_7, & [e_1, e_3, e_5] = \alpha_{236} e_7, \\ [e_2, e_3, e_6] = \alpha_{236} e_7. & & \end{array}$$

Now, by replacing e_5 by $e_5 - \frac{\alpha_{235}}{\alpha_{236}} e_6$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_4] = e_5, & [e_1, e_3, e_4] = e_6, \\ [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_1, e_3, e_5] = \alpha_{236} e_7, & [e_2, e_3, e_6] = \alpha_{236} e_7. \end{array}$$

Depending on whether α_{124} is zero or not, we have the following two algebras:

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_1, e_3, e_4] = e_6, [e_1, e_3, e_5] = [e_2, e_3, e_6] = e_7\rangle.$$

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_1, e_3, e_4] = e_6, [e_1, e_3, e_5] = [e_2, e_3, e_6] = [e_1, e_2, e_4] = e_7\rangle.$$

These algebras are nilpotent of class four and denoted by $A_{3,7,21}$ and $A_{3,7,22}$, respectively.

(b): If $\alpha_{136} \neq 0$, by replacing e_2 by $e_2 - \frac{\alpha_{236}}{\alpha_{136}} e_1$, then the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_4] = e_5, & [e_1, e_3, e_4] = e_6, \\ [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_2, e_3, e_5] = \alpha_{235} e_7, & [e_1, e_3, e_6] = \alpha_{136} e_7. \end{array}$$

Depending on whether α_{124} is zero or not, we have the following two algebras:

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_1, e_3, e_4] = e_6, [e_2, e_3, e_5] = [e_1, e_3, e_6] = e_7\rangle.$$

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_1, e_3, e_4] = e_6, [e_2, e_3, e_5] = [e_1, e_3, e_6] = [e_1, e_2, e_4] = e_7\rangle.$$

These algebras are isomorphic to $A_{3,7,21}$ and $A_{3,7,22}$, respectively. □

Lemma 7. *Let $A/\langle e_7\rangle \cong A_{3,6,4}$. Then A is isomorphic to $A_{3,7,19}$, $A_{3,7,23}$, $A_{3,7,24}$ or $A_{3,7,25}$.*

Proof. Let $A/\langle e_7\rangle \cong A_{3,6,4}$. In this case, the multiplication in A can be written as

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4 + \alpha e_7, & [e_2, e_3, e_4] = e_5 + \beta e_7, & [e_2, e_3, e_5] = e_6 + \gamma e_7, \\ [e_i, e_j, e_k] = \alpha_{ijk} e_7, & & \end{array}$$

where $1 \leq i < j < k \leq 6$, and $\{i, j, k\} \notin \{\{1, 2, 3\}, \{2, 3, 4\}, \{2, 3, 5\}\}$.

Regarding a suitable change of basis, one can assume that $\alpha = \beta = \gamma = 0$, and the Jacobi identity gives us $\alpha_{146} = \alpha_{246} = \alpha_{346} = \alpha_{156} = \alpha_{256} = \alpha_{356} = \alpha_{456} = \alpha_{145} = \alpha_{135} = \alpha_{125} = 0, \alpha_{345} = \alpha_{136}, \alpha_{245} = \alpha_{126}$. Thus, the multiplication in A can be written as

$$\begin{array}{lll} [e_1,e_2,e_3]=e_4, & [e_2,e_3,e_4]=e_5, & [e_2,e_3,e_5]=e_6, \\ [e_1,e_3,e_4]=\alpha_{134}e_7, & [e_1,e_2,e_4]=\alpha_{124}e_7, & [e_2,e_3,e_6]=\alpha_{236}e_7, \\ [e_1,e_3,e_6]=\alpha_{136}e_7, & [e_1,e_2,e_6]=\alpha_{126}e_7, & [e_3,e_4,e_5]=\alpha_{136}e_7, \\ [e_2,e_4,e_5]=\alpha_{126}e_7. & & \end{array}$$

We discuss on the dimension of the center of A. The above multiplication shows that the dimension of the center of A is at most 2.

(i) Let $\dim Z(A) = 2$. In this case, $Z(A) = \langle e_6, e_7\rangle$ and the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1,e_2,e_3]=e_4, & [e_2,e_3,e_4]=e_5, & [e_2,e_3,e_5]=e_6, \\ [e_1,e_3,e_4]=\alpha_{134}e_7, & [e_1,e_2,e_4]=\alpha_{124}e_7. & \end{array}$$

Since $\dim A^2 = 4$, at least one of $\alpha_{134}, \alpha_{124}$ is not equal to zero. Regarding a suitable change of basis,

$$A = \langle e_1,\dots,e_7 : [e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_5, [e_2,e_3,e_5]=e_6, [e_1,e_3,e_4]=e_7\rangle.$$

This algebra is $A_{3,7,19}$.

(ii) Let $\dim Z(A) = 1$. In this case, $Z(A) = \langle e_7\rangle$.

Since e_6 is not belong to $Z(A)$, at least one of $\alpha_{236}, \alpha_{136}, \alpha_{126}$ is not equal to zero.

Up to isomorphism, we have two possibilities:

(a): If $\alpha_{236} \neq 0$, then by replacing e_1 by $e_1 - \frac{\alpha_{136}}{\alpha_{236}}e_2 + \frac{\alpha_{126}}{\alpha_{236}}e_3$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1,e_2,e_3]=e_4, & [e_2,e_3,e_4]=e_5, & [e_2,e_3,e_5]=e_6, \\ [e_1,e_3,e_4]=\alpha_{134}e_7, & [e_1,e_2,e_4]=\alpha_{124}e_7, & [e_2,e_3,e_6]=\alpha_{236}e_7. \end{array}$$

If $\alpha_{234} = \alpha_{134} = 0$, then we have the following algebra:

$$A = \langle e_1,\dots,e_7 : [e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_5, [e_2,e_3,e_5]=e_6, [e_2,e_3,e_6]=e_7\rangle.$$

This algebra is nilpotent of class five and denoted by $A_{3,7,23}$. Otherwise, without loss of generality assume that $\alpha_{134} \neq 0$. Applying the transformations

$e'_1 = \frac{\alpha_{236}}{\alpha_{134}}e_1, e'_2 = e_2 - \frac{\alpha_{124}}{\alpha_{134}}e_3, e'_3 = e_3, e'_4 = \frac{\alpha_{236}}{\alpha_{134}}e_4, e'_5 = \frac{\alpha_{236}}{\alpha_{134}}e_5, e'_6 = \frac{\alpha_{236}}{\alpha_{134}}e_6, e'_7 = \frac{\alpha_{236}^2}{\alpha_{134}}e_7$, we obtain

$$\begin{aligned} A = \langle e_1,\dots,e_7 : [e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_5, [e_2,e_3,e_5]=e_6, \\ [e_2,e_3,e_6]=[e_1,e_3,e_4]=e_7\rangle. \end{aligned}$$

This algebra is nilpotent of class five and denoted by $A_{3,7,24}$.

(b): If $\alpha_{136} \neq 0$, then by replacing e_2 by $e_2 - \frac{\alpha_{126}}{\alpha_{136}}e_3 - \frac{\alpha_{236}}{\alpha_{136}}e_1$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1,e_2,e_3]=e_4, & [e_2,e_3,e_4]=e_5, & [e_2,e_3,e_5]=e_6, \\ [e_1,e_3,e_4]=\alpha_{134}e_7, & [e_1,e_2,e_4]=\alpha_{124}e_7, & [e_1,e_3,e_6]=\alpha_{136}e_7, \\ [e_3,e_4,e_5]=\alpha_{136}e_7. & & \end{array}$$

Now, by replacing e_1 by $e_1 - \frac{\alpha_{134}}{\alpha_{136}} e_5$ and e_4 by $e_4 - \frac{\alpha_{134}}{\alpha_{136}} e_6$ the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_4] = e_5, & [e_2, e_3, e_5] = e_6, \\ [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_1, e_3, e_6] = \alpha_{136} e_7, & [e_3, e_4, e_5] = \alpha_{136} e_7. \end{array}$$

Finally, by replacing e_1 by $e_1 - \frac{\alpha_{124}}{\alpha_{136}} e_5$, e_2 by $e_2 + e_3$ and e_7 by $\alpha_{136} e_7$, the following algebra is estimated:

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_2, e_3, e_5] = e_6, [e_1, e_3, e_6] = [e_3, e_4, e_5] = e_7 \rangle.$$

This algebra is nilpotent of class five and denoted by $A_{3,7,25}$. □

Lemma 8. *Let $A/\langle e_7 \rangle \cong A_{3,6,5}$. Then A is isomorphic to $A_{3,7,19}$, $A_{3,7,26}$, or $A_{3,7,27}$.*

Proof. Let $A/\langle e_7 \rangle \cong A_{3,6,5}$. In this case, the multiplication in A can be written as

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4 + \alpha e_7, & [e_2, e_3, e_4] = e_5 + \beta e_7, & [e_2, e_3, e_5] = e_6 + \gamma e_7, \\ [e_1, e_3, e_4] = e_6 + \lambda e_7, & [e_i, e_j, e_k] = \alpha_{ijk} e_7, & \end{array}$$

where $1 \leq i < j < k \leq 6$, and $\{i, j, k\} \notin \{\{1, 2, 3\}, \{2, 3, 4\}, \{2, 3, 5\}, \{1, 3, 4\}\}$.

Regarding a suitable change of basis, one can assume that $\alpha = \beta = \gamma = 0$, and the Jacobi identity gives us $\alpha_{146} = \alpha_{246} = \alpha_{346} = \alpha_{156} = \alpha_{256} = \alpha_{356} = \alpha_{456} = \alpha_{245} = \alpha_{145} = \alpha_{125} = \alpha_{126} = 0, \alpha_{135} = \alpha_{236}, \alpha_{345} = \alpha_{136}$. Thus, the multiplication in A can be written as

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_4] = e_5, & [e_2, e_3, e_5] = e_6, \\ [e_1, e_3, e_4] = e_6 + \lambda e_7, & [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_1, e_3, e_5] = \alpha_{236} e_7, \\ [e_2, e_3, e_6] = \alpha_{236} e_7, & [e_1, e_3, e_6] = \alpha_{136} e_7, & [e_3, e_4, e_5] = \alpha_{136} e_7. \end{array}$$

We discuss on the dimension of the center of A. The above multiplication shows that the dimension of the center of A is at most 2.

(i) Let $\dim Z(A) = 2$. In this case, $Z(A) = \langle e_6, e_7 \rangle$ and the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_4] = e_5, & [e_2, e_3, e_5] = e_6 \\ [e_1, e_3, e_4] = e_6 + \lambda e_7, & [e_1, e_2, e_4] = \alpha_{124} e_7. & \end{array}$$

Now, $A/\langle e_6 \rangle$ is a 6-dimensional nilpotent 3-Lie algebra with derived subalgebra of dimension 2 or 3. If the dimension of the derived subalgebra is 2, then the algebra is isomorphic to $A_{3,6,5} \oplus F(1)$, and otherwise, we have the following algebra:

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_2, e_3, e_5] = e_6, [e_1, e_3, e_4] = e_7 \rangle.$$

This algebra is isomorphic to $A_{3,7,19}$.

(ii) Let $\dim Z(A) = 1$. In this case, $Z(A) = \langle e_7 \rangle$.

Since e_6 is not belong to $Z(A)$, at least one of $\alpha_{236}, \alpha_{136}$ is not equal to zero.

If $\alpha_{136} \neq 0$, then by replacing e_1 by $e_1 - \frac{\lambda}{\alpha_{136}} e_5$, the multiplication in A can be written as follows:

$$\begin{array}{lll} [e_1, e_2, e_3] = e_4, & [e_2, e_3, e_4] = e_5, & [e_2, e_3, e_5] = e_6, \\ [e_1, e_3, e_4] = e_6, & [e_1, e_2, e_4] = \alpha_{124} e_7, & [e_1, e_3, e_5] = \alpha_{236} e_7, \\ [e_2, e_3, e_6] = \alpha_{236} e_7, & [e_1, e_3, e_6] = \alpha_{136} e_7, & [e_3, e_4, e_5] = \alpha_{136} e_7. \end{array}$$

Similar to the previous episodes, regarding a suitable change of basis,

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_2, e_3, e_5] = [e_1, e_3, e_4] = e_6, [e_1, e_3, e_6] = [e_3, e_4, e_5] = e_7\rangle.$$

This algebra is nilpotent of class five and is denoted by $A_{3,7,26}$.

Finally, if $\alpha_{234} \neq 0$, then we have the following algebra:

$$A = \langle e_1, \ldots, e_7 : [e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_2, e_3, e_5] = [e_1, e_3, e_4] = e_6, [e_1, e_3, e_5] = [e_2, e_3, e_6] = e_7\rangle.$$

This algebra is nilpotent of class five and is denoted by $A_{3,7,27}$. □

The following theorem is an immediate consequence of Lemmas 6, 7, and 8.

Theorem 4. *Let A be a 7-dimensional nilpotent 3-Lie algebra with the derived subalgebra of dimension 4. Then A is isomorphic to $A_{3,7,18}$, $A_{3,7,19}$, $A_{3,7,20}$, $A_{3,7,21}$, $A_{3,7,22}$, $A_{3,7,23}$, $A_{3,7,24}$, $A_{3,7,25}$, $A_{3,7,26}$, or $A_{3,7,27}$.*

Finally, we present the following final theorem about our results.

Theorem 5. *Let A be a 7-dimensional nilpotent 3-Lie algebra. Then A is isomorphic to one of the following algebras:*

$$H(3,1) \oplus F(3), H(3,2), A_{3,5,1} \oplus F(2), A_{3,6,i} \oplus F(1)(1 \leq i \leq 5), A_{3,7,i}(1 \leq i \leq 27).$$

Proof. Let A be a 7-dimensional nilpotent 3-Lie algebra. If $\dim A^2 = 1$, then by [7, Lemma 3.1], A is isomorphic to $H(3,1) \oplus F(3)$ or $H(3,2)$. If $\dim A^2 \geq 2$, then by Theorems 2, 3, and 4, algebras are obtained. The algebras in this theorem, have been obtained according to the process mentioned in the article. At each stage, after obtaining each algebra, its structure is compared with other previously obtained algebras, and if it is similar to them, then it is removed, and if it is new, then it is named. Therefore, the obtained algebras are not isomorphic. □

We have listed the obtained n-Lie algebras in this article in Table 1.

Table 1: All algebras obtained in this article.

Name	Non-zero multiplication
$A_{3,5,1}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5$
$A_{3,6,1}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6$
$A_{3,6,2}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = [e_1, e_3, e_5] = e_6$
$A_{3,6,3}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_1, e_3, e_4] = e_6$
$A_{3,6,4}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_2, e_3, e_5] = e_6$
$A_{3,6,5}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_5, [e_2, e_3, e_5] = [e_1, e_3, e_4] = e_6$
$A_{3,7,1}$	$[e_1, e_2, e_3] = e_6, [e_3, e_4, e_5] = e_7$
$A_{3,7,2}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_5] = e_6, [e_1, e_3, e_5] = e_7$
$A_{3,7,3}$	$[e_1, e_2, e_3] = e_6, [e_2, e_3, e_6] = [e_1, e_3, e_4] = [e_1, e_2, e_5] = e_7$
$A_{3,7,4}$	$[e_1, e_2, e_3] = e_6, [e_2, e_3, e_6] = [e_3, e_4, e_5] = e_7$
$A_{3,7,5}$	$[e_1, e_2, e_3] = e_6, [e_2, e_3, e_6] = [e_1, e_4, e_5] = e_7$
$A_{3,7,6}$	$[e_1, e_2, e_3] = e_4, [e_2, e_3, e_4] = e_6, [e_1, e_3, e_4] = [e_2, e_3, e_5] = e_7$

$A_{3,7,7}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_6, [e_2,e_3,e_5]=e_7$
$A_{3,7,8}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_6, [e_1,e_3,e_5]=e_7$
$A_{3,7,9}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_6, [e_2,e_3,e_6]=[e_1,e_3,e_5]=e_7$
$A_{3,7,10}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_6, [e_2,e_3,e_6]=[e_1,e_3,e_5]=[e_1,e_3,e_4]=e_7$
$A_{3,7,11}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_6, [e_2,e_3,e_6]=[e_1,e_3,e_5]=[e_1,e_2,e_4]=e_7$
$A_{3,7,12}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_5]=e_6, [e_2,e_3,e_4]=[e_1,e_3,e_5]=e_7$
$A_{3,7,13}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_5]=e_6, [e_2,e_3,e_6]=[e_1,e_3,e_4]=e_7$
$A_{3,7,14}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_5]=e_6, [e_1,e_3,e_6]=[e_3,e_4,e_5]=e_7$
$A_{3,7,15}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_5]=e_6, [e_3,e_5,e_6]=[e_1,e_3,e_4]=e_7$
$A_{3,7,16}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_5]=e_6, [e_3,e_5,e_6]=[e_1,e_2,e_4]=e_7$
$A_{3,7,17}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=[e_1,e_3,e_5]=e_6, [e_2,e_3,e_6]=-[e_3,e_4,e_5]=e_7$
$A_{3,7,18}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_5, [e_1,e_3,e_4]=e_6, [e_1,e_2,e_4]=e_7$
$A_{3,7,19}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_5, [e_1,e_3,e_4]=e_6, [e_2,e_3,e_5]=e_7$
$A_{3,7,20}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_5, [e_1,e_3,e_4]=e_6, [e_1,e_2,e_4]=[e_2,e_3,e_5]=e_7$
$A_{3,7,21}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_5, [e_1,e_3,e_4]=e_6, [e_1,e_3,e_5]=[e_2,e_3,e_6]=e_7$
$A_{3,7,22}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_5, [e_1,e_3,e_4]=e_6, [e_1,e_2,e_4]=[e_1,e_3,e_5]=[e_2,e_3,e_6]=e_7$
$A_{3,7,23}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_5, [e_2,e_3,e_5]=e_6, [e_2,e_3,e_6]=e_7$
$A_{3,7,24}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_5, [e_2,e_3,e_5]=e_6, [e_2,e_3,e_6]=[e_1,e_3,e_4]=e_7$
$A_{3,7,25}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_5, [e_2,e_3,e_5]=e_6, [e_3,e_4,e_5]=[e_1,e_3,e_6]=e_7$
$A_{3,7,26}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_5, [e_2,e_3,e_5]=[e_1,e_3,e_4]=e_6, [e_3,e_4,e_5]=[e_1,e_3,e_6]=e_7$
$A_{3,7,27}$	$[e_1,e_2,e_3]=e_4, [e_2,e_3,e_4]=e_5, [e_2,e_3,e_5]=[e_1,e_3,e_4]=e_6, [e_1,e_3,e_5]=[e_2,e_3,e_6]=e_7$

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References

[1] M. A. Alvarez, *Degenerations of 8-dimensional 2-step nilpotent Lie algebras*, Algebr. Represent. Theory, (5) **24** (2021), 1231–1243.

[2] R. Bai, G. Song and Y. Zhang, *On classification of n-Lie algebras*, Front. Math. China, (4) **6** (2011), 581–606.

[3] N. Cantarini and V. G. Kac, *Classification of simple linearly compact n-Lie superalgebras*, Comm. Math. Phys., (3) **298** (2010), 833–853.

[4] R. Carles and Y. Diakité, *Sur les variétés d'algèbres de Lie de dimension 7. (French) [Varieties of Lie algebras of dimension 7]*, J. Algebra, (1) **91** (1984), 53–63.

[5] S. Cicalo, W. A. de Graaf and C. Schneider, *Six-dimensional nilpotent Lie algebras*, Linear Algebra Appl., (1) **436** (2012), 163–189.

[6] H. Darabi, M. Eshrati and F. Saeedi, *The algebraic classification of low-dimensional nilpotent n-Lie algebras*, Asian-Eur. J. Math., (3) **18** (2025), 2450110.

[7] H. Darabi and M. Imanparast, *On classification of 9-dimensional nilpotent 3-ary algebras of class two*, Bull. Iranian Math. Soc., **47** (2021), 929–937.

[8] H. Darabi, F. Saeedi and M. Eshrati, *A characterization of finite dimensional nilpotent Filippov algebras*, J. Geom. Phys., **101** (2016), 100–107.

[9] M. Eshrati, F. Saeedi and H. Darabi, *On the multiplier of nilpotent n-Lie algebras*, J. Algebra, **450** (2016), 162–172.

[10] V. T. Filippov, *n-Lie algebras*, Sib. Math. J, (6) **26** (1985), 879–891.

[11] M. P. Gong, *Classification of Nilpotent Lie Algebras of Dimension 7 (Over Algebraically Closed Fields and R)*, Ph.D. Thesis, University of Waterloo, Waterloo, (1998).

[12] W. A. de Graaf, *Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2*, J. Algebra (2) **309** (2007), 640–653.

[13] F. Grunewald and J. O'Halloran, *Varieties of nilpotent Lie algebras of dimension less than six*, J. Algebra, (2) **112** (1988), 315–325.

[14] A. S. Hegazi and H. Abdelwahab, *Classification of five-dimensional nilpotent Jordan algebras*, Linear Algebra Appl., **494** (2016), 165–218.

[15] Z. Hoseini, F. Saeedi and H. Darabi, *On Classification of (n+5)-dimensional nilpotent n-Lie algebras of class two*, Bull. Iranian Math. Soc., (4) **45** (2019), 939–949.

[16] N. Ismailov, I. Kaygorodov and Y. Volkov, *The geometric classification of Leibniz algebras*, Internat. J. Math., (5) **29** (2018), 1850035.

[17] M. Jamshidi, F. Saeedi and H. Darabi, *On classification of (n+6)-dimensional nilpotent n-Lie algebras of class two with* $n \geq 4$, Arab J. Math. Sci., (2) **27** (2021), 139–150.

[18] S. M. Kasymov, *Theory of n-Lie algebras*, Algebra Logic, (3) **26** (1987), 155–166.

[19] I. Kaygorodov, M. Khrypchenko and S. A. Lopes, *The algebraic and geometric classification of nilpotent anticommutative algebras*, J. Pure Appl. Algebra, (8) **224** (2020), 106337.

[20] I. Kaygorodov, M. Khrypchenko and Y. Popov, *The algebraic and geometric classification of nilpotent terminal algebras*, J. Pure Appl. Algebra, (6) **225** (2021), 106625.

[21] I. Kaygorodov and Y. Volkov, *Degenerations of Filippov algebras*, J. Math. Phys., (2) **61** (2020), 021701.

[22] I. Kaygorodov and Y. Volkov, *Degenerations of noncommutative Heisenberg algebras*, Comm. Algebra, (10) **51** (2023), 4204–4213.

[23] W. Ling, *On Structure of n-Lie Algebras*, Ph.D. thesis, University-GHS-Siegen, (1993).

[24] G. Mazzola, *The algebraic and geometric classification of associative algebras of dimension five*, Manuscripta Math., (1) **27** (1979), 81–101.

[25] V. V. Morozov, *Classification of nilpotent Lie algebras of sixth order*, Izv. Vyssh. Uchebn. Zaved. Mat., **4** (1958), 161–171.

[26] A. P. Pozhidaev, *On simple n-Lie algebras*, Algebra Logic, (3) **38** (1999), 181–192.

[27] B. Ren and D. J. Meng, *Some 2-step nilpotent Lie algebras I*, Linear Algebra Appl., **338** (2001), 77–98.

[28] B. Ren and L. S. Zhu, *Classification of 2-step nilpotent Lie algebras of dimension 8 with 2-dimensional center*, Comm. Algebra, (6) **39** (2011), 2068–2081.

[29] B. Ren and L. S. Zhu, *Classification of 2-step nilpotent Lie algebras of dimension 9 with 2-dimensional center*, Czechoslovak Math. J., (3) **67** (2017), 953–965.

[30] C. Seeley, *7-dimensional nilpotent Lie algebras*, Trans. Amer. Math. Soc., (2) **335** (1993), 479–496.

[31] K. A. Umlauf, *Ueber den Zusammenhang der endlichen continuirlichen Transformationsgruppen*, insbesondere der Gruppen vom Range Null. Ph.D. Thesis, University of Leipzig, (1891), (In German).

[32] Z. Yan and S. Deng, *The classification of two step nilpotent complex Lie algebras of dimension 8*, Czechoslovak Math. J., (3) **63** (2013), 847–863.