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Generalized dual Leonardo quaternion numbers

Fügen Torunbalcı Aydın[†*]

[†]*Department of Mathematical Engineering Yildiz Technical University, Faculty of Chemical and Metallurgical Engineering, Davutpasa Campus, İstanbul, Turkey*
Emails: faydinyildiz.edu.tr; ftorunaygmail.com

Abstract. In this paper, we introduce dual k-Leonardo quaternions which we call generalized dual Leonardo quaternion numbers. Some algebraic properties of these quaternions such as recurrence relation, generating function, Binet's formula, Cassini identity, sum formulas will also be obtained.

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1 Introduction

The quaternions are a number system that extends the complex numbers. They were first described by Irish mathematician William Rowan Hamilton in 1843. Hamilton [6] introduced the set of quaternions which can be represented as

$$H = \{ q = q_0 + i\, q_1 + j\, q_2 + k\, q_3 \mid \ q_0,\, q_1,\, q_2,\, q_3 \in \mathbb{R} \}$$

where,

$$i^2 = j^2 = k^2 = -1\,, \quad i\ j = -j\ i = k\,, \quad j\ k = -k\ j = i\,, \quad k\ i = -i\ k = j\,.$$

Horadam [7] has defined complex Fibonacci and Lucas quaternions as follows

$$F_n = F_n + i\, F_{n+1} + j\ F_{n+2} + k\, F_{n+3}$$

and

$$L_n = L_n + i\, L_{n+1} + j\ L_{n+2} + k\, L_{n+3}$$

*Corresponding author

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where,

$$i^2 = j^2 = k^2 = -1\,, \quad i\,j = -j\,i = k\,, \quad j\,k = -k\,j = i\,, \quad k\,i = -i\,k = j\,.$$

In 2012, Halıcı [5] gave generating functions and Binet's formulas for Fibonacci and Lucas quaternions.
In 2006, Majernik [10] defined dual quaternions as follows:

$$H_{\mathbb{D}} = \left\{ \begin{array}{c} Q = a + b\,i + c\,j + d\,k \;|\; a\,,\ b\,,\ c\,,\ d \in \mathbb{R},\ i^2 = j^2 = k^2 = i\,j\,k = 0, \\ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0 \end{array} \right\}.$$

In 2009, Ata and Yaylı [1], studied on dual quaternions and dual projective spaces.
In 2015, Nurkan and Güven [12], identified dual Fibonacci and Lucas quaternions with dual number coefficients.
In 2016, Yüce and Torunbalcı Aydın [16] defined dual Fibonacci quaternions with real number coefficients as follows:

$$H_{\mathbb{D}} = \{\, Q_n = F_n + i\,F_{n+1} + j\;F_{n+2} + k\,F_{n+3} \;|\, F_n \text{ is the } n-th \text{ Fibonacci number}\},$$

where,

$$i^2 = j^2 = k^2 = i\;j\;k = 0\,, \quad i\;j = -j\;i = j\;k = -k\;j = k\;i = -i\;k = 0.$$

Recent developments on Leonardo nımbers, their generalizations and other intriguing properties can be found in [4]. In 2019, Catarino and Borges [2], defined a new type of number related to Fibonacci numbers, called Leonardo numbers.
In 2020, Catarino and Borges [3], defined incomplete Leonardo numbers.In 2021, Kürüz, Dağdeviren and Catarino [9] worked on Leonardo pisano hybrinomials.
In 2021, Yasemin and Koçer [15], were obtained some properties of Leonardo numbers.
In 2023, Nurkan and Güven [11], introduced a new quadruple number sequence by means of Leonardo numbers, which called ordered Leonardo quadruple numbers as follows

$$Q_{\mathbb{L}} = \{\, Q_n = Le_n + i\,Le_{n+1} + j\;Le_{n+2} + k\,Le_{n+3} \;|\, Le_n \text{ is the } n-th \text{ Leonardo number}\},$$

where,

$$i^2 = j^2 = k^2 = i\;j\;k = 0\,, \quad i\;j = -j\;i = j\;k = -k\;j = k\;i = -i\;k = 0.$$

In 2019, Shannon [13], worked on the generalization of Leonardo numbers.
In 2021, Soykan [14], studied on the generalized Leonardo numbers.
In 2022, Kuhapatanakul and Chobsorn [8], for a fixed positive integer k, define the generalized Leonardo sequence $\mathcal{L}_{k,n}, n \geq 0$ as follows

$$\mathcal{L}_{k,n} = \mathcal{L}_{k,n-1} + \mathcal{L}_{k,n-2} + k,\ n \geq 2. \tag{1}$$

with the initial conditions $\mathcal{L}_{k,0} = \mathcal{L}_{k,1} = 1$ Also, there are many identities between generalized Leonardo numbers and Fibonacci and Lucas numbers as follows

$$\mathcal{L}_{k,n} = (k+1)\,F_{n+1} - k\,, \tag{2}$$
$$\mathcal{L}_{k,n} = (k+1)(L_n - F_{n-1}) - k\,, \tag{3}$$
$$\sum_{i=0}^{n} \mathcal{L}_{k,i} = \mathcal{L}_{k,n+2} - k(n+1) - 1, \tag{4}$$
$$\sum_{i=0}^{n} \mathcal{L}_{k,2i} = \mathcal{L}_{k,2n+1} - kn, \tag{5}$$
$$\sum_{i=0}^{n} \mathcal{L}_{k,2i+1} = \mathcal{L}_{k,2n+2} - k(n+2). \tag{6}$$

The following relations have been calculated by me for use in this article.

$$\mathcal{L}_{k,n+1} + \mathcal{L}_{k,n-1} = (k+1)\,L_{n+1} - 2k\,, \tag{7}$$
$$\mathcal{L}_{k,n+2} + \mathcal{L}_{k,n-2} = (k+1)\,L_{n+1}\,, \tag{8}$$
$$\mathcal{L}^2_{k,n} + \mathcal{L}^2_{k,n-1} = (k+1)\,\mathcal{L}_{k,2n} - 2k\mathcal{L}_{k,n+1} + k(k-1)\,, \tag{9}$$
$$\mathcal{L}^2_{k,n+1} - \mathcal{L}^2_{k,n-1} = (k+1)\,\mathcal{L}_{k,2n+1} - 2k\mathcal{L}_{k,n} - k(k-1)\,. \tag{10}$$

2 Generalized dual Leonardo quaternion numbers

In this section, we define the generalized dual Leonardo numbers. Then, we obtain generating function, Binet's formula, summation formulas, Cassini's identity and the other identities.

Definition 1. *The n-th generalized dual Leonardo quaternion* $(\mathbb{Q}_n)$ *as follows*

$$\mathbb{Q}_n = \mathcal{L}_{k,n} + \mathbf{i}\,\mathcal{L}_{k,n+1} + \mathbf{j}\,\mathcal{L}_{k,n+2} + \mathbf{k}\,\mathcal{L}_{k,n+3}. \tag{11}$$

with the initial conditions $\mathbb{Q}_0 = \mathbb{Q}_1 = 1$

where,

$$\mathbf{i}^{\,2} = \mathbf{j}^{\,2} = \mathbf{k}^{\,2} = \mathbf{i}\,\mathbf{j}\,\mathbf{k} = 0\,,\;\; \mathbf{i}\;\mathbf{j} = -\mathbf{j}\;\mathbf{i} = \mathbf{j}\;\mathbf{k} = -\mathbf{k}\;\mathbf{j} = \mathbf{k}\;\mathbf{i} = -\mathbf{i}\;\mathbf{k} = 0. \tag{12}$$

Let $\mathbb{Q}_n$ and $\mathbb{Q}_m$ be two generalized dual Leonardo quaternion sequences $(\mathbb{Q}_n)$ and $(\mathbb{Q}_m)$ respectively, as follows:

$$\mathbb{Q}_n = \mathcal{L}_{k,n} + \mathbf{i}\,\mathcal{L}_{k,n+1} + \mathbf{j}\,\mathcal{L}_{k,n+2} + \mathbf{k}\,\mathcal{L}_{k,n+3}$$

and

$$\mathbb{Q}_m = \mathcal{L}_{k,m} + \mathbf{i}\,\mathcal{L}_{k,m+1} + \mathbf{j}\,\mathcal{L}_{k,m+2} + \mathbf{k}\,\mathcal{L}_{k,m+3}\,. \tag{13}$$

Then, the addition and subtraction of the the generalized dual Leonardo quaternion sequences is defined by

$$\begin{aligned}\mathbb{Q}_n \pm \mathbb{Q}_m =& (\mathcal{L}_{k,n} \pm \mathcal{L}_{k,m}) + \mathbf{i}\,(\mathcal{L}_{k,n+1} \pm \mathcal{L}_{k,m+1}) \\ &+ \mathbf{j}\,(\mathcal{L}_{k,n+2} \pm \mathcal{L}_{k,m+2}) + \mathbf{k}\,(\mathcal{L}_{k,n+3} \pm \mathcal{L}_{k,m+3}.\end{aligned} \tag{14}$$

Multiplication of two the generalized dual Leonardo quaternion sequences is defined by

$$\begin{aligned}\mathbb{Q}_n\mathbb{Q}_m =& (\mathcal{L}_{k,n}\,\mathcal{L}_{k,m}) + \mathbf{i}\,(\mathcal{L}_{k,n}\,\mathcal{L}_{k,m+1} + \mathcal{L}_{k,n+1}\,\mathcal{L}_{k,m}) \\ &+ \mathbf{j}\,(\mathcal{L}_{k,n}\,\mathcal{L}_{k,m+2} + \mathcal{L}_{k,n+2}\,\mathcal{L}_{k,m}) \\ &+ \mathbf{k}\,(\mathcal{L}_{k,n}\,\mathcal{L}_{k,m+3} + \mathcal{L}_{k,n+3}\,\mathcal{L}_{k,m}).\end{aligned} \tag{15}$$

The scaler and the vector part of $\mathbb{Q}_n$ which is the $n-th$ term of the generalized dual Leonardo quaternion $(\mathbb{Q}_n)$ are denoted by

$$S_{\mathbb{Q}_n} = \mathcal{L}_{k,n} \text{ and } V_{\mathbb{Q}_n} = \mathbf{i}\,\mathcal{L}_{k,n+1} + \mathbf{j}\,\mathcal{L}_{k,n+2} + \mathbf{k}\,\mathcal{L}_{k,n+3}. \tag{16}$$

Thus, the generalized dual Leonardo quaternion $\mathbb{Q}_n$ is given by $\mathbb{Q}_n = S_{\mathbb{Q}_n} + V_{\mathbb{Q}_n}$. Then, relation Eq.(16) is defined by

$$\mathbb{Q}_n\mathbb{Q}_m = S_{\mathbb{Q}_n}\ S_{\mathbb{Q}_m} + S_{\mathbb{Q}_n}\ V_{\mathbb{Q}_m} + S_{\mathbb{Q}_m}\,V_{\mathbb{Q}_n},\quad V_{\mathbb{Q}_n} \times V_{\mathbb{Q}_m} = 0\,. \tag{17}$$

The conjugate of generalized dual Leonardo quaternion $\mathbb{Q}_n$ is denoted by $\overline{\mathbb{Q}}_n$ and it is

$$\overline{\mathbb{Q}}_n = \mathcal{L}_{k,n} - \mathbf{i}\,\mathcal{L}_{k,n+1} - \mathbf{j}\,\mathcal{L}_{k,n+2} - \mathbf{k}\,\mathcal{L}_{k,n+3}. \tag{18}$$

The norm of $\mathbb{Q}_\mathbf{n}$ is defined as

$$\|\mathbb{Q}_n\|^2 = \mathbb{Q}_n\ \overline{\mathbb{Q}}_n = (\mathcal{L}_{k,n})^2. \tag{19}$$

Then, we give the following theorems including the properties of generalized dual Leonardo quaternion numbers.

Theorem 1. *Let* $(\mathbb{Q}_n)$ *generalized dual Leonardo quaternion number. For any integer* $n \geq 0$,

$$\mathbb{Q}_n + \mathbb{Q}_{n+1} + k\,\mathcal{Q} = \mathbb{Q}_{n+2} \tag{20}$$

where, $\mathcal{Q} = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Proof. (20): By using the $n-th$ and $(n+1)-th$ terms in Eq.(11) and Eq.(1), we have

$$\begin{aligned}\mathbb{Q}_n + \mathbb{Q}_{n+1} &= (\mathcal{L}_{k,n} + \mathcal{L}_{k,n+1}) + \mathbf{i}\,(\mathcal{L}_{k,n+1} + \mathcal{L}_{k,n+2}) \\ &\quad + \mathbf{j}\,(\mathcal{L}_{k,n+2} + \mathcal{L}_{k,n+3}) + \mathbf{k}\,(\mathcal{L}_{k,n+3} + \mathcal{L}_{k,n+4}) \\ &= \mathcal{L}_{k,n+2} + \mathbf{i}\,\mathcal{L}_{k,n+3} + \mathbf{j}\,\mathcal{L}_{k,n+4} + \mathbf{k}\,\mathcal{L}_{k,n+5} - k(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= \mathbb{Q}_{n+2} - k\,\mathcal{Q}.\end{aligned}$$

□

Theorem 2. *Let* $(\mathbb{Q}_n)$ *generalized dual Leonardo quaternion number. For any integer* $n \geq 0$*,*

$$\mathbb{Q}_n = 2\,\mathbb{Q}_{n-1} - \mathbb{Q}_{n-3}\,. \tag{21}$$

Proof. (21): In Eq.(20), if we replace $n \to n-1$, then we have

$$\mathbb{Q}_{n+1} = \mathbb{Q}_{n-1} + \mathbb{Q}_n + k\,\mathcal{Q}$$

By subtracting Eq.(20) from this equation, we get

$$\mathbb{Q}_{n+2} = 2\,\mathbb{Q}_{n+1} - \mathbb{Q}_{n-1}$$

If we take $n \to n-2$ in last equality, we find recurrence relation as follows

$$\mathbb{Q}_n = 2\,\mathbb{Q}_{n-1} - \mathbb{Q}_{n-3}\,.$$

□

Theorem 3. *Let* $\mathcal{L}_{k,n}$ *and* $\mathbb{Q}_n$ *be the* $n-th$ *terms of generalized Leonardo number* $(\mathcal{L}_{k,n})$ *and the generalized dual Leonardo quaternion sequence* $(\mathbb{Q}_n)$*, respectively.*

For $n \geq 0$ *and* $m \geq 1$*, the following hold:*

$$\mathbb{Q}_n - \mathbf{i}\,\mathbb{Q}_{n+1} - \mathbf{j}\,\mathbb{Q}_{n+2} - \mathbf{k}\,\mathbb{Q}_{n+3} = \mathcal{L}_{k,n} \tag{22}$$

$$\begin{aligned} \mathbb{Q}_n\mathbb{Q}_m + \mathbb{Q}_{n+1}\mathbb{Q}_{m+1} = {} & (k+1)\,[2\,\mathbb{Q}_{n+m+2} + k] - k\,\mathbb{Q}_{n+2} - k\,\mathbb{Q}_{m+2} \\ & - k(\mathbf{i}+\mathbf{j}+\mathbf{k})\,[\mathcal{L}_{k,n+2} + \mathcal{L}_{k,m+2} - 2(k+1)] \\ & - (k+1)\mathcal{L}_{k,n+m+2}\,. \end{aligned} \tag{23}$$

Proof. (22): By using Eq.(11), we get

$$\begin{aligned} \mathbb{Q}_n - \mathbf{i}\,\mathbb{Q}_{n+1} - \mathbf{j}\,\mathbb{Q}_{n+2} - \mathbf{k}\,\mathbb{Q}_{n+3} = {} & \mathcal{L}_{k,n} + i\,\mathcal{L}_{k,n+1} + j\,\mathcal{L}_{k,n+2} + k\,\mathcal{L}_{k,n+3} \\ & - \mathbf{i}\,(\mathcal{L}_{k,n+1} + i\,\mathcal{L}_{k,n+2} + j\,\mathcal{L}_{k,n+3} + k\,\mathcal{L}_{k,n+4}\,) \\ & - \mathbf{j}\,(\mathcal{L}_{k,n+2} + i\,\mathcal{L}_{k,n+3} + j\,\mathcal{L}_{k,n+4} + k\,\mathcal{L}_{k,n+5}\,) \\ & - \mathbf{k}(\mathcal{L}_{k,n+3} + i\,\mathcal{L}_{k,n+4} + j\,\mathcal{L}_{k,n+5} + k\,\mathcal{L}_{k,n+6}\,) \\ = {} & \mathcal{L}_{k,n}\,. \end{aligned}$$

(23): By using Eq.(15)

$$\begin{aligned} \mathbb{Q}_n\mathbb{Q}_m = {} & (\mathcal{L}_{k,n}\,\mathcal{L}_{k,m}) + \mathbf{i}\,(\mathcal{L}_{k,n}\,\mathcal{L}_{k,m+1} + (\mathcal{L}_{k,n+1}\,\mathcal{L}_{k,m}) \\ & + \mathbf{j}\,(\mathcal{L}_{k,n}\,\mathcal{L}_{k,m+2} + \mathcal{L}_{k,n+2}\,\mathcal{L}_{k,m}) \\ & + \mathbf{k}\,(\mathcal{L}_{k,n}\,\mathcal{L}_{k,m+3} + \mathcal{L}_{k,n+3}\,\mathcal{L}_{k,m}). \end{aligned} \tag{24}$$

and

$$\mathbb{Q}_{n+1}\mathbb{Q}_{m+1} = (\mathcal{L}_{k,n+1}\,\mathcal{L}_{k,m+1}) + \mathbf{i}\,(\mathcal{L}_{k,n+1}\,\mathcal{L}_{k,m+2} + (\mathcal{L}_{k,n+2}\,\mathcal{L}_{k,m+1}) \tag{25}$$

$$+\,\mathbf{j}\,(\mathcal{L}_{k,n+1}\,\mathcal{L}_{k,m+3}+\mathcal{L}_{k,n+3}\,\mathcal{L}_{k,m+1})$$
$$+\,\mathbf{k}\,(\mathcal{L}_{k,n+1}\,\mathcal{L}_{k,m+4}+\mathcal{L}_{k,n+4}\,\mathcal{L}_{k,m+1}).$$

Finally, adding equations Eq.(24) and Eq.(25) side by side, we obtain

$$\begin{aligned}
\mathbb{Q}_n\mathbb{Q}_m+\mathbb{Q}_{n+1}\mathbb{Q}_{m+1} &= (k+1)(\mathcal{L}_{k,n+m+2}+k)-k\mathcal{L}_{k,n+2}-k\mathcal{L}_{k,m+2}\\
&\quad+\mathbf{i}\,[2(k+1)(\mathcal{L}_{k,n+m+3}+k)-k\mathcal{L}_{k,n+2}-k\mathcal{L}_{k,n+3}\\
&\quad-k\mathcal{L}_{k,m+2}-k\mathcal{L}_{k,m+3}]\\
&\quad+\mathbf{j}\,[2(k+1)(\mathcal{L}_{k,n+m+4}+k)-k\mathcal{L}_{k,n+2}-k\mathcal{L}_{k,n+4}\\
&\quad-k\mathcal{L}_{k,m+2}-k\mathcal{L}_{k,m+4}]\\
&\quad+\mathbf{k}\,[\,2(k+1)(\mathcal{L}_{k,n+m+5}+k)-k\mathcal{L}_{k,n+2}-k\mathcal{L}_{k,n+5}\\
&\quad-k\mathcal{L}_{k,m+2}-k\mathcal{L}_{k,m+5}\,]\\
&=(k+1)[\,2\mathcal{L}_{k,n+m+2}+2\mathbf{i}\mathcal{L}_{k,n+m+3}+2\mathbf{j}\mathcal{L}_{k,n+m+4}\\
&\quad+2\mathbf{k}\mathcal{L}_{k,n+m+5}\,]+k(k+1)(1+2\mathbf{i}+2\mathbf{j}+2\mathbf{k})\\
&\quad-k\,[\,\mathcal{L}_{k,n+2}+\mathbf{i}(\mathcal{L}_{k,n+3}+\mathcal{L}_{k,n+2})+\mathbf{j}(\mathcal{L}_{k,n+4}+\mathcal{L}_{k,n+5})\\
&\quad+\mathbf{k}(\mathcal{L}_{k,n+2}+\mathcal{L}_{k,n+5})\,]\\
&\quad-k\,[\,\mathcal{L}_{k,m+2}+\mathbf{i}\mathcal{L}_{k,m+4}+\mathbf{j}(\mathcal{L}_{k,m+2}+\mathcal{L}_{k,m+4})\\
&\quad+\mathbf{k}(\mathcal{L}_{k,m+2}+\mathcal{L}_{k,m+5})\,]\\
&=2(k+1)\,\mathbb{Q}_{n+m+2}+k(k+1)(2\mathcal{Q}-1)\\
&\quad+k(k+1)(1+2\mathbf{i}+2\mathbf{j}+2\mathbf{k})-(k+1)\mathcal{L}_{k,n+m+2},
\end{aligned}$$

or

$$\begin{aligned}
&=(k+1)\,[2\,\mathbb{Q}_{n+m+2}+k\,]-k\,\mathbb{Q}_{n+2}-k\,\mathbb{Q}_{m+2}\\
&\quad-k(\mathbf{i}+\mathbf{j}+\mathbf{k})\,[\mathcal{L}_{k,n+2}+\mathcal{L}_{k,m+2}-2(k+1)]\\
&\quad-(k+1)\mathcal{L}_{k,n+m+2}.
\end{aligned}$$

□

Theorem 4. *Let the $n-th$ terms of the dual Lucas quaternion sequence (T_n) and the generalized dual Leonardo quaternion number sequence $\mathbb{Q}_n$ be T_n and $\mathbb{Q}_n$,respectively.*

The following relations are satisfied

$$\mathbb{Q}_{n-1}+\mathbb{Q}_{n+1}=(k+1)\,T_{n+1}-2k\,\mathcal{Q}\,, \tag{26}$$
$$\mathbb{Q}_{n+2}-\mathbb{Q}_{n-2}=(k+1)\,T_{n+1}\,. \tag{27}$$

Proof. (26):From equations Eq.(11), Eq.(7), it follows that

$$\mathbb{Q}_{n-1}+\mathbb{Q}_{n+1}=(\mathcal{L}_{k,n-1}+\mathcal{L}_{k,n+1})+\mathbf{i}\,(\mathcal{L}_{k,n}+\mathcal{L}_{k,n+2})+\mathbf{j}\,(\mathcal{L}_{k,n+1}+\mathcal{L}_{k,n+3})$$

$$
\begin{aligned}
&+\mathbf{k}\,(\mathcal{L}_{k,n+2}+\mathcal{L}_{k,n+4})\\
&=(k+1)L_{n+1}+i\,L_{n+2}+j\,L_{n+3}+k\,L_{n+4}-2k(1+\mathbf{i}+\mathbf{j}+\mathbf{k})\\
&=(k+1)T_{n+1}-2k\,\mathcal{Q}\,.
\end{aligned}
$$

(27): From equations Eq.(11), Eq.(8), it follows that

$$
\begin{aligned}
\mathbb{Q}_{n+2}-\mathbb{Q}_{n-2}&=(\mathcal{L}_{k,n+2}-\mathcal{L}_{k,n-2})+\mathbf{i}\,(\mathcal{L}_{k,n+3}-\mathcal{L}_{k,n-1})+\mathbf{j}\,(\mathcal{L}_{k,n+4}-\mathcal{L}_{k,n})\\
&\quad+\mathbf{k}\,(\mathcal{L}_{k,n+5}-\mathcal{L}_{k,n+1})\\
&=(k+1)L_{n+1}+i\,L_{n+2}+j\,L_{n+3}+k\,L_{n+4}\\
&=(k+1)T_{n+1}\,.
\end{aligned}
$$

□

Theorem 5. *Let $\mathbb{Q}_n$ be the generalized dual Leonardo quaternion and $\overline{\mathbb{Q}}_n$ be conjugate of $\mathbb{Q}_n$. Then, we can give the following relations between these quaternions*

$$
\begin{aligned}
\mathbb{Q}_n+\overline{\mathbb{Q}}_n&=2\,\mathcal{L}_{k,n}\,,\\
\mathbb{Q}_n\,\mathbb{Q}_n&=2\,\mathbb{Q}_n\,\mathcal{L}_{k,n}-\mathcal{L}_{k,n}^2\,,\\
\mathbb{Q}_n\,\overline{\mathbb{Q}}_n+\mathbb{Q}_{n-1}\,\overline{\mathbb{Q}}_{n-1}&=(k+1)\mathcal{L}_{k,2n}-2k\,\mathcal{L}_{k,n+1}-k(k+1),\\
\mathbb{Q}_n\,\overline{\mathbb{Q}}_n+\mathbb{Q}_{n+1}\,\overline{\mathbb{Q}}_{n+1}&=(k+1)\mathcal{L}_{k,2n+2}-2k\,\mathcal{L}_{k,n+2}+k(k+1),\\
\mathbb{Q}_{n+1}\,\overline{\mathbb{Q}}_{n+1}-\mathbb{Q}_{n-1}\,\overline{\mathbb{Q}}_{n-1}&=(k+1)\mathcal{L}_{k,2n+1}-2k\,\mathcal{L}_{k,n}-k(k-1),\\
\mathbb{Q}_n^2+\mathbb{Q}_{n-1}^2&=2(k+1)\mathbb{Q}_{2n}-2k\mathbb{Q}_{n+1}+\mathcal{Q}\,(2k(k+1)-2k\mathcal{L}_{k,n+1})\\
&\quad-(k+1)(\mathcal{L}_{k,2n}+1)+2\,k\mathcal{L}_{k,n+1}\,.
\end{aligned}
\tag{28}
$$

Proof. The first four properties in Eq.(28) can be easily proved by using Eq.(11) and Eq.(18). Now, let's calculate the last relation using the second relation in Eq.(28) and Eq.(9)

$$
\begin{aligned}
\mathbb{Q}_n^2+\mathbb{Q}_{n-1}^2&=(2\,\mathbb{Q}_n\,\mathcal{L}_{k,n}-\mathcal{L}_{k,n}^2)+(2\,\mathbb{Q}_{n-1}\,\mathcal{L}_{k,n-1}-\mathcal{L}_{k,n-1}^2)\\
&=(2\mathbb{Q}_n\,\mathcal{L}_{k,n}+2\mathbb{Q}_{n-1}\,\mathcal{L}_{k,n-1})-(\mathcal{L}_{k,n}^2+\mathcal{L}_{k,n-1}^2)\\
&=2(\mathcal{L}_{k,n}^2+\mathcal{L}_{k,n-1}^2)+2\mathbf{i}\,(\mathcal{L}_{k,n}\mathcal{L}_{k,n+1}+\mathcal{L}_{k,n-1}\mathcal{L}_{k,n})\\
&\quad+2\mathbf{j}\,(\mathcal{L}_{k,n}\mathcal{L}_{k,n+2}+\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+1})\\
&\quad+2\mathbf{k}\,(\mathcal{L}_{k,n}\mathcal{L}_{k,n+3}+\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+2})-(\mathcal{L}_{k,n}^2+\mathcal{L}_{k,n-1}^2)\\
&=(k+1)\,[\mathcal{L}_{k,2n}-k\mathcal{L}_{k,n+1}-k\mathcal{L}_{k,n+1}+k(k+1)]\\
&\quad+2\,\mathbf{i}[(k+1)\mathcal{L}_{k,2n+1}-k\mathcal{L}_{k,n+1}-k\mathcal{L}_{k,n+2}+k(k+1)]\\
&\quad+2\,\mathbf{j}[(k+1)\mathcal{L}_{k,2n+2}-k\mathcal{L}_{k,n+1}-k\mathcal{L}_{k,n+3}+k(k+1)]\\
&\quad+2\,\mathbf{k}[(k+1)\mathcal{L}_{k,2n+3}-k\mathcal{L}_{k,n+1}-k\mathcal{L}_{k,n+4+k(k+1)}]\\
&=2(k+1)\,[\mathcal{L}_{k,2n}+\mathbf{i}\mathcal{L}_{k,2n+1}+\mathbf{j}\mathcal{L}_{k,2n+2}+\mathbf{k}(k+1)\,\mathcal{L}_{k,2n+3}]\\
&\quad-k\mathcal{L}_{k,n+1}(2+2\mathbf{i}+2\mathbf{j}+2\mathbf{k})\\
&\quad-2k(\mathcal{L}_{k,n+1}+\mathbf{i}\mathcal{L}_{k,n+2}+\mathbf{j}\mathcal{L}_{k,n+3}+\mathbf{k}\mathcal{L}_{k,n+4})\\
&\quad+2\,k(k+1)(1+\mathbf{i}+\mathbf{j}+\mathbf{k})
\end{aligned}
$$

$$
\begin{aligned}
&- (k+1)\mathcal{L}_{k,2n} + 2\,k\mathcal{L}_{k,n+1} - k(k+1) \\
&= 2(k+1)\mathbb{Q}_{2n} + \mathcal{Q}\left(2k(k+1) - 2k\mathcal{L}_{k,n+1}\right) - 2k\mathbb{Q}_{n+1} \\
&- (k+1)(\mathcal{L}_{k,2n}+1) + 2k\mathcal{L}_{k,n+1}\right).
\end{aligned}
$$

Thus, the proof is completed. □

Theorem 6. *Let $\mathcal{L}_{k,n}$ be the generalized dual Leonardo quaternion number. For any integer $n \geq 0$, summation formulas as follows:*

$$\sum_{s=0}^{n} \mathbb{Q}_s = \mathbb{Q}_{n+2} - (k(n+1)-1)\mathcal{Q} + (-k-1)\,(\mathbf{i}+2\mathbf{j}+4\mathbf{k}), \tag{29}$$

$$\sum_{s=0}^{n} \mathbb{Q}_{2s} = \mathbb{Q}_{2n+1} - k\,n\,\mathcal{Q} - k\,(2\mathbf{i}+\mathbf{j}+2\mathbf{k}), \tag{30}$$

$$\sum_{s=0}^{n} \mathbb{Q}_{2n+1} = \mathbb{Q}_{2n+2} - n\mathcal{Q} - (2+2\mathbf{i}+4\mathbf{j}+6\mathbf{k}), \tag{31}$$

$$(\sum_{s=0}^{p} \mathbb{Q}_{n+s}) + \mathbb{Q}_{n+1} = \mathbb{Q}_{n+p+2} - k\,(p+1)\mathcal{Q}. \tag{32}$$

Proof. (29): From Eq.(11) and $\sum_{i=0}^{n} \mathcal{L}_{k,i} = \mathcal{L}_{k,n+2} - k(n+1) - 1$ [7], we get

$$
\begin{aligned}
\sum_{s=0}^{n} \mathbb{Q}_s &= \sum_{s=0}^{n} (\mathcal{L}_{k,s} + \mathbf{i}\mathcal{L}_{k,s+1} + \mathbf{j}\mathcal{L}_{k,s+2} + \mathbf{k}\mathcal{L}_{k,s+3}) \\
&= \sum_{s=0}^{n} \mathcal{L}_{k,s} + \mathbf{i}\,(\mathcal{L}_{k,n+1} - \mathcal{L}_{k,0} + \sum_{s=0}^{n} \mathcal{L}_{k,s}) \\
&\quad + \mathbf{j}(\mathcal{L}_{k,n+2} + \mathcal{L}_{k,n+1} - \mathcal{L}_{k,0} - \mathcal{L}_{k,1} + \sum_{s=0}^{n} \mathcal{L}_{k,s}) \\
&\quad + \mathbf{k}(\mathcal{L}_{k,n+3} + \mathcal{L}_{k,n+2} + \mathcal{L}_{k,n+1} - \mathcal{L}_{k,0} - \mathcal{L}_{k,1} - \mathcal{L}_{k,2} + \sum_{s=0}^{n} \mathcal{L}_{k,s}) \\
&= (\mathcal{L}_{k,n+2} - k(n+1) - 1) + \mathbf{i}\,(\mathcal{L}_{k,n+1} + \mathcal{L}_{k,n+2} - k(n+1) - 1 - \mathcal{L}_{k,0}) \\
&\quad + \mathbf{j}\,(2\mathcal{L}_{k,n+2} + \mathcal{L}_{k,n+1} - k(n+1) - 1 - \mathcal{L}_{k,0} - \mathcal{L}_{k,1}) \\
&\quad + \mathbf{k}\,(\mathcal{L}_{k,n+3} + 2\mathcal{L}_{k,n+2} + \mathcal{L}_{k,n+1} - k(n+1) - 1 - \mathcal{L}_{k,0} - \mathcal{L}_{k,1} - \mathcal{L}_{k,2}) \\
&= (\mathcal{L}_{k,n+2} + \mathbf{i}\,\mathcal{L}_{k,n+3} + \mathbf{j}\,\mathcal{L}_{k,n+4} + \mathbf{k}\,\mathcal{L}_{k,n+5}) \\
&\quad + [-k(n+1) - 1](1 + \mathbf{i} + \mathbf{j} + \mathbf{k}) - \mathcal{L}_{k,0}(\mathbf{i}+\mathbf{j}+\mathbf{k}) - \mathcal{L}_{k,1}(\mathbf{j}+\mathbf{k}) \\
&\quad - \mathcal{L}_{k,2}(\mathbf{k}) - k(\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}) \\
&= \mathbb{Q}_{n+2} + (-k(n+1) - 1)\mathcal{Q} + (-k-1)(\mathbf{i} + 2\,\mathbf{j} + 4\,\mathbf{k})\,.
\end{aligned}
$$

(32): Using Eq.(29), we obtain

$$\begin{aligned}\sum_{s=0}^{p}\mathbb{Q}_{n+s} =& \sum_{r=0}^{n+p}\mathbb{Q}_r - \sum_{r=0}^{n-1}\mathbb{Q}_r \\ =& \mathbb{Q}_{n+p+2} - \mathbb{Q}_{n+1} - k\,(p+1)(1+\mathbf{i}+\mathbf{j}+\mathbf{k})\,.\end{aligned}$$

Where,

$$\begin{aligned}\sum_{r=0}^{n+p}\mathbb{Q}_r &= \mathbb{Q}_{n+p+2} + (-k\,(n+p+1)-1)\,\mathcal{Q} + (-k-1)\,(\mathbf{i}+2\,\mathbf{j}+4\,\mathbf{k}),\\ \sum_{r=0}^{n-1}\mathbb{Q}_r &= \mathbb{Q}_{n+1} + (-k\,(n)-1)\,\mathcal{Q} + (-k-1)\,(\mathbf{i}+2\,\mathbf{j}+4\,\mathbf{k}).\end{aligned}$$

Proof of other equalities can easily be done using Eq.(4) and Eq.(5) and Eq.(6). □

Theorem 7 (Binet's Formula)**.** *Let $\mathbb{Q}_n$ be the generalized dual Leonardo quaternion number. For any integer $n \geq 0$, the Binet's formula for these quaternions is as follows:*

$$\mathbb{Q}_n = (k+1)\Big(\frac{\hat{\alpha}\;\alpha^{n+1} - \hat{\beta}\;\;\beta^{n+1}}{\alpha-\beta}\Big) - k\mathcal{Q}\,. \tag{33}$$

Where,

$$\hat{\alpha} = 1+\mathbf{i}\,\alpha+\mathbf{j}\,\alpha^2+\mathbf{k}\,\alpha^3,\quad \alpha = \tfrac{1+\sqrt{5}}{2}$$

$$\hat{\beta} = 1+\mathbf{i}\,\beta+\mathbf{j}\,\beta^2+\mathbf{k}\,\beta^3,\quad \beta = \tfrac{1-\sqrt{5}}{2}.$$

Proof. Using Eq.(2) and the Binet's formula of Fibonacci number we obtain that,

$$\begin{aligned}\mathbb{Q}_n =\;& \mathcal{L}_{k,n} + \mathbf{i}\,\mathcal{L}_{k,n+1} + \mathbf{j}\,\mathcal{L}_{k,n+2} + \mathbf{k}\,\mathcal{L}_{k,n+3}\\ =\;& \big(\tfrac{(k+1)\,\alpha^{n+1}-(k+1)\beta^{n+1}}{\alpha-\beta} - k\big) + \mathbf{i}\,\big(\tfrac{(k+1)\,\alpha^{n+2}-(k+1)\beta^{n+2}}{\alpha-\beta} - k\big)\\ &+\mathbf{j}\,\big(\tfrac{(k+1)\,\alpha^{n+3}-(k+1)\beta^{n+3}}{\alpha-\beta} - k\big) + \mathbf{k}\,\big(\tfrac{(k+1)\,\alpha^{n+4}-(k+1)\beta^{n+4}}{\alpha-\beta} - k\big)\\ =\;& \tfrac{(k+1)\,\alpha^{n+1}}{\alpha-\beta}(1+\mathbf{i}\,\alpha+\mathbf{j}\,\alpha^2+\mathbf{k}\,\alpha^3)\\ &-\tfrac{(k+1)\,\beta^{n+1}}{\alpha-\beta}(1+\mathbf{i}\,\beta+\mathbf{j}\,\beta^2+\mathbf{k}\,\beta^3) - k(1+\mathbf{i}+\mathbf{j}+\mathbf{k})\\ =\;& \tfrac{(k+1)\,\hat{\alpha}\;\alpha^{n+1}-(k+1)\,\hat{\beta}\;\beta^{n+1}}{\alpha-\beta} - k\,\mathcal{Q}\,.\end{aligned}$$

□

Theorem 8 (Generating function)**.** *Let* $\mathbb{Q}_n$ *be generalized dual Leonardo quaternion numbers. For the generating function for these numbers is as follows:*

$$g_{\mathbb{Q}_n}(t) = \sum_{n=0}^{\infty} \mathbb{Q}_n\, t^n = \frac{\mathbb{Q}_0+(\mathbb{Q}_1-2\,\mathbb{Q}_0)\,t+(\mathbb{Q}_2-2\,\mathbb{Q}_1)\,t^2}{1-2\,t+t^3}$$
$$= \frac{(1-t+k\,t^2)+\mathbf{i}\,(1+k\,t-t^2)+\mathbf{j}\,[\,(2+k)-t-t^2\,]+\mathbf{k}\,[\,(3+2k)-t-(2+k)\,t^2\,]}{1-2\,t+t^3}. \tag{34}$$

Where, $\mathbb{Q}_0 = 1+\mathbf{i}+(2+k)\,\mathbf{j}+(3+2k)\,\mathbf{k}$, $\mathbb{Q}_1 = 1+\mathbf{i}(2+k)+\mathbf{j}(3+2k)+\mathbf{k}(5+4k)$ *and* $\mathbb{Q}_2 = (2+k)+\mathbf{i}(3+2k)+\mathbf{j}(5+4k)+\mathbf{k}(8+7k)$.

Proof. (34):Using the definition of generating function, we obtain

$$g_{\mathbb{Q}_n}(t) = \mathbb{Q}_0 + \mathbb{Q}_1\,t + \cdots + \mathbb{Q}_n\,t^n + \cdots . \tag{35}$$

Multiplying $(1-2\,t+t^3)$ both sides of Eq.(35) and using recurrence relation Eq.(21), we have

$$\begin{aligned}(1-2\,t+t^3)\,g_{\mathbb{Q}_n}(t) = {} & \mathbb{Q}_0 + (\mathbb{Q}_1 - 2\,\mathbb{Q}_0)\,t \\ & + (\mathbb{Q}_2 - 2\,\mathbb{Q}_1)\,t^2 \\ & + (\mathbb{Q}_3 - 2\,\mathbb{Q}_2 + \mathbb{Q}_0)\,t^3 + \cdots \\ & + (\mathbb{Q}_{k+1} - 2\,\mathbb{Q}_k + \mathbb{Q}_{k-2})\,t^{k+1} + \cdots\end{aligned}$$

where,

$$\begin{aligned}\mathbb{Q}_1 - 2\,\mathbb{Q}_0 &= -1 + k\,\mathbf{i} - \mathbf{j} - \mathbf{k} \\ \mathbb{Q}_2 - 2\,\mathbb{Q}_1 &= k - \mathbf{i} - \mathbf{j} + (-2-k)\,\mathbf{k},\end{aligned}$$

and,

$$\mathbb{Q}_3 - 2\,\mathbb{Q}_2 + \mathbb{Q}_0 = 0 \cdots = 0.$$

Thus, the proof is completed. □

Theorem 9 (Cassini's Identity)**.** *Let* $\mathbb{Q}_n$ *be the generalized dual Leonardo quaternion number. For* $n \geq 1$*, the following equality holds:*

$$\begin{aligned}\mathbb{Q}_{n-1}\,\mathbb{Q}_{n+1} - \mathbb{Q}_n^2 = {} & (k+1)^2(-1)^{n+1}\,(1+\mathbf{i}+3\,\mathbf{j}+4\,\mathbf{k}) \\ & + k(1+\mathbf{i}+\mathbf{j}+\mathbf{k})(\mathcal{L}_{k,n-2}) \\ & + \mathbf{j}(k\,\mathcal{L}_{k,n-2}+k) + \mathbf{k}(k\,\mathcal{L}_{k,n}+k^2) - k\,\mathbb{Q}_{n-1}\,.\end{aligned} \tag{36}$$

Proof. (36): Using Eq.(11) and Eq.(3), we obtain that

$$\begin{aligned}\mathbb{Q}_{n-1}\,\mathbb{Q}_{n+1} - \mathbb{Q}_n\,\mathbb{Q}_n = {} & (\mathcal{L}_{k,n-1}\,\mathcal{L}_{k,n+1} - \mathcal{L}_{k,n}\,\mathcal{L}_{k,n}) \\ & + \mathbf{i}(\mathcal{L}_{k,n-1}\mathcal{L}_{k,n+2} - \mathcal{L}_{k,n}\mathcal{L}_{k,n+1}) \\ & + \mathbf{j}\,[(\mathcal{L}_{k,n-1}\,\mathcal{L}_{k,n+3} - \mathcal{L}_{k,n}\,\mathcal{L}_{k,n+2} \\ & + \mathcal{L}_{k,n+1}\,\mathcal{L}_{k,n+1} - \mathcal{L}_{k,n+2}\,\mathcal{L}_{k,n})]\end{aligned}$$

$$
\begin{aligned}
&+\mathbf{k}\,[(\mathcal{L}_{k,n+2}\,\mathcal{L}_{k,n+1}-\mathcal{L}_{k,n+3}\,\mathcal{L}_{k,n}\\
&+\mathcal{L}_{k,n-1}\,\mathcal{L}_{k,n+4}-\mathcal{L}_{k,n}\,\mathcal{L}_{k,n+3})]\\
=&\,[(k+1)^2\,(-1)^{n+1}-k\mathcal{L}_{k,n-1}+k\mathcal{L}_{k,n-2}]\\
&+\mathbf{i}\,[(k+1)^2\,(-1)^{n+1}-k\mathcal{L}_{k,n}+k\mathcal{L}_{k,n-2}]\\
&+\mathbf{j}\,[3\,(k+1)^2\,(-1)^{n+1}-k\mathcal{L}_{k,n+1}+k\mathcal{L}_{k,n-2}\\
&+k(\mathcal{L}_{k,n}-k\mathcal{L}_{k,n-1})]\\
&+\mathbf{k}\,[4\,(k+1)^2\,(-1)^{n+1}-k\mathcal{L}_{k,n+2}+k\mathcal{L}_{k,n-2}\\
&+k(\mathcal{L}_{k,n+1}-\mathcal{L}_{k,n-1})]\\
=&\,(k+1)^2(-1)^{n+1}\,(1+\mathbf{i}+3\,\mathbf{j}+4\,\mathbf{k})\\
&+k(1+\mathbf{i}+\mathbf{j}+\mathbf{k})(\mathcal{L}_{k,n-2})\\
&+\mathbf{j}(k\,\mathcal{L}_{k,n-2}+k^2)+\mathbf{k}(k\,\mathcal{L}_{k,n}+k^2)-k\,\mathbb{Q}_{n-1}\,.
\end{aligned}
$$

Where using Eq.(3), the following relations are obtained

$$
\begin{aligned}
k\,(\mathcal{L}_{k,n}-\mathcal{L}_{k,n-1})&=(k\,\mathcal{L}_{k,n-2}+k^2)\\
k\,(\mathcal{L}_{k,n+1}-\mathcal{L}_{k,n-1})&=(k\,\mathcal{L}_{k,n}+k^2).
\end{aligned}
$$

□

3 Conclusion

In this paper, we define the generalized dual Leonardo quaternion numbers and give some of their properties. Moreover, we investigate the relations between the dual quaternions and the generalized Leonardo numbers. Furthermore, we give the Binet formula and Cassini's identity for these quaternions.

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