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An introduction to bipolar fuzzy soft hypervector spaces

Omid Reza Dehghan[*]

Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, Bojnord, Iran
Email: dehghan@ub.ac.ir

Abstract. The purpose of this document is to present the concept of bipolar fuzzy soft hypervector spaces and explore their fundamental characteristics. To begin with, a new operation and an external hyperoperation are introduced for bipolar fuzzy soft sets on the hypervector space $\mathcal{V}$, which are connected to the operation and external hyperoperation of $\mathcal{V}$. Then the notion of bipolar fuzzy soft hypervector space is defined, supported by non-trivial examples, and it is investigated whether the new bipolar fuzzy soft sets, constructed by the mentioned operation and hyperoperation, are bipolar fuzzy soft hypervector spaces. Finally, the behavior of bipolar fuzzy soft hypervector spaces under linear transformations is investigated.

Keywords: Bipolar fuzzy set, Soft set, Bipolar fuzzy soft set, Bipolar fuzzy soft hypervector space.
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1 Introduction

Fuzzy set, was introduced by Zadeh [43] in 1965, is a mathematical tool for representing objects whose boundary is vague. In fuzzy sets, membership degrees indicate the degree of belonging of elements to the set or the degree of satisfaction of elements to the property related to the set.

There are several extensions for fuzzy sets; for example, fuzzy sets of type 2, $\mathcal{L}$-fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets. Each of these concepts, while having similarities and differences with the others, has its own uses. In 1994, Zhang [44] introduced the notion of bipolar fuzzy sets for cognitive modeling and multi-agent decision analysis as an expansion of traditional fuzzy sets. In bipolar fuzzy sets, the range of membership function values extends from $[0, 1]$ to $[-1, 0] \times [0, 1]$, allowing for representation of satisfaction towards a property and its counter-property through a pair of membership degrees. This extension shares some com-

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monalities with earlier models in terms of semantics and representations while also introducing unique characteristics.

The theory of soft sets, proposed by Molodtsov [32] in 1999, serves as another mathematical framework for handling uncertainty and modeling complex systems. It offers a different perspective compared to fuzzy sets and bipolar fuzzy sets, providing a versatile tool for addressing uncertainty in various domains. It's fascinating to see how the theory of soft sets has been extensively studied and applied in different branches, including algebraic structures. Researchers like Maji [29], Aktas [6], Acar [3], and Sezgin [39] have made significant contributions by exploring its applications in decision-making problems, soft groups, soft rings, and soft vector spaces, respectively. This demonstrates the versatility and relevance of the theory in diverse fields.

The evolution of soft set theory into more sophisticated variations like fuzzy soft sets and bipolar fuzzy soft sets has paved the way for a deeper understanding and more accurate modeling in various fields. Cagman's [14] introduction of fuzzy soft sets in 2011 added a new dimension to the theory, enabling more precise analysis. The subsequent development of bipolar fuzzy soft sets by Abdollah [1], which combines bipolar fuzzy sets and soft sets, has opened up avenues for advanced decision-making techniques. It's remarkable to see how researchers like Akram [4, 5], Ali [7], Abughazalah [2], Riaz [36], Mahmood [30], and Khan [28] have leveraged bipolar fuzzy soft sets in diverse applications ranging from K-algebras and disease diagnosis to BCI-algebras, topology, and decision-making problems. The wealth of research and the broad spectrum of applications reflect the growing interest and potential of bipolar fuzzy soft sets in solving complex real-world problems.

On the other hand, the development of algebraic hyperstructures marked a significant advancement in algebraic theory and operations. Marty's [31] introduction of hyperoperations in 1934 provided a broader framework by assigning unique subsets of a set to any pair of elements, expanding the traditional notion of operations. Researchers across various fields have delved into the study of algebraic hyperstructures, leading to a rich body of literature. Notable works by Corsini [15], Davvaz [17], and Vougiouklis [41] have contributed to the understanding and applications of these structures. In particular, the concept of hypervector spaces, pioneered by Scafati-Tallini [40] in 1990, has sparked significant interest and research. Scholars like Ameri [9], Sedghi [38], and the author [20,24,25] have further explored and investigated hypervector spaces, leading to valuable insights and advancements in the field.

The interplay between fuzzy sets, soft sets, fuzzy soft sets, and bipolar fuzzy soft sets has significantly influenced the study of algebraic hyperstructures (see the book [16]). Ameri's work [8] on fuzzy hypervector spaces over valued fields in 2005, as well as subsequent studies on their properties ([10–13]), have provided valuable insights into this field. The author continued this research, exploring further properties of fuzzy hypervector spaces and contributing to the understanding of these structures ([18, 19, 21, 22, 27]). Ranjbar's [35] investigation into soft hypervector spaces and fuzzy soft hypervector spaces, as well as Xin's [42] application of intuitionistic fuzzy soft sets to hyper BCK-algebras, have expanded the applications of these concepts in algebraic hyperstructures. The author [23, 26] investigated some results in soft hypervector spaces. Moreover, researchers like Norouzi [34], Sarwar [37], and Muhiuddin [33] have made notable contributions by introducing new directions and methods for studying soft hypermodules, soft fuzzy hypermodules, and bipolar fuzzy soft hypergraphs within the framework of algebraic hyperstructures. The cross-fertilization of ideas from fuzzy sets, soft sets, and related concepts with algebraic hyperstructures has led to a rich tapestry of research and advancements in the field.

Now in this paper, we apply the notion of bipolar fuzzy soft set in hypervector spaces and obtain some basic results which they will be a good basis for the future studies. More precisely, in Section 3, we define important operations on bipolar fuzzy soft sets, based on the operation and the external hyperoperation on the hypervector space $\mathcal{V}$. Then in Section 4, we define a bipolar fuzzy soft hypervector space, with some non-trivial interesting examples, and investigate that the combinations of bipolar fuzzy soft hypervector spaces under the presented operation, are bipolar fuzzy soft hypervector spaces. In fact, new bipolar fuzzy soft hypervector spaces are produced with this method. Moreover, in Section 5, we check out the behaviour of bipolar fuzzy soft hypervector spaces under linear transformations.

2 Preliminaries

In this part, we introduce certain explanations and instances that we will refer to in the future.

Let $\mathcal{U}$ be a non-empty universe set of discourse. In the context of bipolar fuzzy sets [44], a set

$$\mathcal{S} = \{(x, \mu^+_{\mathcal{S}}(x), \mu^-_{\mathcal{S}}(x)),\ x \in \mathcal{U}\},$$

defined on $\mathcal{U}$, where $\mu^+_{\mathcal{S}} : \mathcal{U} \to [0, 1]$ indicating partial satisfaction of a property, and $\mu^-_{\mathcal{S}} : \mathcal{U} \to [-1, 0]$ indicating partial satisfaction of the counter-property.

When $\mu^+_{\mathcal{S}}(x) \neq 0$ and $\mu^-_{\mathcal{S}}(x) = 0$, it implies that element x exhibits only positive satisfaction for $\mathcal{S}$. Conversely, when $\mu^+_{\mathcal{S}}(x) = 0$ and $\mu^-_{\mathcal{S}}(x) \neq 0$, it indicates that x does not satisfy the property of $\mathcal{S}$, but it satisfies the counter-property of $\mathcal{S}$.

To simplify, we can denote the above bipolar fuzzy set $\mathcal{S}$ as $(\mathcal{S}^+, \mathcal{S}^-)$.

For instance, the bipolar fuzzy set:

$$\mathcal{S} = \{(mosquito, 1, 0), (dragonfly, 0.4, 0), (turtle, 0, 0), (snake, 0, -1)\}$$

represents the fuzzy concept of "frog's prey".

In view of Molodsov [32], a soft set over $\mathcal{U}$ is defined as a pair $(\mathcal{F}, \mathcal{A})$, where $\mathcal{A}$ is a set of parameters and $\mathcal{F} : \mathcal{A} \to P(\mathcal{U})$ is a function that corresponds to each parameter $e \in \mathcal{A}$ a subset of $\mathcal{U}$.

A fuzzy soft set $(\mathcal{F}, \mathcal{A})$ over $\mathcal{U}$ is a mathematical framework that extends the concept of soft sets to handle precisely uncertainty and vagueness in data. Here the mapping $\mathcal{F} : \mathcal{A} \to FS(\mathcal{U})$ corresponds to each parameter $e \in \mathcal{A}$ a fuzzy subset $\mathcal{F}_e : \mathcal{U} \to [0, 1]$.

A bipolar fuzzy soft set, introduced by Abdullah [1], is an extension of a fuzzy soft set that incorporates the notion of bipolarity. In a bipolar fuzzy soft set $(\mathcal{F}, \mathcal{A})$, each element $e \in \mathcal{A}$ is associated with two real numbers to represent partial satisfaction of a property and partial satisfaction of the counter-property. This allows for a more flexible and expressive way to handle uncertainty and vagueness. More precisely, $\mathcal{F}$ is a mapping $\mathcal{F} : \mathcal{A} \to BF^{\mathcal{U}}$, where assigns to every element $e \in \mathcal{A}$, a bipolar fuzzy set over $\mathcal{U}$, i.e.

$$\forall e \in \mathcal{A};\ \mathcal{F}(e) = \mathcal{F}_e = \{(x, \mathcal{F}^+_e(x), \mathcal{F}^-_e(x)),\ x \in \mathcal{U}\}.$$

Definition 1 ([40])**.** *A hypervector space* $(\mathcal{V},+,\circ,\mathcal{K})$ *over the field* $\mathcal{K}$ *is an Abelian group* $(\mathcal{V},+)$ *with an external hyperoperation* "$\circ:\mathcal{K}\times\mathcal{V}\longrightarrow P_*(\mathcal{V})$", *where* $P_*(\mathcal{V})=\{A\subseteq\mathcal{V};\ A\neq\emptyset\}$ *and the followings hold:*

1. $a\circ(x+y)\subseteq a\circ x+a\circ y$,
2. $(a+b)\circ x\subseteq a\circ x+b\circ x$,
3. $a\circ(b\circ x)=(ab)\circ x$,
4. $a\circ(-x)=(-a)\circ x=-(a\circ x)$,
5. $x\in 1\circ x$,

for all $a,b\in\mathcal{K}$ *and* $x,y\in\mathcal{V}$. *Note that* $A+B=\{x+y:x\in A,y\in B\}$ *and* $a\circ A=\bigcup\limits_{x\in A}a\circ x$. *If* $a\circ(x+y)=a\circ x+a\circ y$, *then* $\mathcal{V}$ *is said to be strongly right distributive. Similarly, we can define the strongly left distributive hypervector spaces.*

In the continuation of this document, $\mathcal{V}$ represents a hypervector space over the field $\mathcal{K}$, unless stated otherwise.

Example 1 ([11])**.** $\mathcal{V}=(\mathbb{R}^3,+,\circ,\mathbb{R})$ is a strongly left distributive hypervector space over the field $\mathbb{R}$, where $a\circ(x_0,y_0,z_0)$ is a line with the parametric equations $x=ax_0$, $y=ay_0$, $z=t$.

Example 2. Let $\mathcal{K}=\mathbb{Z}_2=\{0,1\}$ be the field of two numbers with the following operations:

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

Then $(\mathbb{Z}_4,+,\circ,\mathbb{Z}_2)$ is a hypervector space, such that it is not strongly left or strongly right distributive, where the operation "$+:\mathbb{Z}_4\times\mathbb{Z}_4\to\mathbb{Z}_4$" and the external hyperoperation "$\circ:\mathbb{Z}_2\times\mathbb{Z}_4\to P_*(\mathbb{Z}_4)$" are defined as follow:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

∘	0	1	2	3
0	$\{0,2\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0,2\}$	$\{1,2,3\}$	$\{0,2\}$	$\{1,2,3\}$

3 Operations on bipolar fuzzy soft sets

One of the first topics that is considered in the study of a set, from an algebraic point of view, is the definition of different operations on that set, in order to identify and study the created algebraic structure. In this section, this basic issue is discussed and some related properties are given.

Some operations have been defined on bipolar fuzzy soft sets by some authors. Here, at first we recall the definitions were defined by Akram [4], and then define new operations on bipolar fuzzy soft sets of hypervector spaces, based on their operation and external hyperoperation.

Let $(\mathcal{F},\mathcal{A})$ and $(\mathcal{G},\mathcal{B})$ be bipolar fuzzy soft sets over $\mathcal{U}$. Then

1. $(\mathcal{F},\mathcal{A}) \sqsubseteq (\mathcal{G},\mathcal{B})$, if $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{F}_e^+(x) \leq \mathcal{G}_e^+(x)$ and $\mathcal{F}_e^-(x) \geq \mathcal{G}_e^-(x)$, $\forall e \in \mathcal{A}$, $x \in \mathcal{U}$.
2. $(\mathcal{F},\mathcal{A}) \sqcap (\mathcal{G},\mathcal{B}) = (\mathcal{F} \sqcap \mathcal{G}, \mathcal{A} \cap \mathcal{B})$, where $(\mathcal{F} \sqcap \mathcal{G})_e^+(x) = \mathcal{F}_e^+(x) \wedge \mathcal{G}_e^+(x)$ and $(\mathcal{F} \sqcap \mathcal{G})_e^-(x) = \mathcal{F}_e^-(x) \vee \mathcal{G}_e^-(x)$, $\forall e \in \mathcal{A} \cap \mathcal{B}$, $x \in \mathcal{U}$.
3. $(\mathcal{F},\mathcal{A}) \sqcap_\varepsilon (\mathcal{G},\mathcal{B}) = (\mathcal{F} \sqcap_\varepsilon \mathcal{G}, \mathcal{A} \cup \mathcal{B})$, where $(\mathcal{F} \sqcap_\varepsilon \mathcal{G})(e) = \mathcal{F}(e)$, for all $e \in \mathcal{A} \setminus \mathcal{B}$, $(\mathcal{F} \sqcap_\varepsilon \mathcal{G})(e) = \mathcal{G}(e)$, for all $e \in \mathcal{B} \setminus \mathcal{A}$, and $(\mathcal{F} \sqcap_\varepsilon \mathcal{G})_e^+(x) = \mathcal{F}_e^+(x) \wedge \mathcal{G}_e^+(x)$, $(\mathcal{F} \sqcap_\varepsilon \mathcal{G})_e^-(x) = \mathcal{F}_e^-(x) \vee \mathcal{G}_e^-(x)$, for all $e \in \mathcal{A} \cap \mathcal{B}$.
4. $(\mathcal{F},\mathcal{A}) \sqcup (\mathcal{G},\mathcal{B}) = (\mathcal{F} \sqcup \mathcal{G}, \mathcal{A} \cup \mathcal{B})$, where $(\mathcal{F} \sqcup \mathcal{G})(e) = \mathcal{F}(e)$, for all $e \in \mathcal{A} \setminus \mathcal{B}$, $(\mathcal{F} \sqcup \mathcal{G})(e) = \mathcal{G}(e)$, for all $e \in \mathcal{B} \setminus \mathcal{A}$, $(\mathcal{F} \sqcup \mathcal{G})_e^+(x) = \mathcal{F}_e^+(x) \vee \mathcal{G}_e^+(x)$ and $(\mathcal{F} \sqcup \mathcal{G})_e^-(x) = \mathcal{F}_e^-(x) \wedge \mathcal{G}_e^-(x)$, for all $e \in \mathcal{A} \cap \mathcal{B}$.
5. $(\mathcal{F},\mathcal{A}) \sqcup_R (\mathcal{G},\mathcal{B}) = (\mathcal{F} \sqcup_R \mathcal{G}, \mathcal{A} \cap \mathcal{B})$, where $(\mathcal{F} \sqcup_R \mathcal{G})_e^+(x) = \mathcal{F}_e^+(x) \vee \mathcal{G}_e^+(x)$ and $(\mathcal{F} \sqcup_R \mathcal{G})_e^-(x) = \mathcal{F}_e^-(x) \wedge \mathcal{G}_e^-(x)$, $\forall e \in \mathcal{A} \cap \mathcal{B}$.
6. $(\mathcal{F},\mathcal{A}) \wedge (\mathcal{G},\mathcal{B}) = (\mathcal{F} \wedge \mathcal{G}, \mathcal{A} \times \mathcal{B})$, where $(\mathcal{F} \wedge \mathcal{G})_{(e_1,e_2)}^+(x) = \mathcal{F}_{e_1}^+(x) \wedge \mathcal{G}_{e_2}^+(x)$, $(\mathcal{F} \wedge \mathcal{G})_{(e_1,e_2)}^-(x) = \mathcal{F}_{e_1}^-(x) \vee \mathcal{G}_{e_2}^-(x)$, $\forall (e_1, e_2) \in \mathcal{A} \times \mathcal{B}$.
7. $(\mathcal{F},\mathcal{A}) \vee (\mathcal{G},\mathcal{B}) = (\mathcal{F} \vee \mathcal{G}, \mathcal{A} \times \mathcal{B})$, where $(\mathcal{F} \vee \mathcal{G})_{(e_1,e_2)}^+(x) = \mathcal{F}_{e_1}^+(x) \vee \mathcal{G}_{e_2}^+(x)$, $(\mathcal{F} \vee \mathcal{G})_{(e_1,e_2)}^-(x) = \mathcal{F}_{e_1}^-(x) \wedge \mathcal{G}_{e_2}^-(x)$, $\forall (e_1, e_2) \in \mathcal{A} \times \mathcal{B}$.

Corresponding to every (hyper) operation over an algebraic (hyper) structure $\mathcal{U}$, one can define an operation over bipolar fuzzy soft sets over $\mathcal{U}$. In this paper, we study this idea in hypervector spaces.

Definition 2. *Let $(\mathcal{F},\mathcal{A})$ and $(\mathcal{G},\mathcal{B})$ represent bipolar fuzzy soft sets defined over a hypervector space $\mathcal{V} = (\mathcal{V}, +, \circ, \mathcal{K})$. Then*

1. *$(\mathcal{F},\mathcal{A}) + (\mathcal{G},\mathcal{B}) = (\mathcal{F} + \mathcal{G}, \mathcal{A} \cap \mathcal{B})$ is called the sum of $(\mathcal{F},\mathcal{A})$ and $(\mathcal{G},\mathcal{B})$, where*

$$(\mathcal{F} + \mathcal{G})_e^+(x) = \sup_{x=y+z} (\mathcal{F}_e^+(y) \wedge \mathcal{G}_e^+(z)),$$

$$(\mathcal{F} + \mathcal{G})_e^-(x) = \inf_{x=y+z} (\mathcal{F}_e^-(y) \vee \mathcal{G}_e^-(z)),$$

for all $e \in \mathcal{A} \cap \mathcal{B}$ and $x \in \mathcal{V}$.

2. *$(\mathcal{F},\mathcal{A}) +_\varepsilon (\mathcal{G},\mathcal{B}) = (\mathcal{F} +_\varepsilon \mathcal{G}, \mathcal{A} \cup \mathcal{B})$ is called the extended sum of $(\mathcal{F},\mathcal{A})$ and $(\mathcal{G},\mathcal{B})$, where*

$$(\mathcal{F} +_\varepsilon \mathcal{G})_e^+(x) = \begin{cases} \mathcal{F}_e^+(x) & x \in \mathcal{A} \setminus \mathcal{B}, \\ \mathcal{G}_e^+(x) & x \in \mathcal{B} \setminus \mathcal{A}, \\ (\mathcal{F} + \mathcal{G})_e^+(x) & x \in \mathcal{A} \cap \mathcal{B}, \end{cases}$$

and

$$(\mathcal{F} +_\varepsilon \mathcal{G})_e^-(x) = \begin{cases} \mathcal{F}_e^-(x) & x \in \mathcal{A} \setminus \mathcal{B}, \\ \mathcal{G}_e^-(x) & x \in \mathcal{B} \setminus \mathcal{A}, \\ (\mathcal{F} + \mathcal{G})_e^-(x) & x \in \mathcal{A} \cap \mathcal{B}, \end{cases}$$

for all $e \in \mathcal{A} \cup \mathcal{B}$ and $x \in \mathcal{V}$.

3. *for every $a \in \mathcal{K}$, the scalar product $a \circ (\mathcal{F}, \mathcal{A}) = (a \circ \mathcal{F}, \mathcal{A})$ is defined by*

$$(a \circ \mathcal{F})^+_e(x) = \begin{cases} \sup_{x \in a \circ t} \mathcal{F}^+_e(t) & \exists t \in \mathcal{V}, x \in a \circ t, \\ 0 & otherwise, \end{cases}$$

and

$$(a \circ \mathcal{F})^-_e(x) = \begin{cases} \inf_{x \in a \circ t} \mathcal{F}^-_e(t) & \exists t \in \mathcal{V}, x \in a \circ t, \\ 0 & otherwise. \end{cases}$$

Proposition 1. *For any bipolar fuzzy soft sets $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ defined on a hypervector space $\mathcal{V} = (\mathcal{V}, +, \circ, \mathcal{K})$, and for all elements $x, y \in \mathcal{V}$ and $e \in \mathcal{A} \cap \mathcal{B}$, the following inequalities hold:*

$$(\mathcal{F} + \mathcal{G})^+_e(x + y) \geq \mathcal{F}^+_e(x) \wedge \mathcal{G}^+_e(y),$$

$$(\mathcal{F} + \mathcal{G})^-_e(x + y) \leq \mathcal{F}^-_e(x) \vee \mathcal{G}^-_e(y).$$

Proof. By Definition 2, it follows that:

$$(\mathcal{F} + \mathcal{G})^+_e(x + y) = \sup_{x+y=t_1+t_2} (\mathcal{F}^+_e(t_1) \wedge \mathcal{G}^+_e(t_2)) \geq \mathcal{F}^+_e(x) \wedge \mathcal{G}^+_e(y),$$

and

$$(\mathcal{F} + \mathcal{G})^-_e(x + y) = \inf_{x+y=s_1+s_2} (\mathcal{F}^-_e(s_1) \vee \mathcal{G}^-_e(s_2)) \leq \mathcal{F}^-_e(x) \vee \mathcal{G}^-_e(y).$$

□

Proposition 2. *If a bipolar fuzzy soft set $(\mathcal{F}, \mathcal{A})$ is defined on the hypervector space $\mathcal{V} = (\mathcal{V}, +, \circ, \mathcal{K})$, the subsequent assertions are valid:*

1. $(\mathcal{F}, \mathcal{A}) \sqsubseteq 1 \circ (\mathcal{F}, \mathcal{A})$,
2. $-(\mathcal{F}, \mathcal{A}) \sqsubseteq (-1) \circ (\mathcal{F}, \mathcal{A})$, *where* $(-\mathcal{F})^+_e(x) = \mathcal{F}^+_e(-x)$ *and* $(-\mathcal{F})^-_e(x) = \mathcal{F}^-_e(-x)$, $\forall e \in \mathcal{A}$, $x \in \mathcal{V}$.

Proof. By Definition 1, $x \in 1 \circ x$ and $-x \in 1 \circ (-x) = (-1) \circ x$. Then by Definition 2, it follows that:

$$(1 \circ \mathcal{F})^+_e(x) = \sup_{x \in 1 \circ t} \mathcal{F}^+_e(t) \geq \mathcal{F}^+_e(x),$$

$$(1 \circ \mathcal{F})^-_e(x) = \inf_{x \in 1 \circ t} \mathcal{F}^-_e(t) \leq \mathcal{F}^-_e(x),$$

$$((-1) \circ \mathcal{F})^+_e(x) = \sup_{x \in (-1) \circ t} \mathcal{F}^+_e(t) \geq \mathcal{F}^+_e(-x) = (-\mathcal{F})^+_e(x),$$

and

$$((-1) \circ \mathcal{F})^-_e(x) = \inf_{x \in (-1) \circ t} \mathcal{F}^-_e(t) \leq \mathcal{F}^-_e(-x) = (-\mathcal{F})^-_e(x).$$

□

Proposition 3. *Let $\{(\mathcal{F}_i, \mathcal{A})\}_{i \in I}$, $\{(\mathcal{G}_j, \mathcal{A})\}_{j \in J}$ be families of bipolar fuzzy soft sets over the hypervector space $\mathcal{V}$. Then the followings are valid:*

1. $\left(\bigsqcup_{i\in I}(\mathcal{F}_i,\mathcal{A})\right)+\left(\bigsqcup_{j\in J}(\mathcal{G}_j,\mathcal{A})\right)=\bigsqcup_{i\in I,j\in J}((\mathcal{F}_i,\mathcal{A})+(\mathcal{G}_j,\mathcal{A}))$.

2. *for all* $a\in\mathcal{K}$, $a\circ\left(\bigsqcup_{i\in I}(\mathcal{F}_i,\mathcal{A})\right)=\bigsqcup_{i\in I}(a\circ(\mathcal{F}_i,\mathcal{A}))$.

Proof. 1) Given that $e\in\mathcal{A}$ and $x\in\mathcal{V}$. Then

$$\begin{aligned}
&\left(\left(\bigsqcup_{i\in I}(\mathcal{F}_i,\mathcal{A})\right)+\left(\bigsqcup_{j\in J}(\mathcal{G}_j,\mathcal{A})\right)\right)^+_e(x)\\
=\ &\sup_{x=y+z}\left(\left(\bigsqcup_{i\in I}(\mathcal{F}_i,\mathcal{A})\right)^+_e(y)\wedge\left(\bigsqcup_{j\in J}(\mathcal{G}_j,\mathcal{A})\right)^+_e(z)\right)\\
=\ &\sup_{x=y+z}\left(\left(\sup_{i\in I}(\mathcal{F}_i)^+_e(y)\right)\wedge\left(\sup_{j\in J}(\mathcal{G}_j)^+_e(z)\right)\right)\\
=\ &\sup_{x=y+z}\ \sup_{i\in I,j\in J}\big((\mathcal{F}_i)^+_e(y)\wedge(\mathcal{G}_j)^+_e(z)\big)\\
=\ &\sup_{i\in I,j\in J}\ \sup_{x=y+z}\big((\mathcal{F}_i)^+_e(y)\wedge(\mathcal{G}_j)^+_e(z)\big)\\
=\ &\sup_{i\in I,j\in J}((\mathcal{F}_i,\mathcal{A})+(\mathcal{G}_j,\mathcal{A}))^+_e(x)\\
=\ &\left(\bigsqcup_{i\in I,j\in J}((\mathcal{F}_i,\mathcal{A})+(\mathcal{G}_j,\mathcal{A}))\right)^+_e(x),
\end{aligned}$$

and

$$\begin{aligned}
&\left(\left(\bigsqcup_{i\in I}(\mathcal{F}_i,\mathcal{A})\right)+\left(\bigsqcup_{j\in J}(\mathcal{G}_j,\mathcal{A})\right)\right)^-_e(x)\\
=\ &\inf_{x=y+z}\left(\left(\bigsqcup_{i\in I}(\mathcal{F}_i,\mathcal{A})\right)^-_e(y)\vee\left(\bigsqcup_{j\in J}(\mathcal{G}_j,\mathcal{A})\right)^-_e(z)\right)\\
=\ &\inf_{x=y+z}\left(\left(\sup_{i\in I}(\mathcal{F}_i)^-_e(y)\right)\vee\left(\sup_{j\in J}(\mathcal{G}_j)^-_e(z)\right)\right)\\
=\ &\inf_{x=y+z}\ \inf_{i\in I,j\in J}\big((\mathcal{F}_i)^-_e(y)\vee(\mathcal{G}_j)^-_e(z)\big)\\
=\ &\inf_{i\in I,j\in J}\ \inf_{x=y+z}\big((\mathcal{F}_i)^-_e(y)\vee(\mathcal{G}_j)^-_e(z)\big)\\
=\ &\inf_{i\in I,j\in J}((\mathcal{F}_i,\mathcal{A})+(\mathcal{G}_j,\mathcal{A}))^-_e(x)\\
=\ &\left(\bigsqcup_{i\in I,j\in J}((\mathcal{F}_i,\mathcal{A})+(\mathcal{G}_j,\mathcal{A}))\right)^-_e(x).
\end{aligned}$$

2) Let $e \in \mathcal{A}$ and $x \in \mathcal{V}$. Then

$$\begin{aligned}
&\left(a \circ \left(\bigsqcup_{i\in I}(\mathcal{F}_i, \mathcal{A})\right)\right)_e^+ (x) \\
= & \begin{cases} \sup\limits_{x\in a\circ t} \left(\bigsqcup\limits_{i\in I}(\mathcal{F}_i, \mathcal{A})\right)_e^+ (t) & \exists t \in \mathcal{V}, x \in a \circ t, \\ 0 & otherwise, \end{cases} \\
= & \begin{cases} \sup\limits_{x\in a\circ t} \left(\sup\limits_{i\in I}(\mathcal{F}_i)_e^+ (t)\right) & \exists t \in \mathcal{V}, x \in a \circ t, \\ 0 & otherwise, \end{cases} \\
= & \begin{cases} \sup\limits_{i\in I} \left(\sup\limits_{x\in a\circ t}(\mathcal{F}_i)_e^+ (t)\right) & \exists t \in \mathcal{V}, x \in a \circ t, \\ 0 & otherwise, \end{cases} \\
= & \sup_{i\in I}(a \circ (\mathcal{F}_i, \mathcal{A}))_e^+ (x) \\
= & \left(\bigsqcup_{i\in I}(a \circ (\mathcal{F}_i, \mathcal{A}))\right)_e^+ (x),
\end{aligned}$$

and

$$\begin{aligned}
&\left(a \circ \left(\bigsqcup_{i\in I}(\mathcal{F}_i, \mathcal{A})\right)\right)_e^- (x) \\
= & \begin{cases} \inf\limits_{x\in a\circ t} \left(\bigsqcup\limits_{i\in I}(\mathcal{F}_i, \mathcal{A})\right)_e^- (t) & \exists t \in \mathcal{V}, x \in a \circ t, \\ 0 & otherwise, \end{cases} \\
= & \begin{cases} \inf\limits_{x\in a\circ t} \left(\inf\limits_{i\in I}(\mathcal{F}_i)_e^- (t)\right) & \exists t \in \mathcal{V}, x \in a \circ t, \\ 0 & otherwise, \end{cases} \\
= & \begin{cases} \inf\limits_{i\in I} \left(\inf\limits_{x\in a\circ t}(\mathcal{F}_i)_e^- (t)\right) & \exists t \in \mathcal{V}, x \in a \circ t, \\ 0 & otherwise, \end{cases} \\
= & \inf_{i\in I}(a \circ (\mathcal{F}_i, \mathcal{A}))_e^- (x) \\
= & \left(\bigsqcup_{i\in I}(a \circ (\mathcal{F}_i, \mathcal{A}))\right)_e^- (x).
\end{aligned}$$

□

4 Bipolar fuzzy soft hypervector spaces

In this part, the concept of a bipolar fuzzy soft hypervector space is introduced, which is constructed using particular bipolar fuzzy subsets of a hypervector space $\mathcal{V} = (\mathcal{V}, +, \circ, \mathcal{K})$, supported by some non-trivial examples. Moreover, it will be shown that intersection, extended intersection, union, restricted union, AND, sum, extended sum and scalar product of bipolar fuzzy soft hypervector spaces are bipolar fuzzy soft hypervector spaces, too. In fact, by the mentioned operations, new bipolar fuzzy soft hypervector spaces are constructed.

Definition 3. *Suppose $\mathcal{V}$ is a hypervector space defined over the field $\mathcal{K}$. A bipolar fuzzy set $\mathcal{A} = (\mathcal{A}^+, \mathcal{A}^-)$ in $\mathcal{V}$ is termed a bipolar fuzzy subhyperspace of $\mathcal{V}$, if the following conditions are satisfied for all $x, y \in \mathcal{V}$ and $a \in \mathcal{K}$:*

1. $\mathcal{A}^+(x-y) \geq \min\{\mathcal{A}^+(x), \mathcal{A}^+(y)\}$, $\mathcal{A}^-(x-y) \leq \max\{\mathcal{A}^-(x), \mathcal{A}^-(y)\}$,
2. $\inf\limits_{t \in a \circ x} \mathcal{A}^+(t) \geq \mathcal{A}^+(x)$, $\sup\limits_{t \in a \circ x} \mathcal{A}^-(t) \leq \mathcal{A}^-(x)$.

Example 3. In the given hypervector space $\mathcal{V} = (\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 1, we can define a bipolar fuzzy subhyperspace $\mathcal{A} = (\mathcal{A}^+, \mathcal{A}^-)$ by specifying "$\mathcal{A}^+ : \mathbb{R}^3 \to [0,1]$" and "$\mathcal{A}^- : \mathbb{R}^3 \to [-1,0]$" with

$$\mathcal{A}^+(x,y,z) = \begin{cases} t_1 & (x,y,z) \in \mathcal{X}, \\ t_2 & (x,y,z) \in \mathcal{Y}, \\ t_3 & \textit{otherwise}, \end{cases}$$

and

$$\mathcal{A}^-(x,y,z) = \begin{cases} s_1 & (x,y,z) \in \mathcal{X}, \\ s_2 & (x,y,z) \in \mathcal{Y}, \\ s_3 & \textit{otherwise}, \end{cases}$$

where $\mathcal{X} = \{0\} \times \{0\} \times \mathbb{R}$, $\mathcal{Y} = (\mathbb{R} \times \{0\} \times \mathbb{R}) \setminus (\{0\} \times \{0\} \times \mathbb{R})$ and $-1 \leq s_1 < s_2 < s_3 \leq 0 \leq t_3 < t_2 < t_1 \leq 1$.

Example 4. To define a bipolar fuzzy subhyperspace $\mathcal{B} = (\mathcal{B}^+, \mathcal{B}^-)$ in the hypervector space $(\mathbb{Z}_4, +, \circ, \mathbb{Z}_2)$ in Example 2, we specify the functions "$\mathcal{B}^+ : \mathbb{Z}_4 \to [0,1]$" and "$\mathcal{B}^- : \mathbb{Z}_4 \to [-1,0]$" as given:

$$\mathcal{B}^+(x) = \begin{cases} t_1 & x \in \{0,2\}, \\ t_2 & x \in \{1,3\}, \end{cases} \qquad \mathcal{B}^-(x) = \begin{cases} s_1 & x \in \{0,2\}, \\ s_2 & x \in \{1,3\}, \end{cases}$$

where $-1 \leq s_1 < s_2 \leq 0 \leq t_2 < t_1 \leq 1$.

Definition 4. *Let $(\mathcal{F}, \mathcal{A})$ be a bipolar fuzzy soft set of a hypervector space $\mathcal{V} = (\mathcal{V}, +, \circ, \mathcal{K})$. The pair $(\mathcal{F}, \mathcal{A})$ is considered a bipolar fuzzy soft hypervector space of $\mathcal{V}$, if $\mathcal{F}(e)$ is a bipolar fuzzy subhyperspace of $\mathcal{V}$, for every $e \in \mathcal{A}$, i.e.*

1. $\mathcal{F}_e^+(x-y) \geq \min\{\mathcal{F}_e^+(x), \mathcal{F}_e^+(y)\}$, $\mathcal{F}_e^-(x-y) \leq \max\{\mathcal{F}_e^-(x), \mathcal{F}_e^-(y)\}$,
2. $\inf\limits_{t \in a \circ x} \mathcal{F}_e^+(t) \geq \mathcal{F}_e^+(x)$, $\sup\limits_{t \in a \circ x} \mathcal{F}_e^-(t) \leq \mathcal{F}_e^-(x)$.

Example 5. Consider the hypervector space $\mathcal{V} = (\mathbb{R}^3, +, \circ, \mathbb{R})$ as illustrated in Example 1. Let $\mathcal{A} = \{a, b\}$ denote a set of parameters. Then the pair $(\mathcal{F}, \mathcal{A})$ forms a bipolar fuzzy soft hypervector space of $\mathcal{V}$, where the functions "$\mathcal{F}_a^+, \mathcal{F}_b^+ : \mathbb{R}^3 \to [0,1]$" and "$\mathcal{F}_a^-, \mathcal{F}_b^- : \mathbb{R}^3 \to [-1,0]$" are defined as follows:

$$\mathcal{F}_a^+(x,y,z) = \begin{cases} 0.7 & (x,y,z) \in \mathcal{X}, \\ 0.3 & (x,y,z) \in \mathcal{Y}, \\ 0 & otherwise, \end{cases}$$

$$\mathcal{F}_a^-(x,y,z) = \begin{cases} -0.8 & (x,y,z) \in \mathcal{X}, \\ -0.4 & (x,y,z) \in \mathcal{Y}, \\ -0.2 & otherwise, \end{cases}$$

$$\mathcal{F}_b^+(x,y,z) = \begin{cases} 0.9 & (x,y,z) \in \mathcal{X}, \\ 0.4 & (x,y,z) \in \mathcal{Y}, \\ 0.1 & otherwise, \end{cases}$$

$$\mathcal{F}_b^-(x,y,z) = \begin{cases} -0.6 & (x,y,z) \in \mathcal{X}, \\ -0.5 & (x,y,z) \in \mathcal{Y}, \\ -0.1 & otherwise, \end{cases}$$

where $\mathcal{X} = \{0\} \times \{0\} \times \mathbb{R}$ and $\mathcal{Y} = (\mathbb{R} \times \{0\} \times \mathbb{R}) \setminus (\{0\} \times \{0\} \times \mathbb{R})$.

Example 6. Consider the hypervector space $\mathcal{V} = (\mathbb{Z}_4, +, \circ, \mathbb{Z}_2)$ in Example 2. Suppose $\mathcal{A} = \{c, d, e\}$ is a set of parameters. Then the pair $(\mathcal{F}, \mathcal{A})$ is a bipolar fuzzy soft hypervector space of $\mathcal{V}$, where "$\mathcal{F}_c^+, \mathcal{F}_d^+, \mathcal{F}_e^+ : \mathbb{Z}_4 \to [0,1]$" and "$\mathcal{F}_c^-, \mathcal{F}_d^-, \mathcal{F}_e^- : \mathbb{Z}_4 \to [-1,0]$" are given by the followings:

$$\mathcal{F}_c^+(x) = \begin{cases} 0.5 & x \in \{0,2\}, \\ 0.3 & x \in \{1,3\}, \end{cases} \qquad \mathcal{F}_c^-(x) = \begin{cases} -0.4 & x \in \{0,2\}, \\ -0.2 & x \in \{1,3\}, \end{cases}$$

$$\mathcal{F}_d^+(x) = \begin{cases} 0.7 & x \in \{0,2\}, \\ 0.2 & x \in \{1,3\}, \end{cases} \qquad \mathcal{F}_d^-(x) = \begin{cases} -0.6 & x \in \{0,2\}, \\ -0.3 & x \in \{1,3\}, \end{cases}$$

$$\mathcal{F}_e^+(x) = \begin{cases} 0.8 & x \in \{0,2\}, \\ 0.4 & x \in \{1,3\}, \end{cases} \qquad \mathcal{F}_e^-(x) = \begin{cases} -0.7 & x \in \{0,2\}, \\ -0.5 & x \in \{1,3\}. \end{cases}$$

Proposition 4. *Let $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ be bipolar fuzzy soft hypervector spaces of $\mathcal{V} = (\mathcal{V}, +, \circ, \mathcal{K})$. Then the following combinations are bipolar fuzzy soft hypervector spaces of $\mathcal{V}$.*

1. $(\mathcal{F}, \mathcal{A}) \sqcap (\mathcal{G}, \mathcal{B})$,
2. $(\mathcal{F}, \mathcal{A}) \sqcap_\varepsilon (\mathcal{G}, \mathcal{B})$,
3. $(\mathcal{F}, \mathcal{A}) \sqcup (\mathcal{G}, \mathcal{B})$, *if* $\mathcal{A} \cap \mathcal{B} = \emptyset$,
4. $(\mathcal{F}, \mathcal{A}) \sqcup_R (\mathcal{G}, \mathcal{B})$,
5. $(\mathcal{F}, \mathcal{A}) \wedge (\mathcal{G}, \mathcal{B})$.

Proof. We check the conditions of Definition 4, for every item.

1. Let $e \in \mathcal{A} \cap \mathcal{B}$, $x, y \in \mathcal{V}$ and $a \in \mathcal{K}$. Then

$$\begin{aligned}
(\mathcal{F} \sqcap \mathcal{G})^+_e(x-y) &= \min\{\mathcal{F}^+_e(x-y), \mathcal{G}^+_e(x-y)\} \\
&\geq \min\{\left(\mathcal{F}^+_e(x) \wedge \mathcal{F}^+_e(y)\right), \left(\mathcal{G}^+_e(x) \wedge \mathcal{G}^+_e(y)\right)\} \\
&= \min\{\left(\mathcal{F}^+_e(x) \wedge \mathcal{G}^+_e(x)\right), \left(\mathcal{F}^+_e(y) \wedge \mathcal{G}^+_e(y)\right)\} \\
&= \min\{(\mathcal{F} \sqcap \mathcal{G})^+_e(x), (\mathcal{F} \sqcap \mathcal{G})^+_e(y)\},
\end{aligned}$$

$$\begin{aligned}
(\mathcal{F} \sqcap \mathcal{G})^-_e(x-y) &= \max\{\mathcal{F}^-_e(x-y), \mathcal{G}^-_e(x-y)\} \\
&\leq \max\{\left(\mathcal{F}^-_e(x) \vee \mathcal{F}^-_e(y)\right), \left(\mathcal{G}^-_e(x) \vee \mathcal{G}^-_e(y)\right)\} \\
&= \max\{\left(\mathcal{F}^-_e(x) \vee \mathcal{G}^-_e(x)\right), \left(\mathcal{F}^-_e(y) \vee \mathcal{G}^-_e(y)\right)\} \\
&= \max\{(\mathcal{F} \sqcap \mathcal{G})^-_e(x), (\mathcal{F} \sqcap \mathcal{G})^+_e(y)\},
\end{aligned}$$

$$\begin{aligned}
\inf_{t \in a \circ x} (\mathcal{F} \sqcap \mathcal{G})^+_e(t) &= \inf_{t \in a \circ x} \left(\mathcal{F}^+_e(t) \wedge \mathcal{G}^+_e(t)\right) \\
&= \left(\inf_{t \in a \circ x} \mathcal{F}^+_e(t)\right) \wedge \left(\inf_{t \in a \circ x} \mathcal{G}^+_e(t)\right) \\
&\geq \mathcal{F}^+_e(x) \wedge \mathcal{G}^+_e(x) \\
&= (\mathcal{F} \sqcap \mathcal{G})^+_e(x),
\end{aligned}$$

$$\begin{aligned}
\sup_{t \in a \circ x} (\mathcal{F} \sqcap \mathcal{G})^-_e(t) &= \sup_{t \in a \circ x} \left(\mathcal{F}^-_e(t) \vee \mathcal{G}^-_e(t)\right) \\
&= \left(\sup_{t \in a \circ x} \mathcal{F}^+_e(t)\right) \vee \left(\sup_{t \in a \circ x} \mathcal{G}^+_e(t)\right) \\
&\leq \mathcal{F}^-_e(x) \vee \mathcal{G}^-_e(x) \\
&= (\mathcal{F} \sqcap \mathcal{G})^-_e(x).
\end{aligned}$$

2. It is similar to the proof of part (1).

3. It is clear, since

$$(\mathcal{F} \sqcup \mathcal{G})_e = \begin{cases} \mathcal{F}_e & e \in \mathcal{A} \setminus \mathcal{B}, \\ G_e & e \in \mathcal{B} \setminus \mathcal{A}. \end{cases}$$

4. The proof is obvious.

5. For all $(e_1, e_2) \in \mathcal{A} \times \mathcal{B}$, $x, y \in \mathcal{V}$ and $a \in \mathcal{K}$, it follows that:

$$\begin{aligned}
(\mathcal{F} \wedge \mathcal{G})^+_{(e_1,e_2)}(x-y) &= \min\{\mathcal{F}^+_{e_1}(x-y), \mathcal{G}^+_{e_2}(x-y)\} \\
&\geq \min\{\left(\mathcal{F}^+_{e_1}(x) \wedge \mathcal{F}^+_{e_1}(y)\right), \left(\mathcal{G}^+_{e_2}(x) \wedge \mathcal{G}^+_{e_2}(y)\right)\} \\
&= \min\{\left(\mathcal{F}^+_{e_1}(x) \wedge \mathcal{G}^+_{e_2}(x)\right), \left(\mathcal{F}^+_{e_1}(y) \wedge \mathcal{G}^+_{e_2}(y)\right)\} \\
&= \min\{(\mathcal{F} \wedge \mathcal{G})^+_{(e_1,e_2)}(x), (\mathcal{F} \wedge \mathcal{G})^+_{(e_1,e_2)}(y)\},
\end{aligned}$$

$$(\mathcal{F} \wedge \mathcal{G})^-_{(e_1,e_2)}(x-y) = \max\{\mathcal{F}^-_{e_1}(x-y), \mathcal{G}^-_{e_2}(x-y)\}$$

$$
\begin{aligned}
&\leq \max\{\left(\mathcal{F}^-_{e_1}(x) \vee \mathcal{F}^-_{e_1}(y)\right), \left(\mathcal{G}^-_{e_2}(x) \vee \mathcal{G}^-_{e_2}(y)\right)\} \\
&= \max\{\left(\mathcal{F}^-_{e_1}(x) \vee \mathcal{G}^-_{e_2}(x)\right), \left(\mathcal{F}^-_{e_1}(y) \vee \mathcal{G}^-_{e_2}(y)\right)\} \\
&= \max\{(\mathcal{F} \wedge \mathcal{G})^-_{(e_1,e_2)}(x), (\mathcal{F} \wedge \mathcal{G})^-_{(e_1,e_2)}(y)\},
\end{aligned}
$$

$$
\begin{aligned}
\inf_{t \in a \circ x} (\mathcal{F} \wedge \mathcal{G})^+_{(e_1,e_2)}(t) &= \inf_{t \in a \circ x} \left(\mathcal{F}^+_{e_1}(t) \wedge \mathcal{G}^+_{e_2}(t)\right) \\
&= \left(\inf_{t \in a \circ x} \mathcal{F}^+_{e_1}(t)\right) \wedge \left(\inf_{t \in a \circ x} \mathcal{G}^+_{e_2}(t)\right) \\
&\geq \mathcal{F}^+_{e_1}(x) \wedge \mathcal{G}^+_{e_2}(x) \\
&= (\mathcal{F} \wedge \mathcal{G})^+_{(e_1,e_2)}(x),
\end{aligned}
$$

$$
\begin{aligned}
\sup_{t \in a \circ x} (\mathcal{F} \wedge \mathcal{G})^-_{(e_1,e_2)}(t) &= \sup_{t \in a \circ x} \left(\mathcal{F}^-_{e_1}(t) \vee \mathcal{G}^-_{e_2}(t)\right) \\
&= \left(\sup_{t \in a \circ x} \mathcal{F}^-_{e_1}(t)\right) \vee \left(\sup_{t \in a \circ x} \mathcal{G}^-_{e_2}(t)\right) \\
&\leq \mathcal{F}^-_{e_1}(x) \vee \mathcal{G}^-_{e_2}(x) \\
&= (\mathcal{F} \wedge \mathcal{G})^-_{(e_1,e_2)}(x).
\end{aligned}
$$

□

Proposition 5. *The sum of two bipolar fuzzy soft hypervector spaces $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ of $\mathcal{V} = (\mathcal{V}, +, \circ, \mathcal{K})$, i.e. $(\mathcal{F}, \mathcal{A}) + (\mathcal{G}, \mathcal{B})$, is a bipolar fuzzy soft hypervector space of $\mathcal{V}$.*

Proof. Consider $e \in \mathcal{A} \cap \mathcal{B}$, $x, y \in \mathcal{V}$, $a \in \mathcal{K}$ and

$$\mathcal{A}_x = \{(x_1, x_2) \in \mathcal{V} \times \mathcal{V};\ x = x_1 + x_2\},$$

$$\mathcal{A}_y = \{(y_1, y_2) \in \mathcal{V} \times \mathcal{V};\ y = y_1 + y_2\}.$$

Hence

1) If $\mathcal{A}_x = \emptyset$ or $\mathcal{A}_y = \emptyset$, then obviously $(\mathcal{F} + \mathcal{G})^+_e(x - y) \geq (\mathcal{F} + \mathcal{G})^+_e(x) \wedge (\mathcal{F} + \mathcal{G})^+_e(y)$. Otherwise, if $\mathcal{A}_x \neq \emptyset$ or $\mathcal{A}_y \neq \emptyset$, then there exist $x_1, x_2, y_1, y_2 \in \mathcal{V}$, such that $x = x_1 + x_2$ and $y = y_1 + y_2$. Thus

$$\mathcal{A}_{x-y} = \{(t_1, t_2) \in \mathcal{V} \times \mathcal{V};\ x - y = t_1 + t_2\} \neq \emptyset,$$

and so

$$
\begin{aligned}
(\mathcal{F} + \mathcal{G})^+_e(x - y) &= \sup_{x-y=t_1+t_2} \left(\mathcal{F}^+_e(t_1) \wedge \mathcal{G}^+_e(t_2)\right) \\
&\geq \mathcal{F}^+_e(x_1 - y_1) \wedge \mathcal{G}^+_e(x_2 - y_2) \\
&\geq \left(\mathcal{F}^+_e(x_1) \wedge \mathcal{F}^+_e(y_1)\right) \wedge \left(\mathcal{G}^+_e(x_2) \wedge \mathcal{G}^+_e(y_2)\right) \\
&= \left(\mathcal{F}^+_e(x_1) \wedge \mathcal{G}^+_e(x_2)\right) \wedge \left(\mathcal{F}^+_e(y_1) \wedge \mathcal{G}^+_e(y_2)\right).
\end{aligned}
$$

Hence,

$$\begin{aligned}(\mathcal{F}+\mathcal{G})^+_e(x-y) &\geq \left(\sup_{x=x_1+x_2}\left(\mathcal{F}^+_e(x_1)\wedge\mathcal{G}^+_e(x_2)\right)\right)\\ &\quad\wedge\left(\sup_{y=y_1+y_2}\left(\mathcal{F}^+_e(y_1)\wedge\mathcal{G}^+_e(y_2)\right)\right)\\ &= (\mathcal{F}+\mathcal{G})^+_e(x)\wedge(\mathcal{F}+\mathcal{G})^+_e(y).\end{aligned}$$

For the negative part, if $\mathcal{A}_x=\emptyset$ or $\mathcal{A}_y=\emptyset$, then clearly $(\mathcal{F}+\mathcal{G})^-_e(x-y)\leq(\mathcal{F}+\mathcal{G})^-_e(x)\vee(\mathcal{F}+\mathcal{G})^-_e(y)$. Otherwise, if $\mathcal{A}_x\neq\emptyset$ or $\mathcal{A}_y\neq\emptyset$, then similar to the previous part, it follows that

$$\begin{aligned}(\mathcal{F}+\mathcal{G})^-_e(x-y) &= \inf_{x-y=t_1+t_2}\left(F^-_e(t_1)\vee\mathcal{G}^-_e(t_2)\right)\\ &\leq \mathcal{F}^-_e(x_1-y_1)\vee\mathcal{G}^-_e(x_2-y_2)\\ &\leq \left(\mathcal{F}^-_e(x_1)\vee\mathcal{F}^-_e(y_1)\right)\vee\left(\mathcal{G}^-_e(x_2)\vee\mathcal{G}^-_e(y_2)\right)\\ &= \left(\mathcal{F}^-_e(x_1)\vee\mathcal{G}^-_e(x_2)\right)\vee\left(\mathcal{F}^-_e(y_1)\vee\mathcal{G}^+_e(y_2)\right),\end{aligned}$$

and so,

$$\begin{aligned}(\mathcal{F}+\mathcal{G})^-_e(x-y) &\leq \left(\inf_{x=x_1+x_2}\left(\mathcal{F}^-_e(x_1)\vee\mathcal{G}^-_e(x_2)\right)\right)\\ &\quad\vee\left(\inf_{y=y_1+y_2}\left(\mathcal{F}^-_e(y_1)\vee\mathcal{G}^-_e(y_2)\right)\right)\\ &= (\mathcal{F}+\mathcal{G})^-_e(x)\vee(\mathcal{F}+\mathcal{G})^-_e(y).\end{aligned}$$

2) If $\mathcal{A}_x=\emptyset$, then clearly $\inf\limits_{t\in a\circ x}(\mathcal{F}+\mathcal{G})^+_e(t)\geq(\mathcal{F}+\mathcal{G})^+_e(x)$. If $\mathcal{A}_x\neq\emptyset$, and $x=x_1+x_2$, for some $x_1,x_2\in\mathcal{V}$, then $t\in a\circ(x_1+x_2)\subseteq a\circ x_1+a\circ x_2$, for all $t\in a\circ x$. Thus $t=\acute{t}_1+\acute{t}_2$, for some $\acute{t}_1\in a\circ x_1$, $\acute{t}_2\in a\circ x_2$. Hence

$$\begin{aligned}\sup_{t=t_1+t_2}\left(\mathcal{F}^+_e(t_1)\wedge\mathcal{G}^+_e(t_2)\right) &\geq \mathcal{F}^+_e(\acute{t}_1)\wedge\mathcal{G}^+_e(\acute{t}_2)\\ &\geq \left(\inf_{r_1\in a\circ x_1}\mathcal{F}^+_e(r_1)\right)\wedge\left(\inf_{r_2\in a\circ x_2}\mathcal{G}^+_e(r_2)\right)\\ &\geq \mathcal{F}^+_e(x_1)\wedge\mathcal{G}^+_e(x_2).\end{aligned}$$

It follows that,

$$(\mathcal{F}+\mathcal{G})^+_e(t)\geq\sup_{x=x_1+x_2}\left(\mathcal{F}^+_e(x_1)\wedge\mathcal{G}^+_e(x_2)\right)=(\mathcal{F}+\mathcal{G})^+_e(x),$$

and so, $\inf\limits_{t\in a\circ x}(\mathcal{F}+\mathcal{G})^+_e(t)\geq(\mathcal{F}+\mathcal{G})^+_e(x)$.

Now, for the negative part, if $\mathcal{A}_x=\emptyset$, then obviously $\sup\limits_{t\in a\circ x}(\mathcal{F}+\mathcal{G})^-_e(t)\leq(\mathcal{F}+\mathcal{G})^-_e(x)$, and if $\mathcal{A}_x\neq\emptyset$ and $x_1,x_2\in\mathcal{V}$ such that $x=x_1+x_2$, then for all $t\in a\circ x$, $t=\acute{t}_1+\acute{t}_2$, for some $\acute{t}_1\in a\circ x_1$ and $\acute{t}_2\in a\circ x_2$. Then

$$(\mathcal{F}+\mathcal{G})^-_e(t) = \inf_{t=t_1+t_2}\left(\mathcal{F}^-_e(t_1)\vee\mathcal{G}^-_e(t_2)\right)$$

$$
\begin{aligned}
&\leq \mathcal{F}_e^-(\acute{t}_1) \vee \mathcal{G}_e^-(\acute{t}_2)\\
&\leq \left(\sup_{s_1 \in a\circ x_1} \mathcal{F}_e^-(s_1)\right) \vee \left(\sup_{s_2 \in a\circ x_2} \mathcal{G}_e^-(s_2)\right)\\
&\leq \mathcal{F}_e^-(x_1) \vee \mathcal{G}_e^-(x_2).
\end{aligned}
$$

Thus $(\mathcal{F}+\mathcal{G})_e^-(t) \leq \inf_{x=x_1+x_2} (\mathcal{F}_e^-(x_1) \vee \mathcal{G}_e^-(x_2)) = (\mathcal{F}+\mathcal{G})_e^-(x)$, and so $\sup_{t\in a\circ x} (\mathcal{F}+\mathcal{G})_e^-(t) \leq (\mathcal{F}+\mathcal{G})_e^-(x)$.
Therefore, the proof is completed. □

Corollary 1. *The extended sum of two bipolar fuzzy soft hypervector spaces $(\mathcal{F},\mathcal{A})$ and $(\mathcal{G},\mathcal{B})$ of $\mathcal{V} = (\mathcal{V},+,\circ,\mathcal{K})$, i.e. $(\mathcal{F},\mathcal{A}) +_\varepsilon (\mathcal{G},\mathcal{B})$, is a bipolar fuzzy soft hypervector space of $\mathcal{V}$.*

Proof. It is obvious, by Proposition 5. □

Proposition 6. *For all $a \in \mathcal{K}$, the scalar product of a bipolar fuzzy soft hypervector space $(\mathcal{F},\mathcal{A})$ over $\mathcal{V} = (\mathcal{V},+,\circ,\mathcal{K})$, i.e. $(a\circ\mathcal{F},\mathcal{A})$, is a bipolar fuzzy soft hypervector space of $\mathcal{V}$, if $\mathcal{V}$ is strongly right distributive.*

Proof. Let $e \in \mathcal{A}$, $x, y \in \mathcal{V}$ and $b \in \mathcal{K}$. Then

1. If there does not exist $t_1 \in \mathcal{V}$ such that $x \in a\circ t_1$, or there does not exist $t_2 \in \mathcal{V}$ such that $y \in a \circ t_2$, then it is clear that

$$(a\circ\mathcal{F})_e^+(x-y) \geq \min\{(a\circ\mathcal{F})_e^+(x), (a\circ\mathcal{F})_e^+(y)\},$$

and

$$(a\circ\mathcal{F})_e^-(x-y) \leq \max\{(a\circ\mathcal{F})_e^-(x), (a\circ\mathcal{F})_e^-(y)\}.$$

But, if $x \in a\circ t_1$ and $y \in a\circ t_2$, for some $t_1, t_2 \in \mathcal{V}$, then $x - y \in a\circ t_1 - a\circ t_2 = a\circ(t_1 - t_2)$, and so

$$
\begin{aligned}
(a\circ\mathcal{F})_e^+(x-y) &= \sup_{x-y\in a\circ t} \mathcal{F}_e^+(t)\\
&\geq \mathcal{F}_e^+(t_1 - t_2)\\
&\geq \mathcal{F}_e^+(t_1) \wedge \mathcal{F}_e^+(t_2),
\end{aligned}
$$

thus

$$
\begin{aligned}
(a\circ\mathcal{F})_e^+(x-y) &\geq \left(\sup_{x\in a\circ t_1} \mathcal{F}_e^+(t_1)\right) \wedge \left(\sup_{y\in a\circ t_2} \mathcal{F}_e^+(t_2)\right)\\
&= \min\{(a\circ\mathcal{F})_e^+(x), (a\circ\mathcal{F})_e^+(y)\}.
\end{aligned}
$$

Also, $(a\circ\mathcal{F})_e^-(x-y) = \inf_{x-y\in a\circ t} \mathcal{F}_e^-(t) \leq \mathcal{F}_e^-(t_1 - t_2) \leq \mathcal{F}_e^-(t_1) \vee \mathcal{F}_e^-(t_2)$, hence

$$
\begin{aligned}
(a\circ\mathcal{F})_e^-(x-y) &\leq \left(\inf_{x\in a\circ t_1} \mathcal{F}_e^-(t_1)\right) \vee \left(\inf_{y\in a\circ t_2} \mathcal{F}_e^-(t_2)\right)\\
&= \max\{(a\circ\mathcal{F})_e^-(x), (a\circ\mathcal{F})_e^-(y)\}.
\end{aligned}
$$

2. If there does not exist $t \in \mathcal{V}$ such that $x \in a \circ t$, then $(a \circ \mathcal{F})^+_e(x) = (a \circ \mathcal{F})^-_e(x) = 0$ and clearly the result is obtained. Otherwise, if $x \in a \circ t$, for some $t \in \mathcal{V}$, then for all $s \in b \circ x$ and $r \in a \circ s$, it follows that:

$$\begin{aligned}\mathcal{F}^+_e(r) &\geq \inf_{l \in a \circ s} \mathcal{F}^+_e(l) \geq \mathcal{F}^+_e(s) \geq \inf_{k \in b \circ x} \mathcal{F}^+_e(k) \\ &\geq \mathcal{F}^+_e(x) \geq \inf_{p \in a \circ t} \mathcal{F}^+_e(p) \geq \mathcal{F}^+_e(t).\end{aligned}$$

Thus $\mathcal{F}^+_e(r) \geq \sup_{x \in a \circ t} \mathcal{F}^+_e(t)$, and so $\sup_{r \in a \circ s} \mathcal{F}^+_e(r) \geq \sup_{x \in a \circ t} \mathcal{F}^+_e(t)$. Hence

$$\begin{aligned}\inf_{s \in b \circ x} (a \circ \mathcal{F})^+_e(s) &= \inf_{s \in b \circ x} \left(\sup_{r \in a \circ s} \mathcal{F}^+_e(r) \right) \\ &\geq \sup_{x \in a \circ t} \mathcal{F}^+_e(t) \\ &= (a \circ \mathcal{F})^+_e(x).\end{aligned}$$

Moreover,

$$\begin{aligned}\mathcal{F}^-_e(r) &\leq \sup_{\acute{l} \in a \circ s} \mathcal{F}^-_e(\acute{l}) \leq \mathcal{F}^-_e(s) \leq \sup_{\acute{k} \in b \circ x} \mathcal{F}^-_e(\acute{k}) \\ &\leq \mathcal{F}^-_e(x) \leq \sup_{\acute{p} \in a \circ t} \mathcal{F}^-_e(\acute{p}) \leq \mathcal{F}^-_e(t).\end{aligned}$$

Thus $\mathcal{F}^-_e(r) \leq \inf_{x \in a \circ t} \mathcal{F}^-_e(t)$, and so $\inf_{r \in a \circ s} \mathcal{F}^-_e(r) \leq \inf_{x \in a \circ t} \mathcal{F}^-_e(t)$. Hence

$$\begin{aligned}\sup_{s \in b \circ x} (a \circ \mathcal{F})^-_e(s) &= \sup_{s \in b \circ x} \left(\inf_{r \in a \circ s} \mathcal{F}^-_e(r) \right) \\ &\leq \inf_{x \in a \circ t} \mathcal{F}^-_e(t) \\ &= (a \circ \mathcal{F})^-_e(x).\end{aligned}$$

Therefore, the proof is completed. □

5 Bipolar fuzzy soft hypervector spaces under linear transformations

In this part, the analysis focuses on investigating the impact of linear transformations on the characteristics of bipolar fuzzy soft hypervector spaces through the expansion of the fuzzy sets extension principle to bipolar fuzzy soft sets.

Definition 5. *Suppose $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ denote bipolar fuzzy soft sets over $\mathcal{X}$ and $\mathcal{Y}$, respectively. A fuzzy soft function from $\mathcal{X}$ to $\mathcal{Y}$ is defined by a pair (ϕ, f), where $\phi : \mathcal{X} \to \mathcal{Y}$ is a function from $\mathcal{X}$ to $\mathcal{Y}$ and $f : \mathcal{A} \to \mathcal{B}$ is a function from $\mathcal{A}$ to $\mathcal{B}$. In this case, the image of $(\mathcal{F}, \mathcal{A})$ under (ϕ, f) is the bipolar fuzzy soft set $(\phi, f)(\mathcal{F}, \mathcal{A}) = (\phi(\mathcal{F}), f(\mathcal{A}))$ of $\mathcal{Y}$ defined by:*

$$\phi(\mathcal{F})^+_u(y) = \begin{cases} \sup_{x \in \phi^{-1}(y)} \sup_{e \in f^{-1}(u)} \mathcal{F}^+_e(x) & \phi^{-1}(y) \neq \emptyset, \\ 0 & \phi^{-1}(y) = \emptyset, \end{cases}$$

and

$$\phi(\mathcal{F})^-_u(y) = \begin{cases} \inf\limits_{x\in\phi^{-1}(y)} \inf\limits_{e\in f^{-1}(u)} \mathcal{F}^-_e(x) & \phi^{-1}(y) \neq \emptyset, \\ 0 & \phi^{-1}(y) = \emptyset, \end{cases}$$

for all $u \in f(\mathcal{A})$, $y \in \mathcal{Y}$.
Also, pre-image of $(\mathcal{G}, \mathcal{B})$ *under* (ϕ, f) *is the bipolar fuzzy soft set* $(\phi, f)^{-1}(\mathcal{G}, \mathcal{B}) = (\phi^{-1}(\mathcal{G}), \mathcal{A})$ *of* $\mathcal{X}$*, such that* $\phi^{-1}(\mathcal{G})^+_e(x) = \mathcal{G}^+_{f(e)}(\phi(x))$ *and* $\phi^{-1}(\mathcal{G})^-_e(x) = \mathcal{G}^-_{f(e)}(\phi(x))$*, for all* $e \in \mathcal{A}$, $x \in \mathcal{X}$.

Definition 6. *A fuzzy soft function* $(T : \mathcal{V} \to \mathcal{W},\ f : \mathcal{A} \to \mathcal{B})$ *between hypervector spaces* $\mathcal{V}$ *and* $\mathcal{W}$ *is said to be a bipolar fuzzy soft linear transformation if* T *is a linear transformation, i.e.* $T(x+y) = T(x)+T(y)$ *and* $T(a \circ x) \subseteq a \circ T(x)$*, for all* $x, y \in \mathcal{V}$, $a \in \mathcal{K}$*. If* $T(a \circ x) = a \circ T(x)$*, then* (T, f) *is said to be a bipolar fuzzy soft good transformation.*

Theorem 1. *Let* $(T : \mathcal{V} \to \mathcal{W},\ f : \mathcal{A} \to \mathcal{B})$ *be a bipolar fuzzy soft good transformation. If* $(\mathcal{F}, \mathcal{A})$ *is a bipolar fuzzy soft hypervector space of* $\mathcal{V}$*, then* $(T, f)(\mathcal{F}, \mathcal{A})$ *is a bipolar fuzzy soft hypervector space of* $\mathcal{W}$.

Proof. Let $u \in f(\mathcal{A})$, $\acute{x}, \acute{y} \in \mathcal{W}$, $a \in K$. If $T^{-1}(\acute{x}) = \emptyset$ or $T^{-1}(\acute{y}) = \emptyset$, then the proof is clear. If $T^{-1}(\acute{x}) \neq \emptyset$ and $T^{-1}(\acute{y}) \neq \emptyset$, then $T^{-1}(\acute{x} - \acute{y}) \neq \emptyset$ and it follows that:

$$\begin{aligned} T(\mathcal{F})^+_u(\acute{x} - \acute{y}) &= \sup_{T(t)=\acute{x}-\acute{y}} \sup_{f(e)=u} \mathcal{F}^+_e(t) \\ &\geq \sup_{T(x)=\acute{x}, T(y)=\acute{y}} \sup_{f(e)=u} \mathcal{F}^+_e(x-y) \\ &\geq \sup_{T(x)=\acute{x}, T(y)=\acute{y}} \sup_{f(e)=u} \big(\mathcal{F}^+_e(x) \wedge \mathcal{F}^+_e(y)\big) \\ &= \left(\sup_{T(x)=\acute{x}} \sup_{f(e)=u} \mathcal{F}^+_e(x)\right) \wedge \left(\sup_{T(y)=\acute{y}} \sup_{f(e)=u} \mathcal{F}^+_e(y)\right) \\ &= \min\{T(\mathcal{F})^+_u(\acute{x}), T(\mathcal{F})^+_u(\acute{y})\}, \end{aligned}$$

and

$$\begin{aligned} T(\mathcal{F})^-_u(\acute{x} - \acute{y}) &= \inf_{T(t)=\acute{x}-\acute{y}} \inf_{f(e)=u} \mathcal{F}^-_e(t) \\ &\leq \inf_{T(x)=\acute{x}, T(y)=\acute{y}} \inf_{f(e)=u} \mathcal{F}^-_e(x-y) \\ &\leq \inf_{T(x)=\acute{x}, T(y)=\acute{y}} \inf_{f(e)=u} \big(\mathcal{F}^-_e(x) \vee \mathcal{F}^-_e(y)\big) \\ &= \left(\inf_{T(x)=\acute{x}} \inf_{f(e)=u} \mathcal{F}^-_e(x)\right) \vee \left(\inf_{T(y)=\acute{y}} \inf_{f(e)=u} \mathcal{F}^-_e(y)\right) \\ &= \max\{T(\mathcal{F})^-_u(\acute{x}), T(\mathcal{F})^-_u(\acute{y})\}. \end{aligned}$$

Moreover, if $T^{-1}(\acute{x}) = \emptyset$, then the proof is obvious. If $T^{-1}(\acute{x}) \neq \emptyset$ and $T(x) = \acute{x}$, for some $x \in \mathcal{V}$, then for any $\acute{t} \in a \circ \acute{x}$, $\acute{t} \in a \circ T(x) = T(a \circ x)$, so $\acute{t} = T(t)$, for some $t \in a \circ x$. Thus $T^{-1}(\acute{t}) \neq \emptyset$ and it follows that:

$$T(\mathcal{F})^+_u(\acute{t}) = \sup_{T(s)=\acute{t}} \sup_{f(e)=u} \mathcal{F}^+_e(s)$$

$$
\begin{aligned}
&= \left(\sup_{s\in a\circ x, T(s)=\acute{t}}\ \sup_{f(e)=u} \mathcal{F}_e^+(s)\right) \\
&\quad \vee \left(\sup_{s\in V\setminus a\circ x, T(s)=\acute{t}}\ \sup_{f(e)=u} \mathcal{F}_e^+(s)\right) \\
&\geq \sup_{f(e)=u}\ \sup_{s\in a\circ x, T(s)=\acute{t}} \mathcal{F}_e^+(s) \\
&\geq \sup_{f(e)=u}\ \inf_{s\in a\circ x, T(s)=\acute{t}} \mathcal{F}_e^+(s) \\
&\geq \sup_{f(e)=u}\ \inf_{s\in a\circ x} \mathcal{F}_e^+(s) \\
&\geq \sup_{f(e)=u} \mathcal{F}_e^+(x),
\end{aligned}
$$

Hence, $T(\mathcal{F})_u^+(\acute{t}) \geq \sup_{T(x)=\acute{x}}\ \sup_{f(e)=u} \mathcal{F}_e^+(x) = T(\mathcal{F})_u^+(\acute{x})$, and so $\inf_{\acute{t}\in a\circ\acute{x}} T(\mathcal{F})_u^+(\acute{t}) \geq T(\mathcal{F})_u^+(\acute{x})$.

Moreover,

$$
\begin{aligned}
T(\mathcal{F})_u^-(\acute{t}) &= \inf_{T(s)=\acute{t}}\ \inf_{f(e)=u} \mathcal{F}_e^-(s) \\
&= \left(\inf_{s\in a\circ x, T(s)=\acute{t}}\ \inf_{f(e)=u} \mathcal{F}_e^-(s)\right) \\
&\quad \wedge \left(\inf_{s\in V\setminus a\circ x, T(s)=\acute{t}}\ \inf_{f(e)=u} \mathcal{F}_e^-(s)\right) \\
&\leq \inf_{f(e)=u}\ \inf_{s\in a\circ x, T(s)=\acute{t}} \mathcal{F}_e^-(s) \\
&\leq \inf_{f(e)=u}\ \sup_{s\in a\circ x, T(s)=\acute{t}} \mathcal{F}_e^-(s) \\
&\leq \inf_{f(e)=u}\ \sup_{s\in a\circ x} \mathcal{F}_e^-(s) \\
&\leq \inf_{f(e)=u} \mathcal{F}_e^-(x).
\end{aligned}
$$

Hence,

$$
T(\mathcal{F})_u^-(\acute{t}) \leq \inf_{T(x)=\acute{x}}\ \inf_{f(e)=u} \mathcal{F}_e^-(x) = T(\mathcal{F})_u^-(\acute{x}),
$$

and so, $\sup_{\acute{t}\in a\circ\acute{x}} T(\mathcal{F})_u^-(\acute{t}) \leq T(\mathcal{F})_u^-(\acute{x})$.

Therefore, by Definition 4, $(T, f)(\mathcal{F}, \mathcal{A})$ is a bipolar fuzzy soft hypervector space of $\mathcal{W}$. □

Theorem 2. *Let $(T : \mathcal{V} \to \mathcal{W},\ f : \mathcal{A} \to \mathcal{B})$ be a bipolar fuzzy soft linear transformation. If $(\mathcal{G}, \mathcal{B})$ is a bipolar fuzzy soft hypervector space of $\mathcal{W}$, then $(T, f)^{-1}(\mathcal{G}, \mathcal{B})$ is a bipolar fuzzy soft hypervector space of $\mathcal{V}$.*

Proof. Let $e \in \mathcal{A}$, $x, y \in \mathcal{V}$, and $a \in \mathcal{K}$. Then

$$
T^{-1}(\mathcal{G})_e^+(x-y) = \mathcal{G}_{f(e)}^+(T(x-y))
$$

$$
\begin{aligned}
&= \mathcal{G}^+_{f(e)}(T(x)-T(y)) \\
&\geq \min\{\mathcal{G}^+_{f(e)}(T(x)), \mathcal{G}^+_{f(e)}(T(y))\} \\
&= \min\{T^{-1}(\mathcal{G})^+_e(x), T^{-1}(\mathcal{G})^+_e(y)\},
\end{aligned}
$$

and

$$
\begin{aligned}
T^{-1}(\mathcal{G})^-_e(x-y) &= \mathcal{G}^-_{f(e)}(T(x-y)) \\
&= \mathcal{G}^-_{f(e)}(T(x)-T(y)) \\
&\leq \max\{\mathcal{G}^-_{f(e)}(T(x)), \mathcal{G}^-_{f(e)}(T(y))\} \\
&= \max\{T^{-1}(\mathcal{G})^-_e(x), T^{-1}(\mathcal{G})^-_e(y)\}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\inf_{t\in a\circ x} T^{-1}(\mathcal{G})^+_e(t) &= \inf_{t\in a\circ x} \mathcal{G}^+_{f(e)}(T(t)) \\
&\geq \inf_{\acute{t}\in a\circ T(x)} \mathcal{G}^+_{f(e)}(\acute{t}) \\
&\geq \mathcal{G}^+_{f(e)}(T(x)) \\
&= T^{-1}(\mathcal{G})^+_e(x),
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{t\in a\circ x} T^{-1}(\mathcal{G})^-_e(t) &= \sup_{t\in a\circ x} \mathcal{G}^-_{f(e)}(T(t)) \\
&\leq \sup_{\acute{t}\in a\circ T(x)} \mathcal{G}^-_{f(e)}(\acute{t}) \\
&\leq \mathcal{G}^-_{f(e)}(T(x)) \\
&= T^{-1}(\mathcal{G})^-_e(x).
\end{aligned}
$$

Therefore, by Definition 4, $(T, f)^{-1}(\mathcal{G}, \mathcal{B})$ is a bipolar fuzzy soft hypervector space of $\mathcal{V}$. □

6 Conclusion

Various methods are known for mathematical modeling of imprecise phenomena. Two of these methods are the use of soft sets and bipolar fuzzy sets. Combining these two concepts with each other, i.e. bipolar fuzzy soft sets, leads to more accurate modeling of the discussed concepts. By defining different operations on the discussed set, an algebraic structure is obtained. The article discusses the utilization of bipolar fuzzy soft sets in examining the algebraic structure of hypervector spaces. By introducing the concept of bipolar fuzzy soft hypervector spaces alongside illustrative examples, the study delves into some fundamental aspects and paves the way for future investigations. Based on this information, bipolar fuzzy soft hypervector spaces have several potential motivations and applications in various fields, for future research topics include:

- Equivalent conditions of fuzzy bipolar soft hypervector space,
- Bipolar fuzzy soft hypervector spaces generated by a bipolar fuzzy soft set,
- Normal bipolar fuzzy soft hypervector spaces,
- Cosets of bipolar fuzzy soft hypervector spaces,
- Bipolar fuzzy soft sets on quotient hypervector spaces,
- Finding the applications of the introduced structure, for examples in decision making, pattern recognition, information retrieval, data mining and analysis, artificial intelligence and machine learning,
- Application of bipolar fuzzy soft sets over other algebraic structures/hyperstructures.

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